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Smooth Points in Marcinkiewicz Function Spaces

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We consider the Marcinkiewicz function space M_W and its subspace of order continuous elements M_W^0. We provide criteria for a function in the unit ball of M_W or M_W^0 to be a smooth point.

The description of smooth points in specific spaces have been of interest to many authors since the Banach spaces were born [1]. It is a part of basic knowledge of the isometric structure of Banach spaces with many applications to best approximation, isometries, optimization, projections or local geometry [4, 8, 10, 14, 15]. Let us mention for instance that characterization of smooth points in Lorentz spaces have been done in [4], in Orlicz spaces in [5], in Musielak-Orlicz spaces in [3], in Orlicz-Lorentz spaces in [13], or in the Lorentz spaces $\Gamma_{p,w}$ in [6]. The smooth points in both Lorentz and Marcinkiewicz sequence spaces with decreasing weight have been studied in [10].

In this note, our goal is to characterize the smooth points of the unit ball in Marcinkiewicz function spaces. We will consider here only the case of a decreasing weight function.

Marcinkiewicz and Lorentz spaces play an important role in the theory of Banach spaces. They are key objects in the interpolation theory of linear operators [2, 12]. Marcinkiewicz spaces go back to the theorem on weak type operators [14, Th. 2.b.15] originally due to J. Marcinkiewicz in the 1930-ties.

We will start by agreeing on some notations. Let L^0 be the set of all real-valued $|\cdot|$-measurable functions defined on $(0,\infty)$, where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}$. The *distribution function* d_f of a function $f\in L^0$ is given by $d_f(\lambda)=|\{t>0 : |f(t)|>\lambda\}|$, for all $\lambda\geq 0$. For $f\in L^0$ we define its *decreasing rearrangement* as $f^*(t)=\inf\{s>0 : d_f(s)\leq t\}$, $t>0$. The functions d_f and f^* are right-continuous on $(0,\infty)$. As usual by $f\wedge g$ we denote the essential minimum of $f,g\in L^0$.

A positive decreasing function $w\in L^0$ is called the *weight function* whenever $\lim_{t\to 0^+} w(t)=\infty$, $\lim_{t\to\infty} w(t)=0$, $W(t)=\int_0^t w<\infty$ for all $t>0$, and $W(\infty)=\int_0^\infty w=\infty$.

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The *Marcinkiewicz space* M_W [9, 12] is the space of all functions $f \in L^0$ satisfying

$$\|f\|_W = \sup_{t>0} \frac{\int_0^t f^*(s)ds}{W(t)} < \infty.$$

We also define

$$M_W^0 = \left\{ f \in M_W : \lim_{t\to 0^+,\infty} \frac{\int_0^t f^*}{W(t)} = 0 \right\}.$$

The space M_W equipped with the norm $\|\cdot\|_W$ is a Banach space. The subspace M_W^0 is closed in M_W and it is the subspace of all order continuous elements of M_W which also coincides with the closure of all bounded functions of finite measure supports [9, Theorem 2.3]. It is also well known that M_W^0 is an M-ideal in M_W [9, Theorem 2.4].

Recall that the *Lorentz space* $\Lambda_{1,w}$ is a subset of L^0 such that

$$\|f\|_{1,w} := \int_0^\infty f^* w = \int_0^\infty f^*(t)w(t)dt < \infty.$$

The space $(\Lambda_{1,w}, \|\cdot\|_{1,w})$ is isomorphically isometric to the dual of M_W^0 [12, Theorem 5.4]. The functionals on M_W induced by elements from $\Lambda_{1,w}$ are called *regular*, while the functionals that vanish on M_W^0 are called *singular*. By the M-ideal property of M_W^0 in M_W, every functional $\phi \in (M_W)^*$ has a unique representation $\phi = \psi + \xi$, where $\psi \in \Lambda_{1,w}$, ξ is singular, and $\|\phi\| = \|\psi\|_{1,w} + \|\xi\|$.

Given a Banach space $(X, \|\cdot\|)$, we will denote by S_X and B_X respectively, the unit sphere and the unit ball of the space. Recall that $x \in B_X$ is a smooth point of the ball B_X if x has a unique norm-one supporting functional, that is there is a unique $\phi \in X^*$ such that $\phi(x) = \|x\|$ or alternately $\|\phi\| = \phi(x) = 1$.

The next two theorems are our main results characterizing smooth points in Marcinkiewicz spaces M_W^0 and M_W.

