Journal of Convex Analysis
Volume 18 (2011), No. 3, 811–821

Local U-Convexity

S. Dutta*
Department of Mathematics and Statistics,
Indian Institute of Technology, Kanpur, India
sudipta@iitk.ac.in

Bor-Luh Lin
Department of Mathematics,
University of Iowa, Iowa City, USA
bllin@math.uiowa.edu

Received: September 18, 2009
Revised manuscript received: July 13, 2010

Lau considered the notion of U- convex spaces (originally called U-spaces) and showed that both uniform convexity and uniform smoothness imply U-convexity. Also U-convex spaces are uniformly non-square and hence super-reflexive. In this paper we introduce local U-convexity. It is shown that there are two possible localization of U-convexity. We derive our results quantitatively, that is, by the properties of modulus functions. Relationship to modulus of (local) uniform convexity is established and its consequences are discussed.

Keywords: (Locally) uniformly convex, super-reflexive spaces, U-convexity

2000 Mathematics Subject Classification: 46B20

1. Introduction

Given a Banach space X we denote its unit sphere and unit ball by S_X and B_X respectively. The norm on X is said to be locally uniformly convex at $x \in S_X$ if given $(y_n) \subseteq B_X$ such that $\frac{1}{2}\|x + y_n\| \to 1$ then $y_n \to x$ in norm. In this case we refer x as a LUR point. The space is said to be locally uniformly convex if every $x \in S_X$ is a LUR point. X is called uniformly convex if the convergence above is uniform for all $x \in S_X$. Quantitatively, for $x \in S_X$ and $t \in (0, 2)$ consider the following two modulus

$$\delta(x,t) = \inf_{y \in B_X:\ \|x-y\| \geq t} \left\{ 1 - \frac{1}{2}\|x + y\| \right\},$$

and

$$\delta(t) = \inf_{x \in S_X} \delta(x,t).$$

Then x is a LUR point if and only if $\delta(x,t) > 0$ for all $t \in (0, 2)$ and X is Uniformly convex if and only if $\delta(t) > 0$ for all $t \in (0, 2)$.

*Part of the works in this paper was carried out during May-July 2009, when the first named author was visiting University of Iowa, Iowa City. His visit was supported by the Fellowship from Indo-US Science and Technology Forum.

ISSN 0944-6532 / $ 2.50

In [2, 4] the following generalization of uniform convexity was considered: X is said to be a U-space if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in S_X$ with $\frac{1}{2}\|x+y\| > 1-\delta$, $f(y) > 1-\varepsilon$ for all $f \in \mathcal{D}(x)$, where $\mathcal{D}(x) = \{f \in S_{X^*} : f(x) = \|x\|\}$. It was shown in [4] that U-spaces are uniformly non-square hence super-reflexive. Also the notion of U-space is self dual, that is if X is a U-space so is X^*. In [2] Gao and Lau have shown that U-spaces have uniform normal structure.

Gao in [1] defined modulus of U-convexity which is the same as $U_S(t)$ defined below.

For $x \in S_X$, $f \in S_{X^*}$ and $0 < t < 2$ we will denote the slice $\{y \in B_X : f(y) > f(x) - t\}$ by $S(x, f, t)$ and its complement in B_X by $S(x, f, t)^c$.

The following function may be interpreted as the modulus of local uniform convexity along a given hyperplane.

Definition 1.1. Let X be a Banach space, $x \in S_X$ and $f \in S_{X^*}$. For $t \in (0, 2)$ we define,

$$U(x, f, t) = \inf_{y \in S(x,f,t)^c} \left\{1 - \frac{1}{2}\|x + y\|\right\}.$$

Note that $U(x, f, t) > 0$ for $t \in (0, 2)$ implies and is implied by whenever $(y_n) \in B_X$ is such that $\frac{1}{2}\|x + y_n\| \to 1$ then $f(y_n) \to f(x)$. To see this, first let us assume $U(x, f, t) > 0$ for all $t \in (0, 2)$. Let $(y_n) \subseteq B_X$ be such that $\frac{1}{2}\|x + y_n\| \to 1$. Then for all t, $(y_n) \subseteq S(x, f, t)$ eventually and hence $f(y_n) \to f(x)$. Conversely let $U(x, f, t) = 0$ for some $t \in (0, 1)$. Then we can choose a sequence $(y_n) \subseteq S(x, f, t)^c$ such that $1 - \frac{1}{2}\|x + y_n\| \to 0$. Hence $f(y_n)$ does not converge to $f(x)$.

