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An Integro-Extremization Approach for Non Coercive and Evolution Hamilton-Jacobi Equations

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We devote the *integro-extremization* method to the study of the Dirichlet problem for homogeneous Hamilton-Jacobi equations

$$\begin{cases} F(Du) = 0 & \text{in } \Omega \\ u(x) = \varphi(x) & \text{for } x \in \partial\Omega, \end{cases}$$

with a particular interest for non coercive hamiltonians F, and to the Cauchy-Dirichlet problem for the corresponding homogeneous time-dependent equations

$$\begin{cases} \frac{\partial u}{\partial t} + F(\nabla u) = 0 & \text{in }]0,T[\times\Omega \\ u(0,x) = \eta(x) & \text{for } x \in \Omega \\ u(t,x) = \psi(x) & \text{for } (t,x) \in [0,T] \times \partial\Omega. \end{cases}$$

We prove existence and some qualitative results for viscosity and almost everywhere solutions, under suitably convexity conditions on the hamiltonian F, on the domain Ω and on the boundary datum, without any growth assumptions on F.

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1. Introduction

This paper is devoted to the application of the *integro-extremization* method, introduced in the papers [12], [13] [14], [15], [16], to the study of stationary non coercive and evolution Hamilton-Jacobi equations. The integro-extremization method, roughly speaking, can be described as follows: consider a set of Sobolev functions satisfying almost everywhere a pointwise inequality involving the gradient which makes it compact in the strong L^1 topology; Weiertrass Theorem implies the existence of an element in the set which maximizes (minimizes) the integral. Imposing suitable conditions, such element satisfies the equality in place of the inequality, so that the integro-extremization procedure "transforms inequalities into equalities" and turns out to be a powerful tool to solve first order differential equations and related problems. Originally it was introduced to study non semicontinuous problems of the Calculus of Variations, and in [16] it has been devoted to stationary Hamilton-Jacobi equations.

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Our aim is to apply this procedure in order to find new tools in the analysis of first order partial differential equations of Hamilton-Jacobi type both for existence and for qualitative theories.

Hamilton-Jacobi equations are of particular interest in various aspects of Optimal Control Theory like, for example, the following classical problem consisting in

$$\textit{Minimizing} \int_0^T L(t, x(t), u(t))\, dt + g(x(T)),$$

on absolutely continuous maps $x(\cdot)$ and measurable maps $u(\cdot)$ defined on the interval $[0, T]$, taking values in $\mathbb{R}^m$ and satisfying the conditions

$$x' = f(t, x, u), \quad u(t) \in V \ \textit{a.e.}\ t \in]0, T[, \qquad x(0) = x_0,$$

for some given functions L (the lagrangian), f and g and for a given control set V. Introducing the Hamiltonian

$$H(t, x, p) \doteq \sup \{-p \cdot f(t, x, u) - L(t, x, u);\ u \in V\},$$

we may associate to the original problem the following *dynamic programming equation* of Hamilton-Jacobi type:

$$\begin{cases} \phi_t(t, x) - H(t, x, \nabla_x \phi(t, x)) = 0 & (t, x) \in]0, T[\times \mathbb{R}^m \\ \phi(T, x) = g(x) & x \in \mathbb{R}^m. \end{cases}$$

It is well known (see for example [2]) that if ϕ is a classical solution of this last equation, then the map

$$\Phi(t, x) \doteq \arg\sup \{-\nabla_x \phi(t, x) \cdot f(t, x, u) - L(t, x, u) :\ u \in V\}$$

provides an optimal feedback control for the Optimal Control Problem. The notion of viscosity introduced by Crandall and Lions gave a strong impulse in the study of this class of partial first order equations and we devote our attentions to this kind of solutions considering special classes of Hamilton-Jacobi equations. We manage them by the above mentioned integro-extremization method, claiming that it could provide new insights in this research.

The literature concerning viscosity solutions of Hamilton-Jacobi equations is more than wide. We mention first of all the outstanding monograph [11] and, for example, [3], [4], [6], [7], [9] and [10].

Consider a continuous function $F : \mathbb{R}^n \to \mathbb{R}$, an open bounded subset $\Omega \subset \mathbb{R}^n$ and the following homogeneous stationary Dirichlet problem:

$$\mathcal{P}_\varphi : \quad \begin{cases} F(Du) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where φ is a given boundary datum.

In paper [16] we have considered a hamiltonian $F = F(x, u, \xi)$, depending also on the variables x and u satisfying a standard coercivity condition with respect to the last variable ξ implying that the set of vectors ξ such that $F(x, u, \xi) \leq 0$ is bounded uniformly in (x, u). In the first part of the present paper we devote our interest to the non coercive case, corresponding to a hamiltonian $F = F(\xi)$ such that, setting

$$K \doteq \{\xi \in \mathbb{R}^n : F(\xi) \leq 0\},$$

the set K may be unbounded. By this way, in particular, the solutions may not be globally Lipschitz continuous and, moreover, obviously, the problem can be reformulated as a differential inclusion:

$$\begin{cases} Du \in \partial K & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where K is an open subset of $\mathbb{R}^n$. We stress that this problem has been widely studied (see for example [8] and references quoted there) but with the usual boundedness assumption on K.

We look for a precise class of solutions, namely continuous Sobolev functions taking the value φ on the boundary and solving the equation in $\mathcal{P}_\varphi$ almost everywhere and in viscosity sense. Hence we deal with generalized solutions (or almost everywhere solutions) with the additional properties provided by viscosity, namely uniqueness, stability and continuous dependence on boundary data.

We need convexity assumptions and, actually, we consider also the relaxed problem $\mathcal{P}_\varphi^{**}$ associated to $\mathcal{P}_\varphi$:

$$\mathcal{P}_\varphi^{**}: \quad \begin{cases} F^{**}(Du) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where F^{**} is the lower convex envelope of F. In addition we require convexity of the domain Ω and that φ is a concave function defined on the whole domain Ω. Under these conditions, assuming that F is coercive in one direction and that

$$\{\xi \in \mathbb{R}^n : \ F(\xi) > 0\} = \{\xi \in \mathbb{R}^n : \ F^{**}(\xi) > 0\},$$

we provide well posedness results for both non convex and relaxed problem, i.e. theorems ensuring existence and uniqueness of a maximal viscosity solution and its continuous dependence on the boundary datum φ.

With respect to classical results (see in addition, for example, [11], [3], [4], [1], [5] and literature quoted there) we do not require coercivity in F and consider nonconvex hamiltonians F. Moreover our method allows to develop a well-posedness theory, that is to say existence, uniqueness and continuous dependence on boundary data.

In the second part of the paper we consider the associated Cauchy-Dirichlet problem for the evolution case:

$$\mathcal{P}: \quad \begin{cases} \frac{\partial u}{\partial t} + F(\nabla u) = 0 & \text{in }]0, T[\times\Omega \\ u(0, x) = \eta(x) & \text{for } x \in \Omega \\ u(t, x) = \psi(x) & \text{for } (t, x) \in [0, T] \times \partial\Omega. \end{cases}$$

Due to the linear term in the time derivative, the problem is intrinsically non coercive, so that the ideas used in the first part can be invoked also in this case. For these reasons the last sections of the paper consists in a natural extension of previous arguments and we propose them as a applications of the theory developed in the first part of the article.

We require that the hamiltonian $F : \mathbb{R}^n \to \mathbb{R}$ is convex and bounded from below, without any other assumptions. Unfortunately our method forces us to consider only the homogeneous case, corresponding to a function F depending only on the spatial gradient ∇u of the competing maps, excluding more general classes of problems; in addition we still require that the set Ω is convex and bounded and that the initial-boundary data η and ψ are traces on $\{0\} \times \Omega$ and on $[0, T] \times \partial\Omega$, respectively, of a concave function φ which we assume to to be a subsolution of the equation in $\mathcal{P}$.

Under these conditions we are able to show that there exists a unique maximal concave function $\overline{u}$ defined on $[0, T] \times \overline{\Omega}$ satisfying the initial-boundary conditions and solving the equation almost everywhere and in viscosity sense. In addition we show that the theory remains valid if we drop the Ω-boundary condition $u\lfloor_{\partial\Omega}= \psi\lfloor_{\partial\Omega}$ and consider simply the Cauchy problem.

Also in this case, with respecto to classical literature, we remove any coercivity assumption on F, providing by this way original results. The arguments used to treat the evolution case are very close to the ones of the first part of the paper; however we perform the proofs for the sake of completeness, so that two parts of the paper may be read indipendently.

2. Notations and preliminaries

In this paper $\mathbb{R}^m$ is the euclidean m-dimensional space and we denote respectively by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ the inner product and the euclidean norm, while $\{e_1, \dots, e_m\}$ is the canonical basis. A vector $\xi \in \mathbb{R}^m$ is written as $\xi = (\xi_1, \dots, \xi_m)$. For $x \in \mathbb{R}^m$ and $r > 0$, $B(x, r)$ is the open ball in $\mathbb{R}^m$ of center x and radius r; given $E \subseteq \mathbb{R}^m$, $\text{meas}(E)$ is the Lebesgue measure, ∂E is the boundary, E^c is the complement, χ_E is the characteristic function and $\text{co}(E)$ is the convex hull of E; by $\text{dist}(x, E)$ we mean the distance of the point x from the set E. Given an open bounded subset U of $\mathbb{R}^m$; we use the spaces $C^k(U)$, $\mathcal{D}(U) = C^\infty_c(U)$, $\mathcal{D}'(U)$, $L^p(U)$, $W^{1,p}(U)$, $W^{1,p}_0(U)$, $H^k(U)$, for $k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $1 \le p \le \infty$, with their usual (strong and weak) topologies. Dealing with a Sobolev function we assume to use the precise representative and, given two real valued functions u and v, we set $u \vee v \doteq \sup(u, v)$.

Definition 2.1. For an open set $U \subseteq \mathbb{R}^m$, we denote by $\text{Aff}(U)$ the set of piecewise affine elements of $W^{1,\infty}(U)$. Given $u \in \text{Aff}(U)$ we have

$$Du = \sum_{i=1}^{k} a_i \chi_{U_i},$$

where k is a positive integer, $a_i \in \mathbb{R}^m$ for $i = 1, \dots, k$ and

$$U = \left(\bigcup_{i=1}^{k} U_i \right) \cup N,$$

where $\{U_i, i = 1, \dots, k\}$ is a family of pairwise disjoint open subsets of U and N is a null set. We adopt the following notation:

$$R(Du) \doteq \{a_i, i = 1, \dots, k\}\,. \tag{1}$$

We need the following notions and refer to the monograph [4] (Chapter II) for proofs, general setting and for the definition of viscosity solution of Hamilton-Jacobi equations.

