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Only Solid Spheres Admit a False Axis of Revolution*

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Let $K \subset \mathbb{R}^3$ be a convex body. A point p_0 is a point of revolution for K if every section of K through p_0 has an axis of symmetry that passes through p_0. In particular, every point that lies in an axis of revolution is a point of revolution. A line $L \subset \mathbb{R}^3$ is a *false axis of revolution*, if every point of L is a point of revolution for K but L is not an axis of revolution. The purpose of this paper is to prove that only solid spheres admit a false axis of revolution.

1. Introduction

Let K be a convex body in the Euclidean space $\mathbb{R}^3$. A point $p \in \mathbb{R}^3$ is called a *false centre* of K if p is not a centre of symmetry of K but for any plane H through p, we have that the section $H \cap K$ is either empty or centrally symmetric. The False Centre Theorem claims that a convex set $K \subset \mathbb{R}^n$ with a false centre is an ellipsoid [1], [4], [6] (we also recommend to see [2] and [7]). Following the same spirit we have the following.

Let L be an axis of revolution for a convex body $K \subset \mathbb{R}^3$. Then every point $p_0 \in L$ has the following property: "every section of K through p_0 has an axis of symmetry that passes through p_0". This motivates the following definition:

Definition. Let K be a convex body in the Euclidean 3-space $\mathbb{R}^3$. A point $p_0 \in \mathbb{R}^3$ is a *point of revolution* for K if for every plane H through p_0 that intersects K, the section $K \cap H$ has an orthogonal axis of symmetry that passes through p_0.

So, every point that lies in an axis of symmetry of K is a point of revolution for K and therefore, for a solid sphere, every point of $\mathbb{R}^3$ is a point of revolution.

Definition. Let K be a convex body in the Euclidean 3-space $\mathbb{R}^3$. A line $L \subset \mathbb{R}^3$ is a *false axis of revolution*, if every point of L is a point of revolution for K and L is not an axis of revolution.

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We shall prove in this paper that a convex body with a false axis of revolution must be a solid sphere.

Still there are convex bodies with an isolated point of revolution p_0 in its interior. In fact it was conjectured that a convex body with a point all whose sections through it have axis of symmetry must be a body of revolution or an ellipsoid. This is not true, for the following two examples in which the origin is a point of revolution and every section has exactly two orthogonal axes of symmetry through the origin. In [5] Montejano proved that if this is so, then there is always a circular section through the origin. Nevertheless there is still open the conjecture stated by K. Bezdek [3] that claims that a convex body all whose sections have an axis of symmetry must be either an ellipsoid or a body of revolution.

Example 1.1. Let K be the convex hull of two orthogonal and concentric circles of the same radius.

Example 1.2. Consider an ellipsoid E. Then, through its centre, there are two different concentric circular sections. If this two concentric circular sections are orthogonal, let K be the intersection of two orthogonal copies of E, which coincide on its circular sections.

As we said before, the main purpose of this paper is to prove the following:

Theorem 1.3. *Let $K \subset \mathbb{R}^3$ be a strictly convex body and $L \subset \mathbb{R}^3$ be a line. Suppose that L is a false axis of revolution for K, then K is a solid sphere.*

2. Proof when the false axis of symmetry and K are disjoint.

Lemma 2.1. *For every plane Π passing through L, the section $\Pi \cap K$ is an Euclidean disc.*

Proof. The proof of this lemma is straightforward, since through every point of L passes a line of symmetry for $\Pi \cap K$. □

In what follows, we will say that a line is a *diametral line* of a convex body K if this line contains an affine diameter (diametral chord) of K. Also, we will denote by $[a, b]$ and ab the segment and the line through the points a and b, respectively.