Theorem 1. *Let $f \in S_{M_W^0}$. Then f is a smooth point in M_W^0 if and only if there exists a unique $0 < a < \infty$ such that*

$$1 = \|f\|_W = \frac{\int_0^a f^*}{W(a)}. \tag{1}$$

Theorem 2. *A function $f \in S_{M_W}$ is a smooth point in M_W if and only if*

$$\limsup_{t\to 0} \frac{\int_0^t f^*}{W(t)} < 1 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{\int_0^t f^*}{W(t)} < 1,$$

and there exists a unique $a \in (0,\infty)$ such that

$$\frac{\int_0^a f^*}{W(a)} = 1.$$

In order to prove these theorems we need several lemmas and propositions.

Lemma 3. *Let $f \in S_{M_W}$. If*

$$\limsup_{t\to 0} \frac{\int_0^t f^*}{W(t)} = 1 \quad or \quad \limsup_{t\to\infty} \frac{\int_0^t f^*}{W(t)} = 1$$

then there exists a decomposition $f = f_1 + f_2$ such that $|f_1| \wedge |f_2| = 0$ and $\|f_1\|_W = \|f_2\|_W = 1$.

Proof. *Case 1.* Assume that $t_n \to \infty$ and $\lim_{n\to\infty} \frac{\int_0^{t_n} f^*}{W(t_n)} = 1$. Without loss of generality we can assume that (t_n) is an increasing sequence.

We claim that there exist a sequence of sets (F_n) and a sequence of positive numbers (s_n) such that $F_i \cap F_j = \emptyset$, for all $i \neq j$, $|F_n| \leq s_n$ and for all $n \in \mathbb{N}$,

$$\frac{\int_{F_n} |f|}{W(s_n)} \geq 1 - \frac{1}{2^n}.$$

By [2, Lemma 2.5], we can find a sequence of sets (E_n) such that $E_n \subset E_{n+1}$, $|E_n| = t_n$ and $\int_0^{t_n} f^* = \int_{E_n} |f|$. By the assumption, there exists $n_1 \geq 1$ such that

$$\frac{\int_{E_{n_1}} |f|}{W(t_{n_1})} = \frac{\int_0^{t_{n_1}} f^*}{W(t_{n_1})} \geq 1 - \frac{1}{2}.$$

Set $F_1 = E_{n_1}$ and $s_1 = t_{n_1}$. Then, for $n \geq n_1$,

$$\frac{\int_0^{t_n} f^*}{W(t_n)} = \frac{\int_{E_{n_1}} |f| + \int_{E_n \setminus E_{n_1}} |f|}{W(t_n)} = \frac{\int_{E_{n_1}} |f|}{W(t_n)} + \frac{\int_{E_n \setminus E_{n_1}} |f|}{W(t_n)},$$

so in view of $W(t_n) \to \infty$ we have

$$1 = \lim_{n\to\infty} \frac{\int_0^{t_n} f^*}{W(t_n)} = \lim_{n\to\infty} \frac{\int_{E_n \setminus F_1} |f|}{W(t_n)}.$$

Therefore there exists $n_2 > n_1$ such that

$$\frac{\int_{E_{n_2} \setminus F_1} |f|}{W(t_{n_2})} \geq 1 - \frac{1}{2^2}.$$

Let now $F_2 = E_{n_2} \setminus E_{n_1}$ and $s_2 = t_{n_2}$. So $F_2 \cap F_1 = \emptyset$,

$$\frac{\int_{F_2} |f|}{W(s_2)} \geq 1 - \frac{1}{2^2} \quad \text{and} \quad |F_2| \leq |E_{n_2}| = t_{n_2} = s_2.$$

Proceeding further by induction we shall find a subsequence $(n_k) \in \mathbb{N}$ such that

$$\frac{\int_{E_{n_{k+1}} \setminus E_{n_k}} |f|}{W(t_{n_k})} \geq 1 - \frac{1}{2^k}, \quad k \in \mathbb{N}.$$

Setting $F_k = E_{n_{k+1}} \setminus E_{n_k}$ and $s_k = t_{n_k}$ we get the claim.

Now define the sets

$$G_1 = \left(\bigcup_{n=1}^{\infty} F_{2n-1}\right) \cup \left(\mathbb{R}_+ \setminus \bigcup_{n=1}^{\infty} F_n\right) \quad \text{and} \quad G_2 = \bigcup_{n=1}^{\infty} F_{2n}.$$

Set $f_1 = f\chi_{G_1}$ and $f_2 = f\chi_{G_2}$. Then for all $n \in \mathbb{N}$, in view of the claim and the Hardy-Littlewood inequality,

$$1 - \frac{1}{2^{2n-1}} \le \frac{\int_{F_{2n-1}} |f|}{W(s_{2n-1})} = \frac{\int_0^{|F_{2n-1}|} f_1^*}{W(s_{2n-1})} \le \frac{\int_0^{s_{2n-1}} f_1^*}{W(s_{2n-1})} \le \frac{\int_0^{s_{2n-1}} f^*}{W(s_{2n-1})} \le 1,$$

so

$$1 = \|f\|_W \ge \lim_{n\to\infty} \frac{\int_0^{s_{2n-1}} f_1^*}{W(s_{2n-1})} = 1.$$

Hence $\|f_1\|_W = 1$. Similarly $\|f_2\|_W = 1$ and the first case is complete since $f = f_1 + f_2$.