Definition 2.2. Let $U \subseteq \mathbb{R}^n$ be open, $v \in C^0(U)$ and $x_0 \in U$. We set

$$D^- v(x_0) \doteq \left\{\xi \in \mathbb{R}^n : \liminf_{x \to x_0} \frac{v(x) - v(x_0) - \langle \xi, x - x_0\rangle}{|x - x_0|} \geq 0\right\},$$

$$D^+ v(x_0) \doteq \left\{\xi \in \mathbb{R}^n : \limsup_{x \to x_0} \frac{v(x) - v(x_0) - \langle \xi, x - x_0\rangle}{|x - x_0|} \leq 0\right\}.$$

We call these sets, respectively, *super* and *sub* differentials (or semidifferentials) of v at the point x_0 and set also

$$A^-(v) \doteq \left\{x \in U : D^- v(x) \neq \emptyset\right\}.$$
$$A^+(v) \doteq \left\{x \in U : D^+ v(x) \neq \emptyset\right\},$$

We recall the following fundamental properties of semidifferentials.

Lemma 2.3. *Let $U \subseteq \mathbb{R}^n$ be open, $v \in C^0(U)$ and $x_0 \in U$.*

- (*i*) *$\xi \in D^- v(x_0)$ if and only if there exists a function $\phi \in C^1(U)$ such that $D\phi(x_0) = \xi$ and the function $x \mapsto v(x) - \phi(x)$ has local minimum at the point x_0.*
- (*ii*) *$\xi \in D^+ v(x_0)$ if and only if there exists a function $\phi \in C^1(U)$ such that $D\phi(x_0) = \xi$ and the function $x \mapsto v(x) - \phi(x)$ has local maximum at the point x_0.*
- (*iii*) *$D^+ v(x_0)$ and $D^- v(x_0)$ are closed convex possibly empty subsets of $\mathbb{R}^n$.*
- (*iv*) *If v is differentiable at the point x_0 then $D^+ v(x_0) = D^- v(x_0) = \{Dv(x_0)\}$.*
- (*v*) *If both $D^+ v(x_0)$ and $D^- v(x_0)$ are nonempty, then u is differentiable at x_0 and $D^+ v(x_0) = D^- v(x_0) = \{Dv(x_0)\}$.*
- (*vi*) *The sets $A^-(v)$ and $A^+(v)$ are dense in U.*

We shall need the following two results (see [16]).

Lemma 2.4. *Let U be an open subset of $\mathbb{R}^n$, $p \in [1, \infty]$, $v \in W^{1,p}(U, \mathbb{R}) \cap C^0(U, \mathbb{R})$, $x_0 \in A^-(v)$, $\xi \in D^- v(x_0)$, $r > 0$ and $\rho > 0$ such that $B(x_0, \rho) \subseteq U$. Then there exists a map $\tilde{v} \in W^{1,p}(U, \mathbb{R}) \cap C^0(U, \mathbb{R})$ with the following properties:*

(*i*) $\tilde{v} - v \in W_0^{1,p}(U)$;

(*ii*) $v(x) \leq \tilde{v}(x)$ *for* $x \in U$;

(*iii*) $\tilde{\Lambda} \doteq \{x \in U \colon \tilde{v}(x) > v(x)\}$ *is nonempty and* $\tilde{\Lambda} \subseteq B(x_0, \rho)$;

(*iv*) $$\begin{cases} |D\tilde{v}(x) - \xi| = r, & \text{for a.e. } x \in \tilde{\Lambda} \\ D\tilde{v}(x) = Dv(x), & \text{for a.e. } x \in U \setminus \tilde{\Lambda}; \end{cases}$$

(*v*) $$\int_U \tilde{v}\, dx > \int_U v\, dx.$$

We stress that (v) is a straightforward consequence of (ii) and (iii).

Lemma 2.5. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let $t \in \mathbb{R}^+$ be small, $M > 0$ and $u, v \in W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$ be such that*

$$\|u\|_{C^0(\overline{\Omega})}, \|v\|_{C^0(\overline{\Omega})} \leq M$$

and

$$\|u - v\|_{C^0(\partial\Omega)} \leq t.$$

Then there exist an open subset $\Omega_t \subseteq \Omega$, a map $w_t \in W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$ and a continuous increasing function $\sigma : [0, +\infty[\to [0, +\infty[$, depending only on M and on Ω, with $\sigma(0) = 0$ such that

$$\begin{array}{rl} (i) & \Omega_t \subseteq \Omega_s \quad \textit{if } t > s; \\ (ii) & \operatorname{meas}(\Omega \setminus \Omega_t) \to 0 \quad \textit{as } t \to 0+; \\ (iii) & w_t = u \quad \textit{in } \Omega_t; \\ (iv) & w_t = v \quad \textit{in } \partial\Omega; \\ (v) & |w_t - u|, |w_t - v| \leq \sigma(t) \quad \textit{in } \Omega \setminus \Omega_t; \\ (vi) & \operatorname{dist}(Dw_t(x), \operatorname{co}(\{Du(x), Dv(x)\})) \leq \sigma(t) \quad \textit{for a.e. } x \in \Omega. \end{array}$$

3. The non coercive case: hypotheses

In Section 4 and 5 we assume the following hypotheses.

Hypothesis 3.1. $F : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that

$$\{\xi \in \mathbb{R}^n : F(\xi) > 0\} = \{\xi \in \mathbb{R}^n : F^{**}(\xi) > 0\}, \tag{2}$$

where F^{**} is the lower convex envelope of F.

We assume that the subset of $\mathbb{R}^n$ on which the function F^{**} is non positive is contained in a strip. More precisely, up to rotation and relabeling of the indices, we assume that there exists a positive δ such that

$$F^{**}(\xi) \leq 0 \quad \Longrightarrow \quad |\xi_1| \leq \delta. \tag{3}$$

Remark 3.2. A simple example of a function satisfying Hypothesis 3.1 is, as the reader can easily verify by direct inspection, the following:

$$F(\xi) \doteq \left(|\xi_1|^2 - 1\right)^2 - 2.$$

In Section 5 we shall use the following slightly stronger version of Hypothesis 3.1.

Hypothesis 3.3. The same as in Hypothesis 3.1 assuming $F^{**} = F$ and that such function is uniformly continuous. In place of (3) we impose the the following similar condition: there exists a positive $\tilde{\delta}$ such that

$$F^{**}(\xi) \leq 1 \quad \Longrightarrow \quad |\xi_1| \leq \tilde{\delta}. \tag{4}$$

Hypothesis 3.4. The domain $\Omega \subset \mathbb{R}^n$ is open bounded and convex and we consider, as boundary datum, a map φ defined on a neighbourhood $\tilde{\Omega}$ of Ω, assuming that $\varphi \in W^{1,\infty}(\tilde{\Omega})$ is a concave function such that

$$F^{**}(D\varphi(x)) \leq 0 \quad \text{for a.e. } x \in \tilde{\Omega}.$$

Definition 3.5. We define the sets

$$S_\varphi^\infty \doteq \left\{u \in \varphi + W_0^{1,\infty}(\Omega) \colon F^{**}(Du) \leq 0 \text{ a.e. in } \Omega\right\}, \tag{5}$$

$$S_\varphi^1 \doteq \left\{u \in \left(\varphi + W_0^{1,1}(\Omega)\right) \cap C^0(\overline{\Omega}) \colon F^{**}(Du) \leq 0 \text{ a.e. in } \Omega\right\}. \tag{6}$$

We remark that S_φ^∞ is a subset of S_φ^1 and that, if Hypothesis 3.4 is satisfied, S_φ^∞ is nonempty.

4. The non coercive case: existence and uniqueness of the solution

Throughout this section we assume Hypotheses 3.1 and 3.4.

Lemma 4.1. *We have:*

$$s \doteq \sup\left\{\int_\Omega u(x)\ dx, u \in S_\varphi^1\right\} = \sup\left\{\int_\Omega u(x)\ dx, u \in S_\varphi^\infty\right\} < +\infty. \tag{7}$$

Proof. *Step 1.* By (3) in Hypothesis 3.1 we have

$$\|D_1 u\|_{L^\infty(\Omega)} \leq \delta \quad \forall u \in S_\varphi^1. \tag{8}$$

Poincaré inequality and (8) imply that

$$\int_\Omega |u(x)|\, dx \leq C\|D_1 u\|_{L^1(\Omega)} \leq C\delta \operatorname{meas}(\Omega) \quad \forall u \in S_\varphi^1,$$

for a suitable positive constant C. Hence $\int_\Omega u(x)\, dx$ is bounded both in S_φ^∞ and in S_φ^1 and we have to prove that the two suprema coincide.

Step 2. Set

$$s \doteq \sup\left\{\int_\Omega u(x)\ dx, u \in S_\varphi^\infty\right\}.$$

For every $\epsilon > 0$, consider the set

$$U_\varphi^\epsilon \doteq \left\{v \in W^{1,\infty}(\Omega) \colon F^{**}(Dv) \leq 0 \text{ a.e. in } \Omega \text{ and } \|v - \varphi\|_{C^0(\partial\Omega)} \leq \epsilon\right\}. \tag{9}$$

Proceeding as above, and by elementary computations, we have

$$s^\epsilon \doteq \sup\left\{\int_\Omega u(x)\ dx, u \in U_\varphi^\epsilon\right\} < +\infty$$

and

$$\lim_{\epsilon \to 0+} s^\epsilon = s. \tag{10}$$

Take $u \in S^1_\varphi$ and write $u = \varphi + z$, with $z \in W_0^{1,1}(\Omega)$. Recalling Hypothesis 3.4 and extending by zero the map z outside Ω, we have that u is defined on a neighbourhood $\tilde{\Omega}$ of Ω and that the inequality $F^{**}(Du) \le 0$ is satisfied a.e. in $\tilde{\Omega}$. Consider a regularizing sequence (ρ_k) and, for $x \in \overline{\Omega}$ and $k \in \mathbb{N}$ sufficiently large, define

$$u_k \doteq \rho_k * u.$$

By Jensen inequality, by the convexity of F^{**}, and recalling that the convolution is a convex combination, we have, for $x \in \overline{\Omega}$:

$$\begin{aligned} F^{**}(Du_k(x)) &= F^{**}\left(\int_{B\left(0,\frac{1}{k}\right)} \rho_k(x-y) Du(y)\, dy\right) \\ &\le \int_{B\left(0,\frac{1}{k}\right)} \rho_k(x-y) F^{**}(Du(y))\, dy \le 0, \end{aligned} \tag{11}$$

for all $k \in \mathbb{N}$ large enough so that

$$B\left(x, \frac{1}{k}\right) \subset \tilde{\Omega} \quad \forall x \in \overline{\Omega}.$$

By the uniform convergence of u_k to u on $\overline{\Omega}$, we have that for every $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that

$$\|u_k - \varphi\|_{C^0(\partial\Omega)} \le \epsilon \quad \forall k \ge k_\epsilon. \tag{12}$$

Combining (11) and (12), recalling the Definition (9), we deduce that $u_k \in U^\epsilon_\varphi$ for every $k \ge k_\epsilon$ and, obviously, we have

$$u_k \longrightarrow u \quad \text{in } L^1(\Omega).$$

Step 3. Assume, by contradiction, that there exist $u \in S^1_\varphi$ and $\beta > 0$ such that

$$\int_\Omega u(x)\, dx = s + \beta.$$

By previous step, taking into account (10), there exists a family (v^ϵ) in U^ϵ_φ with the following properties:

$$\int_\Omega v^\epsilon(x)\, dx \le s^\epsilon \longrightarrow s \quad \text{as } \epsilon \to 0+, \tag{13}$$

$$\int_\Omega v^\epsilon(x)\, dx \longrightarrow s + \beta > s \quad \text{as } \epsilon \to 0+. \tag{14}$$

Formulas (13) and (14) provide the contradiction and the lemma is proved. □

By classical arguments (see for example Proposition 2.2 in [4] and Theorem 3.8 in [5]) we obtain the following result.