Lemma 2.2. *Let $p \notin K$ be a point of revolution for K. If L_1 is a diametral line of K passing through p, then the sections which are orthogonal to L_1 are centrally symmetric with centers in L_1.*

Proof. Let $p \notin K$ be a point of revolution for K and let L_1 be a diametral line of K passing through p. Let Π be a plane, $L_1 \subset \Pi$. Since p is a point of revolution for K, there exists a line of symmetry, L_2, of $\Pi \cap K$ passing through p. We affirm that $L_2 = L_1$. Otherwise, we would have that L_1 would be contained in an open half-plane determined by L_2, which contradicts the fact that L_1 is a diametral line of K. Since the aforesaid is true for every Π passing through L_1, the sections which are orthogonal to L_1 are centrally symmetric with centers in L_1. □

Now, let H be a plane containing L which intersects K in a disc of maximum radius.

Lemma 2.3. *If $[a,b]$ is a diametral chord of K, with $ab \cap L \neq \emptyset$, then ab is contained in H.*

Proof. Suppose to the contrary that $[c,d]$ is a diametral chord of K, with $cd \cap L \neq \emptyset$, which is not contained in the plane H. Furthermore, suppose that cd is not orthogonal to L. Notice that, the existence of this chord is clear since there is only one diametral chord of K which intersects L and is orthogonal to it. Now, let $y = cd \cap L$, and consider the diameter $[a,b]$, of $H \cap K$ passing through y. Let H' be the plane $\operatorname{aff}(cd \cup L)$, and let L' be the symmetric image of L with respect to cd in the plane H'. Since every section of K by a plane orthogonal to cd is centrally symmetric (by Lemma 2.2), we have that L' have the same properties as L. Let $H'' = \operatorname{aff}(L' \cup ab)$. Since H, H', and H'' are all different, then (by Lemma 2.1) we have that $H'' \cap K$ is a disc different of $H \cap K$. Let L_a, L_b and L''_a, L''_b be the supporting lines of $H \cap K$ and $H'' \cap K$, through a and b, respectively. Clearly, L_a, L_b, L''_a, L''_b, are all of them orthogonal to ab, then we easily deduce that $[a,b]$ is a diametral chord of K. This is impossible, since a convex body K cannot have two outward normal vectors intersecting each other in an exterior point of K. This contradiction shows that $[c,d] \subset H$. □

Lemma 2.4. *K is centrally symmetric.*

Proof. In order to prove the lemma, we are going to prove that the sections of K with planes containing W are centrally symmetric, where W is the diametral line of K which is orthogonal to H. Let D be a diametral line of K contained in H. In virtue that $L \cap D$ is a point of revolution for K, from Lemma 2.2, the sections of K orthogonal to D are centrally symmetric. In particular, if Π is a plane orthogonal to D and $W \subset \Pi$, then $\Pi \cap K$ is centrally symmetric. Since W is a diametral line of K, W is a diametral line of $\Pi \cap K$. Thus the midpoint of $W \cap K$ is the center of $\Pi \cap K$. Now, by Lemma 2.3 we have that every diameter of $H \cap K$ is a diametral chord of K, then we apply the above arguments and conclude that K is centrally symmetric. □

W.L.G. we may consider that the centre of K is at the origin O. Now, we will proceed to give the proof of the theorem for the case when $L \cap K = \emptyset$.

Proof of Theorem 1.3 when $L \cap K = \emptyset$. First at all, we will prove that the sections of K with planes that contain W are shadow boundaries of K. Let Π be a plane containing W. Let Π_1 be a plane parallel to Π and let D be the diametral chord of K orthogonal to Π. Since $D \cap L$ is a point of revolution for K, from Lemma 2.2, we have $\Pi_1 \cap K$ is centrally symmetric with center in D. Since K is centrally symmetric and the center of $-(\Pi_1 \cap K)$ is in D as well, we have:

$$-(\Pi_1 \cap K) = \alpha v + (\Pi_1 \cap K), \tag{1}$$

where v is a unit vector parallel to D and $\alpha > 0$ is a real number. Thus the shadow boundary of K in the direction of v is contained between the planes Π_1 and $-\Pi_1$. Finally, considering the sequence of planes Π_1, such that $\Pi_1 \to \Pi$, in virtue of (1), we conclude that $\Pi \cap K$ is the shadow boundary of K in direction v.

