Journal of Convex Analysis
Volume 18 (2011), No. 2, 397–401

A Note in Approximative Compactness and Continuity of Metric Projections in Banach Spaces

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Received: August 31, 2009

We give a counterexample to a recent statement in the metric approximation theory and provide a setting where the statement holds.

Keywords: Metric projection, approximative compactness, midpoint locally uniform rotundity, locally uniform rotundity, Chebyshev sets, proximinal sets

Let $(X, \|\cdot\|)$ be a real Banach space. Our notation is standard. We follow, for example, [4]. In this note, if no reference to a different topology on X is made, *convergence* in X means $\|\cdot\|$-convergence. The following concepts in the geometry of Banach spaces are more or less standard. A non-empty subset C of X is said to be *approximately compact* if for every $x \in X$ and every sequence (c_n) in C such that $\|x - c_n\| \to \operatorname{dist}(x, C)$ (such a sequence is called an *approximate sequence for* x in C), then (c_n) has a convergent subsequence. The set C is said to be *proximinal* if, for every $x \in X$, the set $P_C(x) := \{c \in C;\ \|x - c\| = \operatorname{dist}(x, C)\}$ is non-empty (the multivalued mapping $P_C : X \to 2^X$ is called the *metric projection* onto C). The set C is said to be *semi-Chebyshev* if $P_C(x)$ contains at most one point for every $x \in X$. The set C is said to be *Chebyshev* if it is simultaneously proximinal and semi-Chebyshev. In this case we put $P_C(x) = \{\pi_C(x)\}$ for all $x \in X$. A Banach space X is said to be *locally uniformly rotund* (LUR, for short) if for every $x \in S_X$ and every sequence (x_n) in S_X such that $\|x + x_n\| \to 2$, then $x_n \to x$. A Banach space X is said to be *midpoint locally uniformly rotund* (MLUR, for short) if for any x_0, x_n and y_n in S_X, $n \in \mathbb{N}$, such that $\|x_n + y_n - 2x_0\| \to 0$, then $\|x_n - y_n\| \to 0$. Every LUR Banach space is MLUR. Recall, too, that a Banach space X has *property* (H) (sometimes also called *Kadec-Klee property*) if every sequence in S_X that w-converges to a point x in S_X converges (to x). As it is well known, every LUR space has property (H).

*Supported in part by Project MTM2008-05396 and the Universidad Politécnica de Valencia and by Project GV/2010/036.
†Supported in part by Project MTM2008-03211 and the Universidad Politécnica de Valencia and by Project GV/2010/036.

ISSN 0944-6532 / $ 2.50

A Banach space X is *rotund* (also called *strictly convex*) if every point $x \in S_X$ is extremal. Obviously, every MLUR space is rotund.

Let C be a non-empty subset of a Banach space X and let $x^* \in S_{X^*}$ bounded above on C. We denote $S(C, x^*, \delta)$ the δ-section defined by x^* in C, i.e., $S(C, x^*, \delta) := \{x \in C;\ \langle x, x^*\rangle \geq \sup_C x^* - \delta\}$.

The main result in [2] is Theorem 15. We reproduce it here for later reference.

(*) ([2, Theorem 15]). *Let X be a* MLUR *Banach space and C a non-empty closed convex subset of X. Then the two following conditions are equivalent:*

(i) *C is a Chebyshev set and the metric projection π_C is continuous.*

(ii) *C is approximately compact in X.*

We claim that (i) $\Rightarrow$ (ii) in (*) is not correct. We provide later a counterexample (Example 7). As it follows from [7, Appendix II. Theorem 4.1], (ii) $\Rightarrow$ (i) is true for any rotund Banach space. It turns out that (i) $\Rightarrow$ (ii) in (*) holds for the smaller class of strongly convex Banach spaces (see Definition 8), as it has been proved in [12] (see Corollary 10). In particular, it holds for the class of LUR Banach spaces; this will also follow from Proposition 3 below.

Remark 1. Notice that if C is a closed approximately compact subset of a Banach space X, then C is proximinal. As a consequence, if C is a closed approximately compact semi-Chebyshev subset of X then C is Chebyshev, and if $x \in X$ and (c_n) is an approximate sequence in C for x, then $c_n \to \pi_C(x)$. Observe, too, that every proximinal subset of a normed space is closed.

The following result appears in [5] without a proof. In [2, Theorem 13] a "short" argument is provided. In fact, it is a simple corollary of the already mentioned [7, Appendix II. Theorem 4.1] and Remark 1.