Lemma 4.2. *Let $u \in S^\infty_\varphi$. Then*

$$\begin{aligned} F^{**}(\xi) \le 0 \quad \forall \xi \in D^+u(x),\ \forall x \in \Omega; \\ F^{**}(\xi) \le 0 \quad \forall \xi \in D^-u(x),\ \forall x \in \Omega. \end{aligned}$$

Proof. *Step 1.* We start by proving the following

Claim. Given an open set $\Lambda \subset\subset \Omega$, there exists a sequence (u_k) in $C^\infty(\Lambda)$ such that

$$u_k \longrightarrow u \ \text{ uniformly on } \Lambda, \tag{15}$$

$$F^{**}(Du_k(x)) \leq 0 \ \ \forall x \in \Lambda \ \forall k \in \mathbb{N}. \tag{16}$$

Take a regularizing sequence (ρ_k) and set, for k sufficiently large,

$$u_k(x) \doteq (\rho_k * u)(x), \quad x \in \Lambda.$$

Remarking that the convolution is a convex combination, by the convexity of F^{**}, by Jensen inequality and recalling that $u \in S_\varphi^\infty$, we have, for $x \in \Lambda$:

$$\begin{aligned} F^{**}(Du_k(x)) &= F^{**}(\rho_k * Du(x)) \\ &= F^{**}\left(\int_{B(x,\frac{1}{k})} \rho_k(x-y) Du(y)\, dy\right) \\ &\leq \int_{B(x,\frac{1}{k})} \rho_k(x-y) F^{**}(Du(y))\, dy \leq 0, \end{aligned}$$

for k is sufficiently large so that

$$\Lambda + B\left(0, \frac{1}{k}\right) \subset \Omega.$$

Hence the claim is proved.

Step 2. Take now $x_0 \in A^+(u)$, $\xi \in D^+u(x_0)$.

Consider an open set $\Lambda \subset\subset \Omega$ containing x_0 and the sequence (u_k) defined in previous step. By a standard argument (see for example Proposition 2.2 in [4]), by (15), recalling the smoothness of u_k and point (iv) in Lemma 2.2, we deduce the existence of a sequence (x_k) in Λ such that

$$x_k \to x_0 \quad \text{and} \quad Du_k(x_k) \to \xi \ \text{ as } k \to \infty. \tag{17}$$

By the continuity of F^{**}, it follows that

$$F^{**}(Du_k(x_k)) \longrightarrow F^{**}(\xi) \leq 0 \ \text{ as } k \to \infty. \tag{18}$$

Hence, (16), (17) and (18), imply that

$$F^{**}(\xi) \leq 0.$$

By the same argument we obtain that

$$F^{**}(\xi) \leq 0 \ \ \forall x \in A^-(u) \text{ and } \forall \xi \in D^-u(x).$$

□

Definition 4.3. Let $U \subset \mathbb{R}^m$ open, bounded convex and let $u \in W^{1,\infty}_{\rm loc}(U) \cap C^0(\overline{U})$. Consider the set

$$V_u \doteq \{v \text{ concave}, \ v \geq u\}.$$

Then define

$$\hat{u}(x) \doteq \inf\{v(x), \ v \in V_u\}, \quad x \in \Omega.$$

We call $\hat{u}$ the *upper concave envelope* of u (uce(u)).

The following arguments are well known and can be found in [1]; we give a proof for the sake of completeness and for convenience of the reader.

Remark 4.4. We show elementary properties of the upper concave envelope.

(i) The map $\hat{u}$ is concave and then it belongs to $W^{1,\infty}_{\rm loc}(U)$.
(ii) Let $\Lambda \subseteq \partial U$ be a subset of the boundary ∂U of U. If $u\lfloor_\Lambda$ coincides with $\varphi\lfloor_\Lambda$, where $\varphi \in C^0(\overline{U})$ is a concave function, then $\hat{u}\lfloor_\Lambda = u\lfloor_\Lambda$.

Proof. Property (i) is well known. To prove (ii), for every $\epsilon > 0$ one consider the set

$$\tilde{V}^\epsilon_u \doteq \{v \text{ concave}, \ v \geq u, \ u\lfloor_\Lambda \leq \ v\lfloor_\Lambda \leq \ u\lfloor_\Lambda + \epsilon\}.$$

Clearly $\tilde{V}^\epsilon_u$ is nonempty. Indeed consider the gauge function ρ associated to the (convex) set U and, for $\alpha \in \mathbb{R}^+$, define the map

$$w(x) \doteq \varphi(x) + \alpha(1 - \rho(x)) + \epsilon.$$

It is immediate to see that w is concave and that, if α is sufficiently large, $w \geq u$: hence w belongs to $\tilde{V}^\epsilon_u$.

Now set

$$z^\epsilon \doteq \inf\left\{v, \ v \in \tilde{V}^\epsilon_u\right\};$$

and then

$$z \doteq \inf\{z^\epsilon, \ \epsilon > 0\};$$

It is immediate to see that the map z is concave and, since $\tilde{V}^\epsilon_u \subseteq V_u$, for every $\epsilon > 0$, we have $z \geq \hat{u}$. By definition we have $u\lfloor_\Lambda \leq z\lfloor_\Lambda \leq u\lfloor_\Lambda + \epsilon$ for every $\epsilon > 0$ and then z coincides with u on Λ.

The proof will be achieved if we show that $z \leq \hat{u}$. Assume, by contradiction $z > \hat{u}$ on some (open) subset E of Ω and set $w \doteq \inf(\hat{u}, z)$, recalling that the infimum of two concave function is concave. Since we have, necessarily, $\hat{u}\lfloor_\Lambda \geq \varphi\lfloor_\Lambda = u\lfloor_\Lambda = z\lfloor_\Lambda$ on Λ, we would have $w \in \tilde{V}^\epsilon_u$ for every $\epsilon > 0$ and $w < z$ on $E \subseteq U$; a contradiction. □

Lemma 4.5. *Let $U \subset \mathbb{R}^m$ be open, bounded and convex. Let $u \in \mathrm{Aff}(U)$, $\Xi \subseteq \mathbb{R}^m$ closed convex and assume $R(Du) \subseteq \Xi$. Then $\hat{u} = uce(u) \in \mathrm{Aff}(U)$ and $R(D\hat{u}) \subseteq \Xi$.*

Proof. Recalling (1) in Definition 2.1, we write

$$R(Du) = \{a_i, i = 1, \ldots, M\} \subseteq \Xi.$$

The map $\hat{u}$, being concave, is differentiable almost everywhere; in addition it is well known that the convex envelope of a polyhedron is a polyhedron, hence the gradient of

$\hat{u}$ is piecewise constant and then we conclude that $\hat{u}$ belongs to $\mathrm{Aff}(U)$. In particular, recalling Definition 2.1, we may write

$$R(D\hat{u}) = \{b_j, j = 1, \dots, k\},$$

for some $k \in \mathbb{N}$, $b_j \in \mathbb{R}^m$, $j = 1, \dots, k$, and

$$D\hat{u} = \sum_{j=1}^{k} b_j \chi_{U_j}.$$

Our claim is to prove that every vector b_j lies in Ξ.

Take an index j_0 and call V the corresponding (nonempty) open set U_{j_0}. By definition of $\hat{u}$, V is a polyhedron and there exist points $x^0, \dots, x^m$, vertices of V, such that $\hat{u}(x^i) = u(x^i)$, $i = 0, \dots, n$. In addition the affine function which coincides with $\hat{u}$ on V is greater or equal than $\hat{u}$ on U (in terms of convex analysis it defines a supporting hyperplane for the graph of $\hat{u}$). By rotation and translation we may assume $b \doteq b_{j_0} = 0$ and $\hat{u}(x^i) = u(x^i) = 0$ for $i = 0, \dots, n$. Hence we reduce to have $\hat{u} = 0$ on V, $u \leq \hat{u} \leq 0$ on U.

The lemma is so proved if we show that

$$0 \in \mathrm{co}\,\{a_i, i = 1, \dots, M\}.$$

Hence the proof will be achieved by the following statement.

Lemma 4.6. *Let $U \subset \mathbb{R}^m$ be a bounded open convex subset of $\mathbb{R}^m$ and let $u \in \mathrm{Aff}(U)$ such that $u(y) = 0$, for some $y \in \Omega$ and $u \geq 0$ on U. Write*

$$R(Du) = \{a_i, i = 1, \dots, k\}.$$

Then

$$0 \in \mathrm{co}\,\{a_i, i = 1, \dots, k\}.$$

Proof. Assume, by contradiction, that the thesis is false. The range of Du is finite; hence $\mathrm{co}(R(Du))$ is compact and consequently, by Hahn-Banach Theorem, there exists a direction $\mathbf{n}$ in $\mathbb{R}^m$ such that $a_i \cdot \mathbf{n} > 0$ for all $i = 1, \dots, k$. By translation and rotation we may assume $\mathbf{n} = e_1 = (1, 0, \dots, 0)$, so that $\frac{\partial u}{\partial x_1} > 0$ a.e. in $U = \mathrm{int}(U)$. Hence the conditions $u(y) = 0$ and $u \geq 0$ on U provide the contradiction. □

Lemma 4.7. *Let $U \subset \mathbb{R}^m$ be open, bounded and convex; let Ξ be a closed convex subset of $\mathbb{R}^m$ and let $u \in W^{1,\infty}_{loc}(U)$ be such that $Du(x) \in \Xi$ for a.e. $x \in U$. Then $\hat{u} \equiv uce(u)$ belongs to $W^{1,\infty}_{loc}(U)$ and $D\hat{u}(x) \in \Xi$ for a.e. $x \in U$.*

Proof. Let (u_k) be a sequence in $\mathrm{Aff}(U)$ such that $u_k \leq u$ for every $k \in \mathbb{N}$, $u_k \to u$ uniformly in U, $Du_k \to Du$ in $L^2_{\mathrm{loc}}(U)$, $Du_k \to Du$ a.e. in U and

$$Du_k \in \Xi + B(0, \sigma_k) \quad \text{a.e. in } \Omega, \tag{19}$$

where

$$\sigma_k \overset{k \to \infty}{\longrightarrow} 0. \tag{20}$$

To see that (19)–(20) can be satisfied, notice that, by the use of mollifiers and convolution, u can be approximated uniformly and in $H^1_{\rm loc}(U)$ by a smooth function v such that $Dv \in \Xi$ in U and Dv is pointwise close to Du in U. Then, in turn, v can be approximated uniformly and in $H^1_{\rm loc}(U)$ by a function w in $\mathrm{Aff}(U)$ such that Dw (where it exists) is uniformly close to Dv in U.