From here we have that there are two supporting planes of K at the extreme points of $W \cap K$, H_1, H_2 which are parallel to H. Now, we will prove that K is a body of revolution with axis W. Let Γ be a plane parallel to H which intersects the interior of K. Now, let $C = \operatorname{bd} K \cap H$ and C' be the orthogonal projection of $\operatorname{bd}(\Gamma \cap K)$ on the plane H. Consider a line $\Lambda \subset H$ through O, and let a, b be the points of intersection of Λ with C. Also, let a', b' be the corresponding points of intersection of Λ with C'. Since we know that the tangent lines to C and C' through the points a, b, a', b' are all parallel we obtain that C' is homothetic to C, that is, C' is a circle. Hence, $\Gamma \cap K$ is a disc with center in W. Consequently K is a body of revolution with axis W.

Finally, consider an arbitrary plane Π containing L and intersecting K, and let Φ be the circular section $\Pi \cap K$. Since K has W as a line of revolution, then there is only one sphere Σ with center at W and such that $\operatorname{bd} \Phi \subset \Sigma$. While rotating through W the circle $\operatorname{bd} \Phi$, the resulting circles remain always at Σ and also at $\operatorname{bd} K$, because both have W as a line of revolution. Since the above is true for every plane Π containing L, we conclude that K is a solid sphere. □

Remark 2.5. Notice that this proof works for the case when L is tangent to K, that is, L intersects K only in its boundary.

3. Proof when the false axis of symmetry intersects int K.

In this case we were able to remove the hypothesis of strict convexity, that is, we prove:

Theorem 3.1. *Let $K \subset \mathbb{R}^3$ be a convex body and $L \subset \mathbb{R}^3$ be a line such that $L \cap \operatorname{int} K \neq \emptyset$. Suppose that L is a false axis of revolution for K, then K is a solid sphere.*

Proof of Theorem 3.1. Let Π be a plane passing trough L. Since each point p in L is a revolution point of K, there exists a line of symmetry of $\Pi \cap K$ passing trough p, say L_p. If $L_p \neq L$ for all $p \in L$, then $\Pi \cap K$ is a circle. If $L_p = L$ for some $p \in L$, then $\Pi \cap K$ is symmetric with respect to L. Consequently, we have three possibilities:

(1) For each plane Π, $L \subset \Pi$, the section $\Pi \cap K$ is symmetric with respect to L.
(2) For each plane Π, $L \subset \Pi$, the section $\Pi \cap K$ is a circle.
(3) There exists two different planes Π_1, Π_2, passing through L, such that the section $\Pi_1 \cap K$ is a circle and L is not a line of symmetry of it and the section $\Pi_2 \cap K$ has L as a line of symmetry but is not a circle.

A given point $x \in \operatorname{bd} K$ is said to be *regular* if there is exactly one supporting plane of K passing through x. The following lemma will be often used in what follows.

Lemma 3.2. *Let K be a convex body in the Euclidean 3-space $\mathbb{R}^3$ and let $p_0 \in \operatorname{bd} K$ be a regular point which is also a point of revolution for K. Then K is a body of revolution and the axis of revolution passes through p_0 and is orthogonal to the supporting plane of K at p_0.*

Proof. Let Γ be a supporting plane of K through p_0 and let A be a line through p_0 orthogonal to Γ. We shall prove that A is an axis of revolution by proving that every plane H through A is a plane of symmetry for K. Let H be a plane through A and let

$l \subset \Gamma$ be a line through p_0 and orthogonal to H. If Δ is a plane through l, then $K \cap \Delta$ has $H \cap \Delta$ as an axis of symmetry. This is so because, by hypothesis, the section $K \cap \Delta$ has an axis of symmetry that passes through p_0 and hence is orthogonal to its supporting line l. Since this holds for every plane Δ through l, then H is a plane of symmetry for K. Since this holds for every plane H through A, then A is an axis of revolution. □

Case **(1)**. We will assume now that the condition **(1)** holds.

Lemma 3.3. *Theres is no disc D in* $\operatorname{bd} K$ *such that the the plane of D is orthogonal to L and D is passing through some of the points* $\{q_1, q_2\} = L \cap K$.