Proposition 2. *Let X be a Banach space and C a semi-Chebyshev closed subset of X. If C is approximately compact, then C is a Chebyshev subset of X and the metric projection π_C is continuous.*

Proposition 3. *Let X be a* LUR *Banach space and let C be a non-empty closed convex subset of X. Then C is proximinal if and only if C is approximately compact.*

Proof. Assume that C is proximinal. Since X is rotund, C is Chebyshev. Fix $x \in X \setminus C$. Without loss of generality we may assume $x = 0$. Let $d := \operatorname{dist}(0, C)$. Let (x_n) be an approximate sequence in C for 0. Notice that

$$d \leq \left\| \frac{x_n + \pi_C(0)}{2} \right\| \leq \frac{\|x_n\| + \|\pi_C(0)\|}{2} \to d. \tag{1}$$

For $n \in \mathbb{N}$, put $p_n := \|x_n\|^{-1} d x_n$. Then, from (1) we get

$$\left\| \frac{p_n + \pi_C(0)}{2} \right\| = \left\| \frac{d x_n + \|x_n\| \pi_C(0)}{2\|x_n\|} \right\| \to d. \tag{2}$$

Since X is LUR, we get $p_n \to \pi_C(0)$, hence $x_n \to \pi_C(0)$, so C is approximately compact. The converse is trivial (see Remark 1). □

Definition 4 ([9]). A Banach space X is said to be *nearly strongly convex* if given $x \in S_X$, $x^* \in S_{X^*}$ such that $\langle x, x^*\rangle = 1$ and $\{x_n\} \subset B_X$ a sequence in B_X such that $\langle x^*, x_n\rangle \to \langle x^*, x\rangle = 1$, then the set $\{x_n;\ n \in \mathbb{N}\}$ is relatively compact.

Observe that this is equivalent to say that, for every $x^* \in S_{X^*}$ that attains its norm, $\alpha(S(B_X, x^*, \delta)) \to 0$ when $\delta \downarrow 0$, where $\alpha(S)$ denotes the Kuratowski index of non-compactness, i.e., the infimum of all $d > 0$ such that S can be covered by a finite number of sets of $\|\cdot\|$-diameter less than d.

Let X be a Banach space, $x^* \in S_{X^*}$ and $\alpha \in \mathbb{R}$. It is clear that a hyperplane $\{x \in X : \langle x, x^*\rangle = \alpha\}$ is proximinal if and only if x^* attains its norm. The next statement is an improvement of a result in [3].

Theorem 5. *Let X be a Banach space.*

(*i*) *If every proximinal hyperplane of X is approximately compact, then X is nearly strongly convex.*

(*ii*) *If X is nearly strongly convex then every convex and proximinal subset C of X is approximately compact.*

In particular, X is nearly strongly convex if and only if every convex and proximinal subset of X is approximately compact, if and only if every proximinal hyperplane of X is approximately compact.

Proof. (*i*) Let $x \in S_X$ and $x^* \in S_{X^*}$ such that $\langle x, x^*\rangle = 1$. Let $H := \{y \in X;\ \langle y, x^*\rangle = 1\}$. Let (x_n) be a sequence in B_X such that $\langle x_n, x^*\rangle \to 1$. Put $y_n := x_n + \lambda_n x$ such that $\langle y_n, x^*\rangle = 1$ for all $n \in \mathbb{N}$. Then

$$1 = \langle y_n, x^*\rangle = \langle x_n, x^*\rangle + \lambda_n, \quad \text{for all } n \in \mathbb{N},$$

hence $\lambda_n \to 0$. Moreover,

$$1 \le \|y_n\| \le \|x_n\| + |\lambda_n|.\|x\| \le 1 + |\lambda_n| \to 1,$$

hence $\|y_n\| \to 1$, i.e., $\operatorname{dist}(0, y_n) \to \operatorname{dist}(0, H)$. Since H is proximinal, (y_n) has a convergent subsequence, so it does (x_n). This proves that $\{x_n;\ n \in \mathbb{N}\}$ is relatively compact.

(*ii*) Assume now that X is nearly strongly convex. Let C be a proximinal convex subset of X. Fix $x \in X \setminus C$. Without loss of generality we may assume $x = 0$. Let $d := \operatorname{dist}(0, C)$ (> 0). Let (x_n) be an approximate sequence in C for 0. Take $x^* \in S_{X^*}$ that separates C and dB_X. Let $c \in P_C(0)$. Notice that

$$d = \|c\| = \langle c, x^*\rangle \le \langle x_n, x^*\rangle \le \|x_n\| \to d.$$

So x^* is a supporting functional of B_X at $c/\|c\|$, and $\langle x_n/\|x_n\|, x^*\rangle \to 1$. Since X is nearly strongly convex, $\{x_n/\|x_n\|;\ n \in \mathbb{N}\}$ and therefore $\{x_n;\ n \in \mathbb{N}\}$ are relatively compact sets. It follows that (x_n) has a convergent subsequence. □

Remark 6. Note that a nearly strongly convex Banach space has property (H). Indeed, let (x_n) be a sequence in S_X that w-converges to some $x \in S_X$. Let $x^* \in S_{X^*}$ such that $\langle x, x^* \rangle = 1$. Since $\langle x_n, x^* \rangle \to \langle x, x^* \rangle = 1$ we get that $\{x_n;\ n \in \mathbb{N}\}$ is relatively compact. Every subsequence of (x_n) has a cluster point (and, because of the w-convergence, this point is unique, namely x). This implies $x_n \to x$.