For every index $k \in \mathbb{N}$ consider the upper concave envelope $\hat{u}_k$ of u_k. By Lemma 4.5 and by (19) we have $\hat{u}_k \in W^{1,\infty}_{\rm loc}(U)$,

$$D\hat{u}_k \in \Xi + \overline{B(0,\sigma_k)} \ \text{ a.e. in } U \ \forall k \in \mathbb{N} \tag{21}$$

and, for every open subset $V \subset\subset U$,

$$\int_V |D\hat{u}_k(x)|^2 \, dx \le (\|Du\|_{L^\infty(V)} + 1)^2 \operatorname{meas}(V) \ \ \forall k \in \mathbb{N}. \tag{22}$$

Formula (22) implies that there exists $v \in W^{1,\infty}_{\rm loc}(U)$ such that, passing if necessary to subsequences,

$$\hat{u}_k \rightharpoonup v \ \text{ in } H^1_{\rm loc}(U) \text{ as } k \to \infty. \tag{23}$$

The convergence (23) imply also that

$$\hat{u}_k \overset{k\to\infty}{\longrightarrow} v \ \text{ in } \mathcal{D}'(U);$$

hence we have

$$\frac{\partial^2 \hat{u}_k}{\partial \xi^2} \overset{k\to\infty}{\longrightarrow} \frac{\partial^2 v}{\partial \xi^2} \ \text{ in } \mathcal{D}'(U) \ \forall \xi \in \mathbb{R}^m\colon \ |\xi| = 1.$$

The concavity of $\hat{u}_k$, for every $k \in \mathbb{N}$, implies that

$$\frac{\partial^2 \hat{u}_k}{\partial \xi^2} \le 0 \ \text{ in } \mathcal{D}'(U) \ \forall k \in \mathbb{N} \ \forall \xi \in \mathbb{R}^m\colon \ |\xi| = 1,$$

so that

$$\frac{\partial^2 v}{\partial \xi^2} \le 0 \ \text{ in } \mathcal{D}'(U) \ \forall \xi \in \mathbb{R}^m\colon \ |\xi| = 1. \tag{24}$$

Inequality (24) means that v is concave.

Being $u_k \le u$ for every k, we deduce $\hat{u}_k \le \hat{u}$ for every k and, consequently, $v \le \hat{u}$. On the other hand convergence (23) and the concavity of $\hat{u}_k$ for every k imply that $\hat{u}_k \to v$ pointwise. Since $u_k \to u$ pointwise and $u_k \le \hat{u}_k \to v$ as $k \to \infty$ we deduce that $u \le v$. By the definition of $\hat{u}$ as the smallest concave function greater than u we have $\hat{u} \le v$. Hence we have $\hat{u} = v$.

Finally from the convergence $D\hat{u}_k \rightharpoonup D\hat{u}$ in $L^2_{\rm loc}(U)$ (i.e. (23)), from (21), (20), and from Mazur Lemma, we deduce that, for every open subset $V \subset\subset U$,

$$D\hat{u} \in \bigcap_k \left(\Xi + \overline{B(0,\sigma_k)}\right) = \Xi \ \text{ a.e. in } V.$$

The arbitrariness of V yields the statement. □

Lemma 4.8. *Let $I = [a,b]$ be a compact interval and let $f \in W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$ be a concave function. Let M be a positive constant such that $|f(a)|, |f(b)| \leq M$ and let $m = \max_I |f|$. Then there exists a positive constant $\gamma = \gamma(\text{meas}(I), M, m)$ such that*

$$\int_I |f'(t)|\, dt \leq \gamma.$$

Proof. Straightforward. □

Lemma 4.9. *Let Ω be an open bounded convex subset of $\mathbb{R}^n$ and let $u \in W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$ be a concave function. Assume that there exists a positive δ such that*

$$\|D_1 u\|_{L^\infty(\Omega)} \leq \delta \tag{25}$$

and let M be a positive constant such that

$$\|u\lfloor_{\partial\Omega}\|_{C^0(\partial\Omega)} \leq M.$$

Then there exists a positive constant $\Gamma = \Gamma(n, M, \delta, \Omega)$, depending only on n, M, δ, Ω such that

$$\|u\|_{W^{1,1}(\Omega)} \leq \Gamma. \tag{26}$$

Proof. Take $y \in \partial\Omega$ and consider the map

$$g_{1,y}(t) \doteq u(y + te_1), \quad t \in I_{1,y},$$

where $I_{1,y}$ is the compact interval

$$I_{1,y} \doteq \left\{t \in \mathbb{R} : y + te_1 \in \overline{\Omega}\right\}.$$

Clearly $g_{1,y}$ is concave and we have, for almost every line $y + te_1$,

$$g'_{1,y}(t) = D_1 u(y + te_1) \ \text{ for a.e. } t \in I_{1,y}.$$

By (25), we have

$$|g'_{1,y}(t)| \leq \delta \ \text{ for a.e. } t \in I_{1,y}. \tag{27}$$

As a consequence of (27), by elementary computations, we find a positive number $\gamma = \gamma(\Omega, M, \delta)$, depending only on Ω, M, δ, but independent on y, such that, for every $y \in \partial\Omega$,

$$|g_{1,y}(t)| \leq \gamma \ \ \forall t \in I_{1,y}. \tag{28}$$

Recalling the continuity of u, inequality (28) implies that

$$|u(x)| \leq \gamma \ \ \forall x \in \overline{\Omega}. \tag{29}$$

Consider now any index $j \in \{2, \dots, n\}$, any $y \in \partial\Omega$ and the functions

$$g_{j,y}(t) \doteq u(y + te_j), \quad t \in I_{j,y},$$

where $I_{j,y}$ is the interval

$$I_{j,y} = \left\{t \in \mathbb{R} : y + te_j \in \overline{\Omega}\right\}.$$

Clearly $g_{j,y}$ is concave and, by (29), we have, $\forall y \in \partial\Omega$,

$$|g_{j,y}(t)| \leq \gamma \quad \forall t \in I_{1,y}. \tag{30}$$

By Lemma 4.8 and (30) there exists positive constants $\delta_j = \delta_j(\gamma, \Omega, M, \delta)$, $j = 2, \dots, n$ such that

$$\int_{I_{1,y}} |g'_{j,y}(t)|\, dt \leq \delta_j \quad \forall j = 1, \dots, n. \tag{31}$$

Applying Fubini-Tonelli Theorem we obtain from (31) that

$$\int_\Omega |D_j u(x)|\, dx \leq L, \quad \forall j = 1, \dots, n;$$

where L is a suitable positive constant depending only on Ω, M, δ.

From these last inequalities we deduce

$$\int_\Omega |Du(x)|\, dx \leq nL,$$

which gives (26). □

Remark 4.10. Clearly the above lemma applies to any element of S^1_φ satisfying.

Theorem 4.11. *Assume Hypotheses* 3.1 *and* 3.4. *Let* S^1_φ *as in Definition* 3.5 *and* s *as in Lemma* 4.1. *Then there exists a unique* $\overline{u} \in S^1_\varphi$ *with the following properties.*

$$\overline{u} \quad \textit{is concave}; \tag{32}$$

$$\int_\Omega \overline{u}(x)\ dx = s; \tag{33}$$

$$\overline{u} \geq u \quad \textit{on } \overline{\Omega} \textit{ for every } u \in S^1_\varphi; \tag{34}$$

$$F^{**}(\xi) \leq 0 \quad \forall \xi \in D^{\pm}\overline{u}(x)\ \forall x \in \Omega. \tag{35}$$

Proof. *Step 1.* First of all remark that, given two elements u and v in S^∞_φ, we have that $u \vee v \in S^\infty_\varphi$. This fact follows trivially from Stampacchia's Theorem and from the definition of S^∞_φ (see *Step 2* below).

Take a sequence (u_k) in S^∞_φ such that

$$\int_\Omega u_k(x)\, dx \to s \quad \text{as } k \to \infty.$$

Replacing if necessary u_{k+1} by $u_k \vee u_{k+1}$ we may assume

$$u_k \leq u_{k+1} \quad \text{on } \Omega\ \forall k \in \mathbb{N}. \tag{36}$$

For every index k consider the upper concave envelope $\hat{u}_k$ of u_k and observe that, by Lemma 4.7 and by Remark 4.4, $\hat{u}_k \in \varphi + W^{1,\infty}_0(\Omega)$. Setting

$$\Xi = \{\xi \colon F^{**}(\xi) \leq 0\},$$

we deduce, from Lemma 4.7 again, that

$$F^{**}(D\hat{u}_k(x)) \leq 0 \ \text{ for a.e. } x \in \Omega.$$

Hence, recalling (5), $\hat{u}_k$ lies in S_φ^∞. In addition, as a consequence of (36), we have

$$\hat{u}_k \leq \hat{u}_{k+1} \ \text{ on } \Omega \ \forall k \in \mathbb{N}; \tag{37}$$

then, necessarily, we have

$$\int_\Omega \hat{u}_k(x)\, dx \to s \ \text{ as } k \to \infty. \tag{38}$$

Applying Lemma 4.9 we obtain a positive constant Γ such that

$$\|\hat{u}_k\|_{W^{1,1}(\Omega)} \leq \Gamma \ \ \forall k \in \mathbb{N}.$$

By Rellich Theorem we deduce that the sequence $(\hat{u}_k)$ is precompact in $L^1(\Omega)$ and then, by monotonicity (37), there exists $\overline{u} \in L^1(\Omega)$ such that

$$\hat{u}_k \longrightarrow \overline{u} \ \text{ in } L^1(\Omega) \text{ and almost everywhere in } \Omega. \tag{39}$$

Obviously, by (38) and (39), we have

$$\int_\Omega \overline{u}(x)\, dx = s;$$

hence (33) is proved.

Step 2. We prove now pointwise maximality (34).