Case 2. Assume that $t_n \to 0$ and $\lim_{n\to\infty} \frac{\int_0^{t_n} f^*}{W(t_n)} = 1$. Without loss of generality we can assume that (t_n) is a decreasing sequence.

We will show by induction that we can find a sequence $F_k \subset (0,\infty)$ of disjoint sets, $(s_k) \subset (0,\infty)$ and $(n_k) \subset \mathbb{N}$, $n_k \to \infty$ such that for all $k \in \mathbb{N}$, $|F_k| \le s_k$ and

$$1 \ge \frac{\int_{F_k} |f|}{W(s_k)} \ge 1 - \frac{1}{2^{n_{k-1}-1}},$$

where $n_0 = 1$. As in case 1, we can find a sequence of sets (E_n) such that $E_n \supset E_{n+1}$, $|E_n| = t_n$ and $\int_0^{t_n} f^* = \int_{E_n} |f|$. Without loss of generality assume that for all $n \in \mathbb{N}$,

$$1 \ge \frac{\int_0^{t_n} f^*}{W(t_n)} = \frac{\int_{E_n} |f|}{W(t_n)} \ge 1 - \frac{1}{2^n}. \tag{2}$$

We can write

$$\frac{\int_{E_1} |f|}{W(t_1)} = \frac{\int_{E_1 \setminus E_n} |f|}{W(t_1)} + \frac{\int_{E_n} |f|}{W(t_1)} \tag{3}$$

so

$$\lim_{n\to\infty} \frac{\int_{E_n} |f|}{W(t_1)} = 0.$$

Choose $n_1 \ge 1$ such that

$$\frac{\int_{E_{n_1}} |f|}{W(t_1)} < \frac{1}{2^2}. \tag{4}$$

Hence from (2), (3) and (4),

$$\frac{\int_{E_1 \setminus E_{n_1}} |f|}{W(t_1)} \ge 1 - \frac{1}{2} - \frac{1}{2^2} = 1 - \frac{3}{2^2}.$$

Let $F_1 = E_1 \setminus E_{n_1}$ and $s_1 = t_1$. Then $|F_1| \le s_1$ and

$$1 \ge \frac{\int_{E_1} |f|}{W(t_1)} \ge \frac{\int_{F_1} |f|}{W(t_1)} \ge 1 - \frac{3}{2^2}.$$

In view of (2), for $n > n_1$,

$$\frac{\int_{E_{n_1} \setminus E_n} |f|}{W(t_{n_1})} = \frac{\int_{E_{n_1}} |f|}{W(t_{n_1})} - \frac{\int_{E_n} |f|}{W(t_{n_1})} \ge 1 - \frac{1}{2^{n_1}} - \frac{\int_{E_n} |f|}{W(t_{n_1})}.$$

Note that $\lim_{n\to\infty} \frac{\int_{E_n} |f|}{W(t_{n_1})} = 0$, so there exists $n_2 > n_1$ such that

$$\frac{\int_{E_{n_2}} |f|}{W(t_{n_1})} < \frac{1}{2^{2n_1}}.$$

Hence

$$1 \ge \frac{\int_{E_{n_1}} |f|}{W(t_{n_1})} \ge \frac{\int_{E_{n_1}} |f|}{W(t_{n_1})} - \frac{\int_{E_{n_2}} |f|}{W(t_{n_1})} \ge 1 - \frac{1}{2^{n_1}} - \frac{1}{2^{2n_1}} \ge 1 - \frac{2^{n_1} \cdot 2}{2^{2n_1}} = 1 - \frac{1}{2^{n_1 - 1}}.$$

Letting $F_2 = E_{n_1} \setminus E_{n_2}$ and $s_2 = t_{n_1}$, we get that $|F_2| < s_2$, $F_1 \cap F_2 = \emptyset$ and

$$\frac{\int_{F_2} |f|}{W(s_2)} \ge 1 - \frac{1}{2^{n_1 - 1}}.$$

Proceeding further by induction we prove the claim.