The following example was provided by M. A. Smith [8] in order to prove that MLUR does not imply property (H), answering in the negative a question in [1, p. 31]. We shall prove below that it also provides a counterexample to (*) [2, Thm. 15].

Example 7. There exists an *MLUR* Banach space X and a non-empty closed convex Chebyshev subset H of X (indeed, a closed proximinal hyperplane) such that π_H is continuous and H is not approximately compact.

Proof. Let X be the Banach space introduced in [8]. X is an MLUR space without property (H). By Remark 6, the space X is not nearly strongly convex. In view of Theorem 5, this imply the existence of a proximinal hyperplane H in X that is not approximately compact. The space X is (R), hence the metric projection onto H is single-valued and continuous. Indeed, in [10] (in Chinese, see Math. Rev. 2278791) the following trivial computation appears as one of the main results of the paper: *Suppose that X is a real Banach space, that $\alpha \in \mathbb{R}$ and that $x_0^* \in X^*$ with $x_0^* \neq 0$. If there exists $x_0 \in X$ with $\|x_0\| = 1$ such that $\langle x_0, x_0^* \rangle = \|x_0^*\|$, then*

$$P_A(x) = x + \frac{\alpha - \langle x, x_0^* \rangle}{\|x_0^*\|} F^{-1}(x_0^*), \quad x \in X,$$

where $A := \{y \in X; \langle y, x_0^* \rangle = \alpha\}$ and $F : X \to 2^X$ is the duality mapping, i.e.,

$$F(x) := \{x^* \in X^*;\ \langle x, x^* \rangle = \|x\|.\|x^*\| = \|x\|^2 = \|x^*\|^2\}.$$

In particular, if X is (R) and H is a proximinal hyperplane of X, then the mapping π_H is continuous. □

Definition 8. A Banach space X is said to be *strongly convex* if the following happens: given $x \in S_X$, $x^* \in S_{X^*}$ such that $\langle x, x^* \rangle = 1$, and a sequence (x_n) in B_X such that $\langle x_n, x^* \rangle \to 1$, then (x_n) converges.

It is elementary to prove that X is strongly convex if and only for every $x \in S_X$ and $x^* \in S_{X^*}$ such that $\langle x, x^* \rangle = 1$, we have $\operatorname{diam} S(B_X, x^*, \delta) \to 0$ whenever $\delta \downarrow 0$ (in particular, every strongly convex space is nearly strongly convex, see the paragraph after Definition 4). Recall that if C is a non-empty closed and convex subset of a Banach space X, a point $x \in C$ is *strongly exposed* (by a functional $x^* \in S_{X^*}$) if $\langle x, x^* \rangle = \sup_C x^*$ and, for every sequence (x_n) in C such that $\langle x_n, x^* \rangle \to \langle x, x^* \rangle$, we have $x_n \to x$. It follows then that X is strongly convex if and only if every $x \in S_X$ is strongly exposed in B_X by every $x^* \in S_{X^*}$ such that $\langle x, x^* \rangle = 1$. From this also follows that every LUR space is strongly convex. It is elementary to prove that in rotund Banach spaces, nearly strongly convexity is equivalent to strong convexity.

Corollary 10 below proves that $(i) \Rightarrow (ii)$ in (*) holds for the class of strongly convex Banach spaces. In [11] it is proved the elementary fact that strong convexity implies

MLUR, so the authors of [12] conclude that Corollary 10 follows from (*); fortunately, they give also an alternative proof of their result not based on (*). By formulating the next proposition we observe that Corollary 10 follows easily from results above.

Proposition 9. *If X is a rotund Banach space and C is a non-empty convex closed approximately compact subset of X, then C is Chebyshev and π_C is continuous.*

Proof. It is a straightforward consequence of Remark 1 and Proposition 2. □

Corollary 10 ([12], Corollary 3.6). *Let X be a strongly convex Banach space and A a non-empty closed and convex subset of X. Then A is a Chebyshev set and the metric projection P_A is continuous if and only if A is approximatively compact in X.*

Proof. It is a consequence of Proposition 9 and (*ii*) in Theorem 5. □

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