Let u be any element in S_φ^1; we claim that

$$u(x) \leq \overline{u}(x) \ \ \forall x \in \Omega. \tag{40}$$

Assume, by contradiction, that there exists $v \in S_\varphi^1$ and a nonempty (open) set E such that

$$v > \overline{u} \ \text{ on } E. \tag{41}$$

Set

$$w \doteq \sup(v, \overline{u}).$$

By Stampacchia's Theorem we have

$$Dw = \begin{cases} Dv & \text{on } E, \\ D\overline{u} & \text{on } \Omega \setminus E, \end{cases} \tag{42}$$

and, by direct inspection, (42) gives that

$$w \in S_\varphi^1. \tag{43}$$

As a consequence of (41) we obtain, recalling (33) and (7),

$$\int_\Omega w(x)\,dx > \int_\Omega \overline{u}(x)\,dx = s = \sup\left\{\int_\Omega u(x)\,dx; u \in S^1_\varphi\right\}. \tag{44}$$

Formulas (43) and (44) provide the contradiction and then (40) is proved.

Step 3. Since $(\hat{u}_k)$ is a nondecreasing sequence of concave functions converging a.e. to $\overline{u}$, it follows that $\overline{u}$ is concave.

Step 4. As a consequence of concavity we have $\overline{u} \in W^{1,\infty}_{\text{loc}}(\Omega)$; then $\overline{u}$ is continuous and differentiable almost everywhere in Ω.

By computations analogous to those of Lemma 4.9 we prove that, actually, $\overline{u}$ belongs to $W^{1,1}(\Omega)$ and, in addition, being $\overline{u}_k$ and $\overline{u}$ continuous on Ω, by Dini's Lemma, properties (37) and (39) imply that

$$\hat{u}_k \longrightarrow \overline{u} \;\text{ uniformly on each compact subset of } \Omega. \tag{45}$$

Step 5. We know that $\hat{u}_k \in S^\infty_\varphi$ and then, recalling Lemma 4.2, we have

$$F^{**}(\xi) \le 0 \;\; \forall \xi \in D^\pm \hat{u}_k(x) \; \forall k \in \mathbb{N} \; \forall x \in \Omega. \tag{46}$$

Hence, by a standard argument (see Proposition 2.2. in [4]), the uniform convergence (45) and (46) imply that

$$F^{**}(\xi) \le 0 \;\; \forall \xi \in D^- \overline{u}(x) \; \forall x \in \Omega$$

and that

$$F^{**}(\xi) \le 0 \;\; \forall \xi \in D^+ \overline{u}(x) \; \forall x \in \Omega.$$

Hence (35) is proved.

Finally, being $\overline{u}$ differentiable a.e. in Ω, recalling item *(iv)* in Lemma 2.2, we have

$$F^{**}(D\overline{u}(x)) \le 0 \;\text{ for a.e. } x \in \Omega.$$

Collecting all the properties of $\overline{u}$ proved up to now we deduce that $\overline{u} \in S^1_\varphi$. □

The following result allows us to recover by the integro-extremization method the arguments contained in [10] and [5]. Another result can be found in Theorem 5.12 in [2].

Theorem 4.12. *Assume Hypotheses* 3.1 *and* 3.4. *Let* $\overline{u}$ *be the function given by Theorem* 4.11. *Then* $\overline{u}$ *is the maximal viscosity solution of* $\mathcal{P}^{**}_\varphi$ *and of* $\mathcal{P}_\varphi$.

Remark 4.13. Clearly any other viscosity (sub)solution of the equation cannot exceed $\overline{u}$, hence the theorem states implicity the uniqueness of the maximal solution $\overline{u}$.

Proof. *Step 1.* We observe that condition (35) of Theorem 4.11 implies, in particular, that $\overline{u}$ is a subsolution of the relaxed problem $\mathcal{P}^{**}_\varphi$. In addition property (34) implies uniqueness in the sense specified in the theorem.

Step 2. We prove now that $\overline{u}$ is a viscosity supersolution of the relaxed problem $\mathcal{P}_\varphi^{**}$. Take $x_0 \in A^-(\overline{u})$, $\xi \in D^-\overline{u}(x_0)$ and assume, by contradiction, that

$$F^{**}(\xi) < 0. \tag{47}$$

By the continuity of F^{**} and $\overline{u}$, we infer from (47) the existence of $R > 0$ and $r > 0$ such that $B(x_0, R) \subseteq \Omega$ and

$$F^{**}(\eta) \leq 0 \quad \forall \eta \in \overline{B(\xi, r)}. \tag{48}$$

By Lemma 2.4 there exist $\rho \in]0, R[$, a map

$$w \in \overline{u} + W_0^{1,\infty}(\Omega) = \varphi + W_0^{1,\infty}(\Omega) \tag{49}$$

and a nonempty open set

$$\Lambda \subseteq B(x_0, \rho) \subseteq B(x_0, R) \tag{50}$$

with the following properties:

$$w(x) = \overline{u}(x) \quad \text{for every} \quad x \in \Omega \setminus \Lambda, \tag{51}$$

$$\overline{u}(x) < w(x) \quad \text{for every } x \in \Lambda, \tag{52}$$

$$Dw(x) = D\overline{u}(x) \quad \text{for a.e. } x \in \Omega \setminus \Lambda, \tag{53}$$

$$|Dw(x) - D\overline{u}(x)| = r \quad \text{for a.e. } x \in \Lambda, \tag{54}$$

$$\int_\Omega w(x)\,dx > \int_\Omega \overline{u}(x)\,dx. \tag{55}$$

Conditions (50)–(54), together with (48), ensure that

$$F^{**}(Dw(x)) \leq 0 \quad \text{for a.e. } x \in \Omega. \tag{56}$$

Recalling (49) and Definition 3.5, inequality (56) implies that w is an element of S_φ^1. Hence inequality (55) contradicts the maximality of the integral of $\overline{u}$ (see (7) in Lemma 4.1 and formula (33) in Theorem 4.11). Hence (47) is absurd and then we have

$$F^{**}(\xi) \geq 0 \quad \forall x \in A^-(\overline{u}) \text{ and } \forall \xi \in D^-\overline{u}(x). \tag{57}$$

This proves the claim of *Step 2* and, in particular, collecting (35) and (57), we have, actually, that

$$F^{**}(\xi) = 0 \quad \forall x \in A^-(\overline{u}) \text{ and } \forall \xi \in D^-\overline{u}(x). \tag{58}$$

Step 1 and *Step 2* imply that $\overline{u}$ is a viscosity solution of $\mathcal{P}_\varphi^{**}$ (see also [5]).

Step 3. Let us consider now the non convex problem $\mathcal{P}_\varphi$.

Take $x_0 \in A^+(\overline{u})$, $\xi \in D^+\overline{u}(x_0)$ and assume, by contradiction, that

$$F(\xi) > 0.$$

Recalling (2) in Hypothesis 3.1 we have also $F^{**}(\xi) > 0$ and then, invoking (35), we obtain a contradiction.

Take $x_0 \in A^-(\overline{u})$, $\xi \in D^-\overline{u}(x_0)$ and assume, by contradiction, that

$$F(\xi) < 0.$$

Since, by definition, $F^{**} \leq F$, we have $F^{**}(\xi) < 0$ and then, invoking (58), we obtain a contradiction also in this case.

Hence $\overline{u}$ is a viscosity solution of $\mathcal{P}_\varphi$. □

5. The non coercive case: continuous dependence on boundary data

The viscosity solution $\overline{u}$ of $\mathcal{P}_\varphi$ and of $\mathcal{P}^{**}_\varphi$ provided by Theorems 4.11 and 4.12 is unique, in the sense specified in previous section. Hence we may investigate if $\overline{u}$ depends continuously on the boundary datum φ with respect to some topology; in this section we face this problem. Since we have obtained the maximal solution of the non convex problem as a solution of the relaxed one, keeping the notation of previous section, we assume, without loss of generality, that $F = F^{**}$.

Theorem 5.1. *Assume Hypotheses* 3.3. *Let* (φ_k), $k \in \mathbb{N}_0$, *be a bounded sequence in* $W^{1,\infty}(\tilde{\Omega})$ *of concave functions such that*

$$F(D\varphi_k(x)) \leq 0 \quad \textit{for a.e. } x \in \tilde{\Omega} \ \forall k \in \mathbb{N}_0,$$

where $\tilde{\Omega}$ *is a neighbourhood of* Ω. *Assume that*

$$\varphi_k \longrightarrow \varphi_0 \quad \textit{uniformly on } \partial\Omega \textit{ as } k \to \infty.$$

For every $k \in \mathbb{N}_0$ *consider the problem*

$$\mathcal{P}_{\varphi_k}: \quad \begin{cases} F(Du) = 0 & \textit{in } \Omega \\ u = \varphi_k & \textit{on } \partial\Omega, \end{cases}$$

and let $\overline{u}_k$ *be the viscosity solution of* $\mathcal{P}_{\varphi_k}$ *given by Theorems* 4.11 *and* 4.12. *Then*

$$\overline{u}_k \longrightarrow \overline{u}_0 \quad \textit{in } L^1(\Omega).$$

Proof. *Step 1.* For every $\epsilon \in [0,1]$ and for every $k \in \mathbb{N}_0$, introduce the set

$$S^1_{\varphi_k,\epsilon} \doteq \left\{ u \in \left(\varphi_k + W^{1,1}_0(\Omega)\right) \cap C^0(\overline{\Omega}) \colon F(Du(x)) \leq \epsilon \text{ for a.e. } x \in \Omega \right\},$$

remarking that $S^1_{\varphi_k,0} = S^1_{\varphi_k}$ (see Definition 6). By the same argument used in Theorems 4.11 and 4.12 and recalling (4) in Hypothesis 3.3, for every $\epsilon > 0$, we may infer the existence of a map $\overline{u}^\epsilon_k$, unique viscosity solution of the problem

$$\begin{cases} F(Du) - \epsilon = 0 & \text{in } \Omega \\ u = \varphi_k & \text{on } \partial\Omega, \end{cases}$$

such that

$$\overline{u}^\epsilon_k \geq u \quad \text{in } \Omega \ \forall u \in S^1_{\varphi_k,\epsilon}. \tag{59}$$

By Lemma 4.9 and by the assumptions on the sequence (φ_k), there exists a positive constant Γ such that

$$\|\overline{u}_k^\epsilon\|_{W^{1,1}(\Omega)} \leq \Gamma \quad \forall \epsilon \in [0,1] \ \forall k \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$ and take any sequence (ϵ_j) with $\epsilon_j \to 0+$ such that, by Rellich Theorem,

$$u_k^{\epsilon_j} \longrightarrow v \ \text{ in } L^1(\Omega) \text{ and almost everywhere.}$$

Being the limit in $\mathcal{D}'(\Omega)$ of a sequence of concave functions, reasoning as in *Step 2* in the proof of Theorem 4.11, we obtain that v is concave, so that, in particular, it belongs to $W^{1,\infty}_{\text{loc}}(\Omega)$.