Define now the sets G_1, G_2 and the functions f_1, f_2 like in the first case. By the claim and the Hardy-Littlewood inequality, for all $k \in \mathbb{N}$,

$$1 - \frac{1}{2^{n_{k-1} - 1}} \le \frac{\int_{F_k} |f|}{W(s_k)} \le \frac{\int_0^{s_k} f^*}{W(s_k)} \le 1.$$

Hence

$$1 = \|f\|_W \ge \lim_{k\to\infty} \frac{\int_0^{s_{2k-1}} f_1^*}{W(s_{2k-1})} = 1,$$

and so $\|f_1\|_W = 1$. Similarly, $\|f_2\|_W = 1$ and the proof is complete. □

It is well known that M_W contains an isomorphic copy of l_∞ and M_W^0 contains an isomorphic copy of c_0. In fact M_W is not order continuous since $f_n = w\chi_{(0,1/n)} \downarrow 0$, but $\|f_n\|_W = 1$ and so by [11, Theorem 4, page 295], M_W contains an isomorphic copy of l_∞. Applying now [11, Theorem 9, page 298], M_W^0 contains an isomorphic copy of c_0 since it does not satisfy the Fatou property in view of $f_n = w\chi_{(0,n)} \uparrow \chi_{(0,\infty)} \notin M_W^0$. Using Lemma 3, we can now prove something more.

Corollary 4. *M_W contains an isomorphic and isometric copy of l_∞.*

Proof. Let $f = w$. By Lemma 3, there exist $w_1, w_2 \geq 0$ such that $w = w_1 + w_2$, $w_1 \wedge w_2 = 0$ and $\|w_1\|_W = \|w_2\|_W = 1$. By induction, there exists a sequence (w_n) such that $w_i \wedge w_j = 0$ for all $i \neq j$, $w = \sum_{n=1}^{\infty} w_i$ and $\|w_n\|_W = 1$, for all $n \in \mathbb{N}$.

We claim that the closed linear span of (w_n) in M_W is isometric to l_∞. Indeed, let $m \in \mathbb{N}$ and $(\lambda_n)_{n=1}^{\infty} \subset \mathbb{R}$. Then

$$\left\|\sum_{n=1}^{m} \lambda_n w_n\right\|_W \leq \sup_{t>0} \frac{\int_0^t \left(\max_{1\leq n\leq m} |\lambda_n| \sum_{n=1}^m w_n\right)^*}{W(t)} = \max_{1\leq n\leq m} |\lambda_n| \sup_{t>0} \frac{\int_0^t \left(\sum_{n=1}^m w_n\right)^*}{W(t)}$$
$$\leq \max_{1\leq n\leq m} |\lambda_n| \|w\|_W \leq \|(\lambda_n)\|_\infty.$$

Also for all $n = 1, \cdots, m$, and any $m \in \mathbb{N}$,

$$\left\|\sum_{n=1}^{m} \lambda_n w_n\right\|_W \geq |\lambda_n| \cdot \|w_n\|_W = |\lambda_n|.$$

Hence

$$\left\|\sum_{n=1}^{m} \lambda_n w_n\right\|_W \geq \|(\lambda_n)\|_\infty,$$

and the proof is complete. □

Proposition 5. *Let $f \in S_{M_W}$. If*

$$\limsup_{t\to 0} \frac{\int_0^t f^*}{W(t)} = 1 \quad \text{or} \quad \limsup_{t\to\infty} \frac{\int_0^t f^*}{W(t)} = 1$$

then there exist two different norm-one supporting functionals at f.

Proof. By Lemma 3, there is a decomposition of $f = f_1 + f_2$ such that $|f_1| \wedge |f_2| = 0$ and $\|f_1\|_W = \|f_2\|_W = 1$. The two-dimensional subspace spanned by $\{f_1, f_2\}$ is isometric to the two-dimensional space l_∞^2 with supremum norm. In fact, this isometry is given by $Tf_i = e_i$, $i = 1, 2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Thus $Tf = T(f_1 + f_2) = (1, 1)$. The point $(1, 1)$ is not smooth in l_∞^2, and so by the Hahn-Banach theorem, f is not smooth in M_W. □

Lemma 6. *Let $f \in S_{M_W}$ (or $f \in S_{M_W^0}$). If there exist $0 < a < b < \infty$ such that*

$$\|f\|_W = \frac{\int_0^a f^*}{W(a)} = \frac{\int_0^b f^*}{W(b)},$$

then there exist two different regular norm-one supporting functionals at f.