Proof. Contrary to the assertion of the lemma, let us assume that there exits a disc D in $\operatorname{bd} K$ such that the plane of D, say Γ, is orthogonal to L and the point q_1 is in D (the argument is similar if we assume that q_2 is in D). If $q_1 \in \operatorname{int} D$, then q_1 is a regular point of K and, in virtue of Lemma 3.2, K is a body of revolution with axis L but this is in contradiction with the assumption that L is a false axis of revolution of K. Now if $q_1 \in \operatorname{bd} D$, since we are assuming the condition **(1)**, each plane Δ passing through L intersects D in a chord whose image after reflection in L is a chord which is situated also in Γ. Varying the plane Δ trough L, we see that the collection of such chords is a circle $D' \subset \Gamma$ contained in $\operatorname{bd} K$. In virtue of the convexity of K, $\operatorname{conv}(D \cup D') \subset K$ and, consequently, we have q_1 is in the interior of a circle contained in Γ. This is in contradiction with the first part of proof of Lemma 3.3. The claim of Lemma 3.3 follows. □

Lemma 3.4. *Let $M \subset \mathbb{R}^2$ be a convex figure, symmetric with respect to L, and let T be a supporting line of K, orthogonal to L and passing through $q \in L \cap \operatorname{bd} M$. Suppose that there is no segment $E \subset \operatorname{bd} M$ such that $E \subset T$. Then there exists a segment $I \subset L \cap M$, $q \in I$, such that for every $p \in I$ the unique chord of M which has p as its midpoint is the chord orthogonal to L.*

Proof. We consider a coordinate system (x, y) for $\mathbb{R}^2$ such that L is the x-axis, q is the origin and $M \subset \{(x, y) \mid x \leq 0\}$. For each point $p \in L \cap M$, with coordinates $(t, 0)$, we denote by $I(t)$ the chord of M orthogonal to L and by $\mid I(x) \mid$ the length of $I(x)$. Let R be the supremum of the lengths of chords of M orthogonal to L, that is,

$$R = \sup_{x \in L \cap M} \mid I(x) \mid .$$

We denote by Ω_R the set $\{x \in \mathbb{R} : R = \mid I(x) \mid\}$ and let α be the supremum of Ω_R. Since there is no segment $E \subset \operatorname{bd} M$ such that $E \subset T$, we conclude that $\alpha < 0$, furthermore, as $q \in \operatorname{bd} K$, $\mid I(x) \mid$ is a strictly decreasing function for $x > \alpha$.

Consider a point p_0 in $L \cap M$, with coordinates $(t_0, 0)$, such that $\alpha/2 < t_0$. We will see that in the set of the chords of M passing trough p_0, the only chord which has its midpoint in p_0 is $I(t_0)$. We will see this by the absurd. Thus we assume that there exists a chord with end point $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\operatorname{bd} M$, $a_1 < b_1$ (consequently, $ab \neq I(t_0)$) and with p_0 as its midpoint. Since M is symmetric with respect to L, $a' = (a_1, -a_2)$ and $b' = (b_1, -b_2)$ belongs to $\operatorname{bd} M$ and $pa = pa'$ and

$pb = pb'$. From $pa = pb$, we have $pa' = pb'$, that is, the chord $a'b'$ has p as its midpoint. From here we get

$$I(a_1) = I(b_1). \tag{2}$$

On the other hand, since inequalities $b_1 < 0$ and $\alpha/2 < t_0$ and equality $t_0 = (a_1+b_1)/2$ holds, we have

$$\alpha < 2t_0 < 2t_0 - b_1 = a_1. \tag{3}$$

In virtue of the fact that $\mid I(x) \mid$ is a strictly decreasing function for $x > \alpha$ and the condition $a_1 < b_1$, from (3) we conclude $I(a_1) > I(b_1)$. But this is in contradiction with (2). From such contradiction the claim of Lemma 3.4 follows. □