Take a test function $\theta \in \mathcal{D}(\Omega)$ (with $\int_\Omega \theta \, dx = 1$) and observe that, by the convexity of F and by classical semicontinuity results, we have

$$\int_\Omega F(Dv(x))\theta(x)\, dx \leq \liminf_{j\to\infty} \int_\Omega F(D\overline{u}_k^{\epsilon_j}(x))\theta(x)\, dx \leq \epsilon_j \to 0.$$

By the arbitrariness of θ we deduce from this last formula that

$$F(Dv(x)) \leq 0 \ \text{ for a.e. } x \in \Omega. \tag{60}$$

By computations analogous to those of Lemma 4.9 we deduce that, actually, v belongs to $W^{1,1}(\Omega)$, and, being the pointwise limit of a sequence of elements which take the value φ_k on $\partial\Omega$, it follows that v belongs to $\varphi_k + W_0^{1,1}(\Omega)$. In addition it is continuous on $\overline{\Omega}$ and then, recalling (60), we have that

$$v \in S^1_{\varphi_k}. \tag{61}$$

On the other hand we have $S^1_{\varphi_k} \subseteq S^1_{\varphi_k,\epsilon}$ for every $\epsilon \in [0,1]$ and, consequently, recalling the maximality (59) of $\overline{u}_k^\epsilon$, we have

$$\int_\Omega \overline{u}_k \, dx \leq \int_\Omega \overline{u}_k^\epsilon \, dx \quad \forall \epsilon \in [0,1]. \tag{62}$$

Inequality (62) and the $L^1(\Omega)$-convergence $\overline{u}_k^{\epsilon_j} \longrightarrow v$ as $j \to \infty$ imply that

$$\int_\Omega \overline{u}_k \, dx \leq \int_\Omega v \, dx. \tag{63}$$

Recalling (61) and that, by definition, $\overline{u}_k$ is the maximal element of $S^1_{\varphi_k}$, (63) implies that $v = \overline{u}_k$. By the arbitrariness of the subsequence $(\overline{u}_k^{\epsilon_j})$ we conclude that

$$\overline{u}_k^\epsilon \longrightarrow \overline{u}_k \ \text{ in } L^1(\Omega) \text{ as } \epsilon \to 0+ \ \ \forall k \in \mathbb{N}. \tag{64}$$

Step 2. Claim. For every $\epsilon \in [0,1]$ there exists a sequence (u_k) in $S^1_{\varphi_k,\epsilon}$ such that

$$u_k \longrightarrow \overline{u}_0 \ \text{ strongly in } W^{1,1}(\Omega). \tag{65}$$

Since $\overline{u}_0\lfloor_{\partial\Omega}= \varphi_0\lfloor_{\partial\Omega}$ and $\varphi_k \longrightarrow \varphi_0$ uniformly on $\partial\Omega$, remarking that all functions are uniformly bounded in $W^{1,1}(\Omega)$, we may fix $k \in \mathbb{N}$ and apply Lemma 2.5 with u, v replaced by the functions $\overline{u}_0$ and φ_k, obtaining sets $\Omega_k \subseteq \Omega$, a sequence $\sigma_k \to 0+$ and maps $u_k \in W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$ such that:

$$\Omega_k \subseteq \Omega_{k+1} \quad \forall k \in \mathbb{N}; \tag{66}$$

$$\operatorname{meas}(\Omega \setminus \Omega_k) \longrightarrow 0 \quad \text{as } k \to \infty; \tag{67}$$

$$u_k = \overline{u}_0 \quad \text{in } \Omega_k; \tag{68}$$

$$u_k = \varphi_k \quad \text{on } \partial\Omega; \tag{69}$$

$$|u_k - \overline{u}_0|, |u_k - \varphi_k| \leq \sigma_k \quad \text{in } \Omega \setminus \Omega_k; \tag{70}$$

$$\operatorname{dist}(Du_k(x), \operatorname{co}(\{D\overline{u}_0(x), D\varphi_k(x)\})) \leq \sigma_k \quad \text{for a.e. } x \in \Omega. \tag{71}$$

Formula (71) implies that the sequence (u_k) is bounded in $W^{1,1}(\Omega)$; hence (66), (67) and (68) imply (65). Since (69) implies that u_k belongs to $\varphi_k + W^{1,1}(\Omega)$, we have only to prove that there exists $k_\epsilon \in \mathbb{N}$ such that

$$u_k \in S^1_{\varphi_k,\epsilon} \quad \forall k \geq k_\epsilon. \tag{72}$$

By construction we have $u_k = \overline{u}_0$ and $Du_k = D\overline{u}_0$ a.e. in Ω_k; hence

$$F(Du_k(x)) \leq 0 \quad \text{for a.e. } x \in \Omega_k. \tag{73}$$

In order to perform the computation on $\Omega \setminus \Omega_k$ we observe that by (71) there exist measurable functions

$$\lambda_k \colon \Omega \longrightarrow [0,1] \quad k \in \mathbb{N} \tag{74}$$

and

$$\theta_k \colon \Omega \to \mathbb{R}^n \quad \text{with } |\theta_k(x)| \leq \sigma_k \text{ for a.e. } x \in \Omega \setminus \Omega_k,\ k \in \mathbb{N}, \tag{75}$$

such that

$$Du_k(x) = \lambda_k(x) D\overline{u}_0(x) + (1 - \lambda_k(x)) D\varphi_k(x) + \theta_k(x) \quad \text{for a.e. } x \in \Omega \setminus \Omega_k. \tag{76}$$

Formulas (74), (75) and (76) imply that

$$\begin{aligned} F(Du_k) &= F(\lambda_k D\overline{u}_0 + (1-\lambda_k) D\varphi_k + \theta_k) \\ &= [F(\lambda_k D\overline{u}_0 + (1-\lambda_k) D\varphi_k + \theta_k) - F(\lambda_k D\overline{u}_0 + (1-\lambda_k) D\varphi_k)] \\ &\qquad + F(\lambda_k D\overline{u}_0 + (1-\lambda_k) D\varphi_k) \quad \text{a.e. in } \Omega \setminus \Omega_k. \end{aligned} \tag{77}$$

Recalling (75), by the uniform continuity of F, there exists $k_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned} &|F(\lambda_k D\overline{u}_0 + (1-\lambda_k) D\varphi_k + \theta_k) - F(\lambda_k D\overline{u}_0 + (1-\lambda_k) D\varphi_k)| \leq \epsilon \\ &\quad \text{a.e. in } \Omega \setminus \Omega_k,\ \forall k \geq k_\epsilon. \end{aligned} \tag{78}$$

Then, recalling that

$$F(D\overline{u}) \leq 0, \quad F(D\varphi_k) \leq 0 \quad \text{a.e. in } \Omega,$$

we have

$$F(\lambda_k D\overline{u}_0 + (1-\lambda_k) D\varphi_k) \leq \lambda_k F(D\overline{u}_0) + (1-\lambda_k) F(D\varphi_k) \leq 0 \quad \text{a.e. in } \Omega. \tag{79}$$

Collecting (77), (78) and (79) and recalling (73), we obtain that

$$F(D\overline{u}_k) \leq \epsilon \;\text{ a.e. in } \Omega \; \forall k \geq k_\epsilon.$$

This inequality and the other properties of $\overline{u}_k$ imply (72) and then the proof of the claim is finished.

Step 3. Consider the sequence $(\overline{u}_k)$ which, by Lemma 4.9, is bounded in $W^{1,1}(\Omega)$. Invoking Rellich Theorem take a subsequence, still denoted by $(\overline{u}_k)$, such that

$$\overline{u}_k \longrightarrow v \;\text{ in } L^1(\Omega) \text{ and a.e. in } \Omega. \tag{80}$$

Claim. The map v belongs to S^1_φ.

Being the limit in $\mathcal{D}'(\Omega)$ of a sequence of concave functions, v, reasoning as in *Step 2* of Theorem 4.11, turns out to be concave and then it belongs to $W^{1,\infty}_{\text{loc}}(\Omega)$. Reasoning again as in Lemma 4.9, v turns out to be an element of $W^{1,1}(\Omega)$. Clearly we have $v \in \varphi + W^{1,1}_0(\Omega)$, since it is the pointwise limit of a sequence whose elements coincide at the boundary with the elements of the sequence (φ_k), which converges uniformly to φ on $\partial\Omega$. As in *Step 1* take a test function $\theta \in \mathcal{D}(\Omega)$ (with $\int_\Omega \theta(x)\,dx = 1$) and observe that, by the convexity of F and by classical semicontinuity results, we have:

$$\int_\Omega F(Dv(x))\theta(x)\,dx \leq \liminf_{k\to\infty} \int_\Omega F(D\overline{u}_k(x))\theta(x)\,dx \leq 0,$$

since $\overline{u}_k \in S^1_{\varphi_k}$ for every k. By the arbitrariness of θ we deduce form the above formula that

$$F(Dv(x)) \leq 0 \;\text{ for a.e. } x \in \Omega.$$

Hence the claim is proved and, in particular, by item (*iii*) in Theorem 4.12, we have:

$$v \leq \overline{u}_0 \;\text{ in } \Omega. \tag{81}$$

Claim. We have to show that

$$v = \overline{u}_0. \tag{82}$$

Fix an index k and recall (64). Since $\overline{u}_k \longrightarrow v$ in $L^1(\Omega)$ as $k \to \infty$, by a diagonal argument we may find a sequence $\epsilon_k \to 0+$ such that

$$\overline{u}_k^{\epsilon_k} \longrightarrow v \;\text{ in } L^1(\Omega) \text{ as } k \to \infty. \tag{83}$$

By *Step 2* we may construct a sequence (u_k) such that $u_k \in S^1_{\varphi_k,\epsilon}$ $\forall k \in \mathbb{N}$ and $u_k \longrightarrow \overline{u}_0$ strongly in $W^{1,1}(\Omega)$. Hence we have

$$\int_\Omega u_k\,dx \longrightarrow \int_\Omega \overline{u}_0\,dx \;\text{ as } k \to \infty. \tag{84}$$

In addition, by the Definition (59) of $\overline{u}_k^\epsilon$, we have

$$\int_\Omega u_k\,dx \leq \int_\Omega \overline{u}_k^{\epsilon_k}\,dx \;\; \forall k \in \mathbb{N}. \tag{85}$$

Then, collecting (83), (84) and (85), we conclude that

$$\int_\Omega \overline{u}_0 \, dx = \lim_{k\to\infty} \int_\Omega u_k \, dx \leq \lim_{k\to\infty} \int_\Omega \overline{u}_k^{\epsilon_k} \, dx = \int_\Omega v \, dx. \tag{86}$$

Putting together (81) and (86) we obtain that $v = \overline{u}_0$ and the claim is proved.