Proof. By [2, Theorem 2.5], choose the sets $E(a), E(b) \subset (0, \infty)$ such that $|E(a)| = a$, $|E(b)| = b$, $E(a) \subset E(b)$ and

$$\int_0^a f^* = \int_{E(a)} |f|, \qquad \int_0^b f^* = \int_{E(b)} |f|.$$

Define

$$\phi_1(g) = \frac{1}{W(a)} \int_{E(a)} (\operatorname{sign} f) \cdot g, \quad \text{and} \quad \phi_2(g) = \frac{1}{W(b)} \int_{E(b)} (\operatorname{sign} f) \cdot g.$$

Then we have that $\phi_1(f) = \phi_2(f) = 1$. It follows that $\|\phi_1\| = 1$, since

$$|\phi_1(g)| \leq \frac{1}{W(a)} \int_{E(a)} |g| \leq \frac{1}{W(a)} \int_0^a g^* \leq \|g\|_W.$$

Similarly $\|\phi_2\| = 1$, so ϕ_1 and ϕ_2 are norm-one supporting functionals at f. Now consider $g = \chi_{E(b)} \operatorname{sign} f$. Then $\phi_1(g) \neq \phi_2(g)$, so $\phi_1 \neq \phi_2$. □

Lemma 7. *Let $f \in S_{M_W}$ (or $f \in S_{M_W^0}$) and ϕ be a supporting regular functional at f induced by $h \in \Lambda_{1,w}$. If for some $t_0 > 0$, $\int_0^{t_0} f^* < \int_0^{t_0} w$, then there exists $s > 0$ such that $h^*(t_0) = h^*(t_0 + s)$, that is h^* is constant on some right neighborhood of t_0.*

Proof. Assume for a contrary that for all $s > 0$, $h^*(t_0) > h^*(t_0 + s)$. We will show that

$$\int_0^\infty h^* f^* < \int_0^\infty h^* w, \tag{5}$$

which is a contradiction since then

$$\|\phi\| = \phi(f) = \int_0^\infty hf \leq \int_0^\infty h^* f^* < \int_0^\infty h^* w = \|h\|_{1,w} = \|\phi\|.$$

Note that since $\|f\|_W = \sup_{t>0} \frac{\int_0^t f^*}{W(t)} = 1$, we have that for all $t > 0$, $\int_0^t f^* \leq \int_0^t w$. Therefore by Hardy's Lemma, $\int_0^\infty h^* f^* \leq \int_0^\infty h^* w$.

To prove (5), notice that by the integration by parts formula,

$$\begin{aligned} \int_0^\infty h^*(t) f^*(t)\, dt &= \int_0^\infty h^*(t) d\left(\int_0^t f^*(s) ds \right) \\ &= h^*(t) \int_0^t f^*(s) ds \Big|_0^\infty + \int_0^\infty \left(\int_0^t f^*(s) ds \right) d(-h^*(t)). \end{aligned} \tag{6}$$

The measure μ defined on $(0, \infty)$ as $d(-h^*(t)) = d\mu(t)$ is positive on any $(t_0, t_0 + s)$, $s > 0$, by the assumption $h^*(t_0) > h^*(t_0 + s)$ and right-continuity of h^*. Also the assumption $\int_0^{t_0} f^* < \int_0^{t_0} w$ implies that the inequality $\int_0^t f^* < \int_0^t w$ must be satisfied on some interval $(t_0, t_0 + s)$. Since also $\int_0^t f^* \leq \int_0^t w$ for every $t \geq 0$, we must have that

$$\int_0^\infty \left(\int_0^t f^*(s)\, ds \right) d(-h^*(t)) < \int_0^\infty \left(\int_0^t w(s)\, ds \right) d(-h^*(t)).$$

Notice also that

$$\begin{aligned} & h^*(t) \int_0^t f^*(s)\, ds \Big|_0^\infty \\ &= \lim_{t \to \infty} h^*(t) \int_0^t f^*(s)\, ds \leq \lim_{t \to \infty} h^*(t) \int_0^t w(s)\, ds = h^*(t) \int_0^t w(s)\, ds \Big|_0^\infty. \end{aligned}$$

Combining the above two inequalities we get the claim. □

Lemma 8. *Let $f \in S_{M_W}$ (or $f \in S_{M_W^0}$). If there exists a unique number $0 < a < \infty$ such that*

$$1 = \|f\|_W = \frac{\int_0^a f^*}{W(a)},$$

then f has a unique regular norm-one supporting functional.