Now we consider an orthogonal coordinate system (x, y, z) for $\mathbb{R}^3$ such that L is the axis z, q_1 is the origin. Let $l(\theta)$ be the line passing through the origin, in the plane xy and making an angle θ with the axis x and let $\Pi(\theta)$ be the plane determined by $l(\theta)$ and axis z. We denote by $K(\theta)$ the section $\Pi(\theta) \cap K$. In virtue than we are assuming that the condition **(1)** holds, $K(\theta)$ is symmetric with respect to axis z for all θ in $[0, \pi]$. Consequently, varying θ in $[0, \pi]$, we see that plane xy is a supporting plane of K at q_1. We choose the notation for q_1 such that

$$K \subset \{(x, y, z) \mid z \leq 0\}. \tag{4}$$

Lemma 3.5. *There exists a point $p \in L$, close enough to q_1, such that the chords of K which has its midpoint in p are those orthogonal to L.*

Proof. In virtue of Lemma 3.3, for all θ in $[0, \pi]$, except for, perhaps, at most one $\theta_0 \in [0, \pi]$, there are no line segments contained in $\operatorname{bd} K(\theta)$, orthogonal to L and passing through q_1. Since the conditions **(1)** and (4) holds, the conditions of Lemma 3.4 are satisfied for $K(\theta)$, for all θ in $[0, \pi]$, $\theta \neq \theta_0$. Now Lemma 3.5 follows easily from continuity and compactness arguments. □

Lemma 3.6. *For all $\theta \in [0, \pi]$, $\Pi(\theta)$ is a plane of symmetry of K.*

Proof. Let $\theta \in [0, \pi]$. We take a point $p \in L$ given by Lemma 3.5. Let W be the plane passing through p and orthogonal to axis z. En virtue that we are assuming the condition **(1)**, $W \cap K$ is centrally symmetric with center at p. We consider a plane Σ passing through the origin and containing the line $l(\theta + \pi/2)$. Since p is a revolution point, there exists a line of symmetry of $(p + \Sigma) \cap K$ passing through p. We are going to show that such line is $(p + \Sigma) \cap \Pi(\theta)$. We will see this assuming the contrary and we will rich a contradiction. Suppose that there exists a line of symmetry $(p+\Sigma) \cap K$, say Δ, such that is passing through p and $\Delta \neq (p+\Sigma) \cap \Pi(\theta)$. Then there exists a chord I of $(p+\Sigma) \cap K$, $l \perp \Delta$ and it has its midpoint in p. Since $\Delta \neq (p+\Sigma) \cap \Pi(\theta)$ we have $l \neq (p+l(\theta+\pi/2)) \cap K$ and I is not contained in W. But this is in contradiction with Lemma 3.5. Such contradiction shows that $(p+\Sigma) \cap \Pi(\theta)$ is line of symmetry of $(p + \Sigma) \cap K$.

Finally, varying the plane Σ, always having $l(\theta + \pi/2) \subset \Sigma$, we conclude that $\Pi(\theta)$ is plane of symmetry of K. □

From Lemma 3.6 follows that all the sections of K, orthogonal to L, are circles with center at L and, consequently, K is a body of revolution with axis L. This is in contradiction with the assumption that L is a false axis of revolution. Hence such contradiction shows that case **(1)** is impossible.

Cases **(2)***, and* **(3)***.* Since $L \cap \operatorname{int} K \neq \emptyset$, we have that L intersects $\operatorname{bd} K$ in exactly two points, say $\{a, b\} = \operatorname{bd} K \cap L$. It is easy to see that a and b are regular, for that purpose only note that there are at least two different sections which are circles passing through a and b, simultaneously. By Lemma 3.2, there is an axis of revolution through the boundary point a which is normal to K. Analogously, there is an axis of revolution through the boundary point $b \neq a$ which is normal to K. If L is normal to K at a and b, hence L is the axis of revolution for K, but this is a contradiction. This implies that K has two different axis of revolution, one through a and the other through b. It is an easy exercise to prove that a convex body with two different axes of revolution is a solid sphere. We let the simple details to the interested reader. □

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