By the arbitrariness of the chosen subsequence, by (80) and (82), we obtain that the whole sequence $(\overline{u}_k)$ converges to $\overline{u}_0$ in $L^1(\Omega)$. □

Remark 5.2. Clearly, if the hamiltonian F is coercive, so that $F(\xi) \leq 0$ implies $|\xi| \leq C$ for some positive constant C, the solution $\overline{u}$ provided by Theorems 4.11 and 4.12 coincides with the one given by Theorems 1 and 2 in [16].

6. The evolution case: notations and main hypotheses

In this section we maintain the notations used above for the euclidean m-dimensional space $\mathbb{R}^m$ in the cases the cases $m = n$ or $m = n+1$; a point $x' \in \mathbb{R}^n$ is written as $x' = (x_1, \dots, x_n)$, while a point $x \in \mathbb{R}^{n+1}$ is written as $x = (x_0, x') = (x_0, x_1, \dots, x_n)$. For a function $u = u(x) = u(x_0, x_1, \dots, x_n)$ defined on $\mathbb{R}^{n+1}$, we use the following symbols for its derivatives:

$$Du = \left(\frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = \left(\frac{\partial u}{\partial x_0}, \nabla u\right),$$

where

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right).$$

We study the equation

$$\frac{\partial u}{\partial x_0}(x_0, x') + F\left(\nabla'_x u(x_0, x')\right) = 0 \quad (x_0, x') \in \left]0, T\right[\times \Omega$$

subject to the initial condition

$$u(0, x') = \eta(x') \quad x' \in \Omega$$

and to the Ω-boundary condition

$$u(x_0, x') = \psi(x') \quad (x_0, x') \in [0, T] \times \partial\Omega.$$

Actually we formulate the boundary conditions assuming that there exists a function $\varphi : [0, T] \times \overline{\Omega} \longrightarrow \mathbb{R}$ such that

$$\varphi(0, x') = \eta(x') \quad \forall x' \in \Omega, \qquad \varphi(x_0, x') = \psi(x') \quad \forall (x_0, x') \in [0, T] \times \partial\Omega;$$

hence we can write problem $\mathcal{P}$ in the introduction as

$$\begin{cases} \frac{\partial u}{\partial x_0} + F\left(\nabla_{x'} u\right) = 0 & \text{in } \left]0, T\right[\times \Omega \\ u = \varphi & \text{on } (\{0\} \times \Omega) \cup ([0, T] \times \partial\Omega). \end{cases}$$

Introduce the function $G : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ given by

$$G(\xi) = G(\xi_0, \xi_1, \dots, \xi_n) \doteq \xi_0 + F(\xi_1, \dots, \xi_n) \tag{87}$$

and the set

$$\Gamma \doteq (\{0\} \times \Omega) \cup ([0, T] \times \partial\Omega). \tag{88}$$

By virtue of (87) and (88) our problem can be formulated as follows:

$$\mathcal{P}: \quad \begin{cases} G(Du) = 0 & \text{in }]0, T[\times\Omega \\ u = \varphi & \text{on } \Gamma, \end{cases}$$

where the variable x_0 plays the role of time t and we also remark that actually, by this choice, the Ω-boundary datum ψ can depend also on the variable x_0.

We assume the following hypotheses.

Hypothesis 6.1. $F : \mathbb{R}^n \to \mathbb{R}$ is a convex function bounded from below and we set

$$-\gamma \doteq \inf_{\xi' \in \mathbb{R}^n} F(\xi'). \tag{89}$$

Hypothesis 6.2. Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex. We consider a concave map $\varphi \in W^{1,\infty}_{\text{loc}}(]0, T[\times\Omega) \cap C^0([0, T] \times \overline{\Omega})$ assuming that

$$G(D\varphi(x)) \le 0 \ \text{ for a.e. } x \in\,]0, T[\times \Omega. \tag{90}$$

Definition 6.3. We define the set

$$\begin{aligned} S_\varphi \doteq \{u \in W^{1,\infty}_{\text{loc}}(]0, T[\times \Omega) \cap C^0([0, T] \times \overline{\Omega}) : \\ u = \varphi \text{ on } \Gamma,\ G(Du) \le 0 \text{ a.e. in }]0, T[\times \Omega\}, \end{aligned} \tag{91}$$

remarking that, by virtue of Hypothesis 6.2, S_φ is nonempty.

7. The evolution case: existence and uniqueness of the solution

Throughout this section we assume Hypotheses 6.1 and 6.2 and we stress that the arguments are similar to the ones used in previous section; however we perform the proofs for the sake of completeness.

Lemma 7.1. *There exists a positive constant K such that*

$$u(x) \le K \quad \forall x \in [0, T] \times \overline{\Omega},\ \forall u \in S_\varphi. \tag{92}$$

Proof. By definition, for every $u \in S_\varphi$, we have

$$\frac{\partial}{\partial x_0} u(x_0, x') + F(\nabla u(x_0, x') \le 0 \ \text{ for a.e. } (x_0, x') \in\,]0, T[\times \Omega;$$

hence, recalling (89) in Hypothesis 6.1, we have

$$\frac{\partial}{\partial x_0} u(x_0, x') \le -F(\nabla u(x_0, x')) \le \gamma \ \text{ for a.e. } (x_0, x') \in\,]0, T[\times \Omega.$$

By Fubini-Tonelli Theorem we have that, for almost every $x' \in \Omega$ and $x_0 \in [0,T]$,

$$u(x_0, x') = u(0, x') + \int_0^{x_0} \frac{\partial}{\partial x_0} u(t, x')\, dt \le \varphi(0, x') + \gamma T.$$

The continuity of u on $[0,T] \times \overline{\Omega}$ implies the result. □

Definition 7.2. As a consequence of Lemma 7.1 we may set

$$s \doteq \sup \left\{ \int_{[0,T]\times\overline{\Omega}} u(x)\, dx;\ u \in S_\varphi \right\}. \tag{93}$$

Lemma 7.3. *Let $u \in S_\varphi$. Then*

$$G(\xi) \le 0 \quad \forall \xi \in D^- u(x)\ \forall x \in A^-(u);$$
$$G(\xi) \le 0 \quad \forall \xi \in D^+ u(x)\ \forall x \in A^+(u).$$

Proof. *Step 1.* We start by proving the following

Claim. Given an open set $\Lambda \subset\subset\ ]0,T[\ \times \Omega$, there exists a sequence (u_k) in $C^\infty(\Lambda)$ such that

$$u_k \longrightarrow u \ \text{ uniformly on } \Lambda, \tag{94}$$
$$G(Du_k(x)) \le 0 \ \ \forall x \in \Lambda\ \forall k \in \mathbb{N}. \tag{95}$$

Take a regularizing sequence (ρ_k) in $\mathbb{R}^{n+1}$ and set, for k sufficiently large,

$$u_k(x) \doteq (\rho_k * u)(x), \quad x \in \Lambda.$$

Remarking that the convolution is a convex combination, by the convexity of G, by Jensen inequality and recalling that $u \in S_\varphi$, we have, for $x \in \Lambda$:

$$\begin{aligned} G(Du_k(x)) &= G(\rho_k * Du(x)) \\ &= G\left(\int_{B(x,\frac{1}{k})} \rho_k(x-y) Du(y)\, dy \right) \\ &\le \int_{B(x,\frac{1}{k})} \rho_k(x-y) G(Du(y))\, dy \le 0, \end{aligned}$$

for k is sufficiently large so that

$$\Lambda + B\left(0, \frac{1}{k}\right) \subset\ ]0,T[\ \times \Omega.$$

Hence the claim is proved.

Step 2. Take now an arbitrary $y \in A^+(u)$, and any $\xi \in D^+ u(y)$.

Consider an open set $\Lambda \subset\subset\ ]0,T[\ \times \Omega$ containing y and the sequence (u_k) defined in previous step. By a standard argument (see for example Proposition 2.2 in [4]), by (94), recalling the smoothness of u_k and point (iv) in Lemma 2.2, we deduce the existence of a sequence (x_k) in Λ such that

$$x_k \to y \text{ and } Du_k(x_k) \to \xi \quad \text{as } k \to \infty. \tag{96}$$

By the continuity of G, it follows that

$$G(Du_k(x_k)) \longrightarrow G(\xi) \le 0 \quad \text{as } k \to \infty. \tag{97}$$

Hence, (95), (96) and (97), imply that

$$G(\xi) \le 0.$$

By the same argument we obtain that

$$G(\xi) \le 0 \quad \forall \xi \in D^- u(y)\ \forall y \in A^-(u).$$

□

Theorem 7.4. *Assume Hypotheses* 6.1 *and* 6.2. *Let* S_φ *as in Definition* 6.3 *and* s *as in Definition* 7.2. *Then there exists a unique* $\overline{u} \in S_\varphi$ *with the following properties.*

$$\overline{u} \ \ \textit{is concave}; \tag{98}$$

$$\int_{[0,T]\times\overline{\Omega}} \overline{u}(x)\ dx = s; \tag{99}$$

$$\overline{u} \ge u \quad \textit{on } [0,T]\times\overline{\Omega} \textit{ for every } u \in S_\varphi; \tag{100}$$

$$G(\xi) \le 0 \quad \forall \xi \in D^\pm \overline{u}(x)\ \forall x \in A^\pm(\overline{u}). \tag{101}$$

Proof. *Step 1.* First of all remark that, given two elements u and v in S_φ, we have that $u \vee v \in S_\varphi$. This fact follows trivially from Stampacchia's Theorem and from the definition of S_φ (see *Step 2* below).