Proof. Let ϕ, induced by $h \in \Lambda_{1,w}$, be a norm-one supporting functional of f. Then

$$\phi(f) = \int_0^\infty fh = 1 = \|\phi\| = \|h\|_{1,w} = \int_0^\infty h^* w.$$

We have that $\int_0^t f^* < \int_0^t w$, for all $0 < t < a$. So by Lemma 7, h^* is constant on some right neighborhood of t, for all $0 < t < a$, so $h^*(t) = \alpha$, for all $0 < t < a$. We apply the same lemma for $t > a$, so $h^*(t) = \beta$, for all $t \geq a$. But $h \in \Lambda_{1,w}$, so $\lim_{t\to\infty} h^*(t) = 0$, therefore $\beta = 0$. It follows that $h^*(t) = \alpha\chi_{(0,a)}$, hence $h(t)$ is given by

$$h(t) = \alpha(t)\chi_A(t),$$

where A is some set with $|A| = a$ and $\alpha(t)$ is a measurable function such that $|\alpha(t)| = \alpha$.

Since ϕ is a supporting functional, so

$$\begin{aligned} 1 = \|\phi\| = \|h\|_{1,w} = \phi(f) &= \int_0^\infty hf \leq \int_0^\infty h^* f^* \\ &= \int_0^a \alpha f^* = \alpha \int_0^a w = \int_0^\infty \alpha\chi_{(0,a)} w = \int_0^\infty h^* w = \|h\|_{1,w} = 1. \end{aligned}$$

Therefore $\alpha = \frac{1}{W(a)}$. Also, $\int_0^\infty hf = \int_0^\infty h^* f^*$, so

$$\begin{aligned} \int_0^\infty hf &= \int_0^\infty \alpha(t) f(t) \chi_A(t)\, dt = \int_0^\infty \alpha f^*(t) \chi_{(0,a)}(t)\, dt \\ &= \alpha \int_{E(a)} |f| = \int_0^\infty \alpha |f(t)| \chi_{E(a)}(t)\, dt, \end{aligned}$$

where $E(a)$ is the set such that $|E(a)| = a$ and $\int_0^a f^* = \int_{E(a)} |f|$. So we have that

$$\int_0^\infty \alpha(t) f(t) \chi_A(t) dt = \int_0^\infty \alpha |f(t)| \chi_{E(a)}(t) dt. \tag{7}$$

We want to show now that $A = E(a)$ a.e.. We show first that

$$\int_A |f| = \int_0^a f^* = \int_{E(a)} |f|.$$

Since $|\alpha(t)| = \alpha$, we obtain that $\alpha(t) = \alpha \operatorname{sign} \alpha(t)$, therefore we have

$$1 = \int_0^\infty hf = \int_0^\infty \alpha(t) \chi_A(t) f(t)\, dt = \int_0^\infty \alpha \operatorname{sign} \alpha(t) \chi_A(t) f(t)\, dt.$$

Hence

$$\frac{1}{\alpha} = \int_0^\infty \operatorname{sign}\alpha(t)\chi_A(t)f(t)\,dt \le \int_A |f(t)|\,dt \le \sup\left\{\int_B |f(t)|\,dt : |B| = a\right\}$$
$$= \int_0^a f^*(t)\,dt = W(a) = \frac{1}{\alpha}.$$

So we got that

$$\int_A |f(t)|\,dt = \int_0^a f^*(t)\,dt = \int_{E(a)} |f(t)|\,dt = \frac{1}{\alpha}. \tag{8}$$

We show now that for all $t > a$, we have that

$$f^*_-(a) := \lim_{s\to a^-} f^*(s) > f^*(t). \tag{9}$$

Let first $0 < c < a$. Then by the assumption

$$1 = \frac{\int_0^a f^*}{\int_0^a w} = \frac{\int_0^{a-c} f^*}{\int_0^{a-c} w}\frac{\int_0^{a-c} w}{\int_0^a w} + \frac{\int_{a-c}^a f^*}{\int_0^a w} < \frac{\int_0^{a-c} w}{\int_0^a w} + \frac{\int_{a-c}^a f^*}{\int_0^a w},$$

that is

$$\int_{a-c}^a w < \int_{a-c}^a f^*.$$

We have then that for all $0 < c < a$,

$$1 < \frac{\int_{a-c}^a f^*}{\int_{a-c}^a w} \le \frac{f^*(a-c)a}{w(a)a} = \frac{f^*(a-c)}{w(a)},$$

therefore $w(a) < f^*(a-c)$, for all $0 < c < a$, and it follows that

$$f^*_-(a) = \lim_{s\to a^-} f^*(s) \ge w(a). \tag{10}$$

Now let $t > a$. Since $\int_0^a f^* = \int_0^a w$ and $\int_0^t f^* < \int_0^t w$, then for all $t > a$, $\int_a^t f^* < \int_a^t w$. Now by the inequality (10),

$$f^*(t)(t-a) \le \int_a^t f^* < \int_a^t w \le \int_a^t w(a) = w(a)(t-a) \le f^*_-(a)(t-a),$$

and (9) is proven. So by (8) and (9), we have that $A = E(a)$ a.e.. Therefore by (7),

$$\int_0^\infty \alpha(t)f(t)\chi_{E(a)}(t)\,dt = \int_0^\infty \alpha|f(t)|\chi_{E(a)}(t)\,dt.$$