Take a sequence (u_k) in S_φ such that

$$\int_{[0,T]\times\overline{\Omega}} u_k(x)\, dx \longrightarrow s \quad \text{as } k \to \infty.$$

Replacing if necessary u_1 with $u_1 \vee \varphi$ and u_{k+1} by $u_k \vee u_{k+1}$ we may assume

$$\varphi \le u_k \le u_{k+1} \quad \text{on } [0,T]\times\overline{\Omega}\ \forall k \in \mathbb{N}. \tag{102}$$

For every index k consider the upper concave envelope $\hat{u}_k$ of u_k and observe that given any open set $U \subset\subset\,]0,T[\, \times \Omega$, by Lemma 4.7, $\hat{u}_k \in W^{1,\infty}(U) \cap C^0(\overline{U})$; in addition, recalling Remark 4.4, we have $\hat{u}_k\lfloor_\Gamma = \varphi\lfloor_\Gamma$. Setting

$$\Xi = \{\xi\colon G(\xi) \le 0\},$$

we deduce, from Lemma 4.7 again, that

$$G(D\hat{u}_k(x)) \le 0 \quad \text{for a.e. } x \in U. \tag{103}$$

By the arbitrariness of $U \subset\subset]0,T[\times \Omega$, the inequality in (103) holds on the whole $]0,T[\times\Omega$. Hence, recalling (91), $\hat{u}_k$ lies in S_φ. In addition, as a consequence of (102), we have

$$\varphi \leq \hat{u}_k \leq \hat{u}_{k+1} \quad \text{on } [0,T] \times \overline{\Omega}\ \forall k \in \mathbb{N}; \tag{104}$$

then, necessarily, we have

$$\int_\Omega \hat{u}_k(x)\, dx \longrightarrow s \quad \text{as } k \to \infty. \tag{105}$$

Being $\hat{u}_k \in S_\varphi$ for every $k \in \mathbb{N}$, by Lemma 7.1, we have that

$$\hat{u}_k \leq K \quad \text{on } [0,T] \times \overline{\Omega}\ \forall k \in \mathbb{N}. \tag{106}$$

Since for every $x \in]0,T[\times\Omega$ the sequence $(\hat{u}_k(x))_k$ is monotone non decreasing (recall (104), we have

$$\hat{u}_k(x) \overset{k\to\infty}{\longrightarrow} \overline{u}(x) \quad \forall x \in]0,T[\times \Omega, \tag{107}$$

where u is clearly a measurable function. Recall the bounds (106) and (102); observe that we have

$$\varphi \leq \hat{u}_k \leq \hat{u}_{k+1} \leq K \quad \forall k \in \mathbb{N} \text{ on } [0,T] \times \overline{\Omega}.$$

Since $\varphi \in C^0([0,T] \times \overline{\Omega})$, by dominated convergence, we have that $\overline{u} \in L^1(]0,T[\times\Omega)$ and that, recalling (107)

$$\hat{u}_k \overset{k\to\infty}{\longrightarrow} \overline{u} \quad \text{in } L^1(]0,T[\times \Omega). \tag{108}$$

Obviously, by (105) and (108), we have

$$\int_{[0,T]\times\overline{\Omega}} \overline{u}(x)\, dx = s;$$

hence (99) is proved.

Step 2. We prove now pointwise maximality (100).

Let u be any element in S_φ; we claim that

$$u(x) \leq \overline{u}(x) \quad \forall x \in [0,T] \times \overline{\Omega}. \tag{109}$$

Assume, by contradiction, that there exists $v \in S_\varphi$ and a nonempty (open) set $E \subseteq]0,T[\times \Omega$ such that

$$v > \overline{u} \quad \text{on } E. \tag{110}$$

Set

$$w \doteq \sup(v, \overline{u}).$$

By Stampacchia's Theorem we have

$$Dw = \begin{cases} Dv & \text{on } E, \\ D\overline{u} & \text{on }]0,T[\times \Omega \setminus E, \end{cases} \tag{111}$$

and, by direct inspection, (111) gives that

$$w \in S_\varphi. \tag{112}$$

As a consequence of (110) we obtain, recalling (99) and (7.2),

$$\int_{[0,T]\times\overline{\Omega}} w(x)\,dx > \int_{[0,T]\times\overline{\Omega}} \overline{u}(x)\,dx = s = \sup\left\{\int_{[0,T]\times\overline{\Omega}} u(x)\,dx; u \in S_\varphi\right\}. \tag{113}$$

Formulas (112) and (113) provide the contradiction and then (109) is proved.

Step 3. As in *Step 3* of the proof of Theorem 4.11 we obtain easily that $\overline{u}$ is concave.

Step 4. As a consequence of concavity we have $\overline{u} \in W^{1,\infty}_{\mathrm{loc}}(]0,T[\times\Omega)$; then $\overline{u}$ is continuous and differentiable almost everywhere in $]0,T[\times\Omega$; in addition, being $\overline{u}_k$ and $\overline{u}$ continuous on $[0,T]\times\overline{\Omega}$, by Dini's Lemma properties (104) and (108) imply that

$$\hat{u}_k \overset{k\to\infty}{\longrightarrow} \overline{u} \text{ uniformly on } [0,T]\times\overline{\Omega}. \tag{114}$$

Step 5. We know that $\hat{u}_k \in S_\varphi$ and then, recalling Lemma 7.3, we have

$$G(\xi) \le 0 \;\; \forall \xi \in D^\pm u_k(x) \; \forall k \in \mathbb{N} \; \forall x \in A^\pm(u_k). \tag{115}$$

Hence, by a standard argument (see Proposition 2.2 in [4]), the uniform convergence (114) and (115) imply that

$$G(\xi) \le 0 \;\; \forall \xi \in D^-\overline{u}(x) \; \forall x \in A^-(\overline{u})$$

and that

$$G(\xi) \le 0 \;\; \forall \xi \in D^+\overline{u}(x) \; \forall x \in A^+(\overline{u}).$$

Hence (101) is proved.

Finally, being $\overline{u}$ differentiable a.e. in $]0,T[\times\Omega$, recalling item *(iv)* in Lemma 2.2, we have

$$G(D\overline{u}(x)) \le 0 \text{ for a.e. } x \in]0,T[\times\Omega.$$

Collecting all the properties of $\overline{u}$ proved up to now we deduce that $\overline{u} \in S_\varphi$. □

Theorem 7.5. *Assume Hypotheses* 6.1 *and* 6.2. *Let* $\overline{u}$ *be the function given by Theorem* 7.4. *Then* $\overline{u}$ *is the unique maximal viscosity solution of* $\mathcal{P}$.

Proof. *Step 1.* We observe that condition (101) of Theorem 7.4 implies, in particular, that $\overline{u}$ is a subsolution of $\mathcal{P}$. In addition property (100) implies uniqueness in the sense specified in the theorem.

Step 2. We prove now that $\overline{u}$ is a viscosity supersolution of $\mathcal{P}$. Take $y \in A^-(\overline{u})$, $\xi \in D^-\overline{u}(y)$ and assume, by contradiction, that

$$G(\xi) < 0. \tag{116}$$

By the continuity of G and $\overline{u}$, we infer from (116) the existence of $R > 0$ and $r > 0$ such that $B(y,R) \subseteq]0,T[\times\Omega$ and

$$G(\eta) \le 0 \;\; \forall \eta \in \overline{B(\xi,r)}. \tag{117}$$

By Lemma 2.4 there exist $\rho \in]0, R[$, a map $w \in W^{1,\infty}_{\text{loc}}(]0, T[\times \Omega)$ and a nonempty open set

$$\Lambda \subseteq B(y, \rho) \subseteq B(y, R) \tag{118}$$

with the following properties:

$$w(x) = \overline{u}(x) \quad \text{for every} \quad x \in]0, T[\times \Omega \setminus \Lambda, \tag{119}$$

$$\overline{u}(x) < w(x) \quad \text{for every } x \in \Lambda, \tag{120}$$

$$Dw(x) = D\overline{u}(x) \quad \text{for a.e. } x \in]0, T[\times \Omega \setminus \Lambda, \tag{121}$$

$$|Dw(x) - D\overline{u}(x)| = r \quad \text{for a.e. } x \in \Lambda, \tag{122}$$

$$\int_{[0,T]\times\overline{\Omega}} w(x)\, dx > \int_{[0,T]\times\overline{\Omega}} \overline{u}(x)\, dx. \tag{123}$$

Conditions (118)–(122), together with (117), ensure that w coincides with $\overline{u}$ on $\partial(]0, T[\times \Omega)$ and, consequently, with φ on Γ; in addition they imply that

$$G\left(Dw(x)\right) \leq 0 \quad \text{for a.e. } x \in]0, T[\times \Omega. \tag{124}$$

Recalling Definition 6.3, inequality (124) implies that w is an element of S_φ. Hence inequality (123) contradicts the maximality of the integral of $\overline{u}$ (see Definition 7.2, Lemma 7.1 and formula (99) in Theorem 7.4). Hence (116) is absurd and then we have

$$G(\xi) \geq 0 \quad \forall \xi \in D^-\overline{u}(x)\ \forall x \in A^-(\overline{u}). \tag{125}$$

This proves the claim of *Step 2* and, in particular, collecting (101) and (125), we have, actually, that

$$G(\xi) = 0 \quad \forall \xi \in D^-\overline{u}(x)\ \forall x \in A^-(\overline{u}). \tag{126}$$

Step 1 and *Step 2* imply that $\overline{u}$ is a viscosity solution of $\mathcal{P}$. □

8. The Cauchy problem

In this last section we consider briefly the problem treated up to now without the boundary condition on $\partial\Omega$, that is to say

$$\begin{cases} \frac{\partial u}{\partial t} + F(\nabla u) = 0 & \text{in }]0, T[\times\Omega \\ u(0, x) = \eta(x) & \text{for } x \in \Omega. \end{cases}$$

Adopting the notations of previous section we have simply to replace the set Γ by the set

$$\Gamma_0 \doteq \{0\} \times \Omega$$

and to formulate the problem

$$\mathcal{P}_0 : \quad \begin{cases} \frac{\partial u}{\partial x_0} + F\left(\nabla u\right) = 0 & \text{in }]0, T[\times\Omega \\ u = \varphi & \text{on } \Gamma_0. \end{cases}$$

where the datum φ satisfies the same requirements of previous section and $\varphi(0, x) = \eta(x)$ for every $x \in \Omega$.

Definition 8.1. Define the set

$$S_\varphi^0 \doteq \big\{u \in W^{1,\infty}_{\rm loc}(]0,T[\times\Omega) \cap C^0([0,T]\times\overline{\Omega}) : \\ u=\varphi \text{ on } \Gamma_0,\ G(Du)\le 0 \text{ a.e. in }]0,T[\times\Omega\big\}$$

and the supremun

$$s^0 \doteq \sup\left\{\int_{[0,T]\times\overline{\Omega}} u(x)\,dx;\quad u\in S_\varphi^0\right\} < +\infty.$$

The reader can easily recognize that the arguments of Section 3 can be reproduced obtaining the following

Theorem 8.2. *Assume Hypotheses* 6.1 *and* 6.2. *Let* S_φ^0 *and* s^0 *as in Definition* 8.1. *Then there exists a unique* $\overline{u}\in S_\varphi^0$ *with the following properties.*

(*i*) $\overline{u}$ *is concave*;

(*ii*) $\int_{[0,T]\times\overline{\Omega}} \overline{u}(x)\ dx = s^0$;

(*iii*) $\overline{u}\ge u$ *on* $[0,T]\times\overline{\Omega}$ *for every* $u\in S_\varphi^0$;

(*iv*) $G(\xi)\le 0\ \ \forall \xi\in D^\pm\overline{u}(x)\ \forall x\in\,]0,T[\times\Omega$;

(*v*) $\overline{u}$ *is the unique maximal viscosity solution of* $\mathcal{P}_0$.

Proof. The proof can be obtained reproducing step by step the arguments of Section 3. The unique change consists in replacing the set Γ by the set Γ_0; the reader sees immediately that this does not affect any point of the procedure. □

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