We also have that, for a.a.t,

$$\alpha(t)f(t)\chi_{E(a)}(t) \le \alpha|f(t)|\chi_{E(a)}(t),$$

so from both we get that $\alpha(t)f(t) = \alpha|f(t)|$ a.e. on $E(a)$. Therefore $\operatorname{sign}\alpha(t) = \operatorname{sign} f(t)$ a.e. on $E(a)$, and since $A = E(a)$ a.e. and $\alpha = \frac{1}{W(a)}$, it follows that $h(t) = \frac{1}{W(a)}\operatorname{sign} f(t)\chi_{E(a)}(t)$, and ϕ is uniquely determined by h. $\square$

Remark 9. Let $0 < G(t) < F(t)$ for all $t \in (t_1, t_2)$, where $0 < t_1 < t_2 < \infty$. Assume that $F, G : (0, \infty) \to [0, \infty)$ are continuous and there exists $a \in (t_1, t_2)$ such that

$$\max_{t \in [t_1, t_2]} F(t) = F(a) = 1,$$

and for all $t \neq a$, $F(t) < F(a)$. Then

$$\max_{t \in [t_1, t_2]} G(t) < 1.$$

Proof. If G assumes maximum on $[t_1, t_2]$ at $z \in [t_1, t_2]$, then for all $t \in (t_1, t_2)$, $G(t) \leq G(z) < F(z) < F(a) = 1$. By continuity of G, for all $t \in [t_1, t_2]$, $G(t) \leq F(z) < 1$, so $\max_{t \in [t_1, t_2]} G(t) < 1$. If G assumes maximum at t_1 or t_2, say at t_1, then by continuity of G and F, for all $t \in [t_1, t_2]$, $G(t) \leq G(t_1) \leq F(t_1) < F(a) = 1$. □

Now we are ready to prove Theorems 1 and 2.

Proof of Theorem 1. Let (1) be satisfied. The space M_W^0 contains all order continuous elements of M_W, so the dual of M_W^0 coincides with the space of regular functionals, and by Lemma 8, f has a unique supporting functional.

Now let f be a smooth point. If (1) is not satisfied, since $f \in M_W^0$, there exist $0 < a < b < \infty$ such that

$$\frac{\int_0^a f^*}{W(a)} = \frac{\int_0^b f^*}{W(b)} = 1.$$

By Lemma 6 there are more than one norm-one supporting functionals at f, so f is not a smooth point. □

Proof of Theorem 2. If f is a smooth point in M_W, the result follows from Lemma 6 and Proposition 5.

Let now

$$\sup_{t>0} \frac{\int_0^t f^*}{W(t)} = \frac{\int_0^a f^*}{W(a)} = 1,$$

for some unique $a \in (0, \infty)$. Let $\varepsilon > 0$. Then there exist $0 < t_1 < a < t_2 < \infty$ such that for all $0 < t \leq t_1$ and for all $t \geq t_2$,

$$\frac{\int_0^t f^*}{W(t)} \leq 1 - \varepsilon.$$

Let

$$s_1 = \inf\{t : f^*(t) = f^*(a)\} \quad \text{and} \quad s_2 = \sup\{t : f^*(t) = f^*(a)\}.$$

We have $s_2 < \infty$ since $\lim_{t \to \infty} f^*(t) = 0$.

We shall consider two cases. First, suppose $s_1 = 0$. We observe that $W(t)/t$ is a strictly decreasing function on $(0, \infty)$. Then for all $0 \leq t \leq a$, $f^*(t) = f^*(a)$ and

$$1 = \sup_{0<t\leq a} \frac{\int_0^t f^*}{W(t)} = \sup_{0<t\leq a} \frac{t f^*(a)}{W(t)} = \frac{a f^*(a)}{W(a)}.$$

It follows that $f^*(t) < f^*(a)$, for all $t > a$. Indeed, if not, then $f^*(t) = f^*(a)$, for $t \in (0, b)$, for some $b > a$. Then

$$1 \geq \sup_{0<t\leq b} \frac{\int_0^t f^*}{W(t)} = \frac{f^*(a)b}{W(b)} > \frac{f^*(a)a}{W(a)} = 1,$$

which is a contradiction. Let $E(t_i) \subset (0, \infty)$, $i = 1, 2$, be such that $|E(t_i)| = t_i$, $\int_0^{t_i} f^* = \int_{E(t_i)} |f|$, and $E(t_2) \supset E(t_1)$. Let's define $g(t) = f(t)\chi_{E(t_1)\cup E(t_2)^c}$. Denote by $F(t) = \frac{\int_0^t f^*}{W(t)}$ and $G(t) = \frac{\int_0^t g^*}{W(t)}$. Then

$$g^*(t) = \begin{cases} f^*(t), & \text{if } t \in (0, t_1); \\ f^*(t + t_2 - t_1), & \text{if } t \in [t_1, \infty). \end{cases}$$

Notice that $g^*(t) \leq f^*(t)$, for all $t > 0$, and $g^*(t) < f^*(t)$, for all $t \in (t_1, t_2)$. Hence $G(t) < F(t)$, for all $t \in (t_1, t_2)$, so by the previous remark, $\max_{t\in[t_1,t_2]} G(t) < 1$. We also have for $0 < t < t_1$,

$$G(t) = \frac{\int_0^t g^*}{W(t)} = \frac{\int_0^t f^*}{W(t)} \leq 1 - \varepsilon,$$

and for $t > t_2$,

$$G(t) = \frac{\int_0^t g^*}{W(t)} < \frac{\int_0^t f^*}{W(t)} \leq 1 - \varepsilon.$$

Therefore $\|g\|_W < 1$.

Now let $0 < s_1 \leq a \leq s_2 < \infty$. Let $z_1 = \min\{s_1, t_1\}$ and $z_2 = \max\{s_2, t_2\}$. In this case, define $g(t) = f(t)\chi_{E(z_1)\cup E(z_2)^c}$, where $E(z_i) \subset (0, \infty)$, $i = 1, 2$, are such that $|E(z_i)| = z_i$ and $\int_0^{z_i} f^* = \int_{E(z_i)} |f|$. Then

$$g^*(t) = \begin{cases} f^*(t), & \text{if } t \in (0, z_1); \\ f^*(t + z_2 - z_1), & \text{if } t \in [z_1, \infty). \end{cases}$$

Then $g^*(t) \leq f^*(t)$ for all $t > 0$ and $g^*(t) < f^*(t)$ for all $t \in [z_1, z_2]$. So $G(t) < F(t)$ on $[z_1, z_2]$ and by the previous remark, $\max_{t\in[z_1,z_2]} G(t) < 1$. It follows analogously as in the previous case that $\|g\|_W < 1$.

In both cases, $\|g\|_W < 1$ and $f - g$ has support with finite measure and it is bounded. Thus $f - g \in M_W^0$.

Consider $\phi \in (M_W)^*$ a norm-one supporting functional at f. Then ϕ has a unique representation $\phi = \psi + \xi$, where $\psi \in \Lambda_{1,w}$ and ξ is singular, that is $\psi(g) = \int_0^\infty gh$, for all $g \in M_W$ and some unique $h \in \Lambda_{1,w}$, and $\xi(g) = 0$, for all $g \in M_W^0$ [9]. ϕ is a supporting functional, so by the M-ideal property of M_W^0 in M_W we have

$$\|\phi\| = \|\psi\| + \|\xi\| \geq \psi(f) + \xi(f) = \phi(f) = \|\phi\|,$$

therefore ϕ and ξ are supporting functionals. Then $\xi(f - g) = 0$, and so

$$\|\xi\| = \xi(f) = \xi(g) + \xi(f - g) = \xi(g) \leq \|\xi\| \cdot \|g\| < \|\xi\|.$$

Hence $\xi = 0$ and ϕ is a regular functional. By Lemma 8, ϕ is unique and f is a smooth point. □

We finish with the result in sequence spaces that can be proved analogously as Theorem 1. It completes the earlier result on smooth points in Marcinkiewicz sequence spaces in [10]. Let $m_W^0 = d_*(w,1)$ be a subspace of order continuous elements in Marcinkiewicz sequence spaces $m_W = d^*(w,1)$ [10].

Theorem 10. *An element $x \in S_{m_W^0}$ is a smooth point in m_W^0 if and only if there exists $i_0 \in \mathbb{N}$ such that*

$$1 = \frac{\sum_{j=1}^{i_0} x^*(j)}{W(i_0)} > \sup_{n \neq i_0} \frac{\sum_{j=1}^{n} x^*(j)}{W(n)}.$$

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