Starting with $U(x, f, t)$ we can define the following three quantities.

Definition 1.2.

$$\tilde{U}(x, t) = \inf_{f \in S_{X^*}} U(x, f, t).$$

$$U_S(x, t) = \sup_{f \in \mathcal{D}(x)} U(x, f, t).$$

$$U_I(x, t) = \inf_{f \in \mathcal{D}(x)} U(x, f, t).$$

We will show in the next section that for any Banach space X and $t \in (0, 2)$ one has $\tilde{U}(x, t) = \delta(x, t)$.

Our main objects of study in this paper is the modulus $U_S(x, t)$ and $U_I(x, t)$ and their uniform versions, namely,

$$U_S(t) = \inf_{x \in S_X} U_S(x, t); \qquad U_I(t) = \inf_{x \in S_X} U_I(x, t).$$

From Gao's result in [1] it follows that $U_S(t) > 0$ for all $t \in (0, 2)$ implies X is an U-space. The relationship of the modulus of U-convexity to the James' constant, $J(X) = \sup\{\min\{\|x + y\|, \|x - y\|\} \ x, y \in B_X\}$ is described in [6].

It is easy to observe that $U_I(t) > 0$ for all $t \in (0, 2)$ if and only if X is a U-space. We will actually show that for a U-convex space X and $t \in (0, 2)$, $U_S(t) = U_I(t)$.

Thus following Gao, we will be refereing X to be a U-convex space if $U_S(t) > 0$ for all $t \in (0, 2)$.

However, in their local version, $U_S(x,t) \geq U_I(x,t),\ t \in (0,2)$ and in general they are not same. Thus there are at least two possible localizations of the notion of U-convexity. We show in Section 1 that both $U_S(x,\cdot)$ and $U_I(x,\cdot)$ are continuous functions of t. As function of x, these two modulus satisfy the following relation (see Proposition 2.15): Let $x, x_1, x_2 \in S_X$ such that $x = \lambda x_1 + (1-\lambda)x_2,\ \lambda \in (0,1)$. Then $U_S(x,t) \geq \lambda U_I(x_1,t) + (1-\lambda)U_I(x_2,t)$. We illustrate how this relation helps us to determine U_I for ℓ_p spaces.

As mentioned before, Lau in [4] has shown that the dual of a U-convex space is also U-convex. In Section 1 of this paper we will recover this result quantitatively by showing $U_I(t) = U_I^*(t)$ for all t, where $U_I^*(t)$ denote the corresponding modulus for X^*. This shows that uniformly smooth spaces are also U-convex and it was noted in [4]. In the local version, we show that if $x \in S_X$ is a Fréchet smooth point then $U_I(x,t) > 0$ for all $t \in (0,2)$. However, if $x \in S_X$ is just a smooth point then even $U_S(x,t)$ need not be positive (Example 2.13).

In Section 2 we define two new quantities $d(x,t)$ and $s(x,t)$ which are, respectively the modulus of denting point and modulus of strongly exposed point. Our main theorem in Section 2 (Theorem 3.5) establishes the relations between $\delta(x,t)$, $U_I(x,t)$ and these two modulus.

Let $x \in S_X$ and $f \in \mathcal{D}(x)$. If f is a LUR point then x is Fréchet smooth point and the converse is not true in general. As a corollary to Theorem 3.5, we show that f is LUR if and only if x is Fréchet smooth and $U_I(f,t) > 0$ for all $t \in (0,2)$.

Both $d(x,t)$ and $s(x,t)$ have their respective uniform version, namely, $d(t)$ and $s(t)$. The condition $d(t) > 0$ for all $t \in (0,2)$ may be interpreted as every point $x \in S_X$ is uniformly denting. In this case a standard argument shows that the dentability index $D(X)$ is finite and thus X has a uniformly convex renormimg (see [5] for definition and the renorming result) and in particular, X is super-reflexive. If $s(t) > 0$ for all t then X is 'uniformly strongly exposed', meaning for all $x \in S_X$, given t there exists $f \in \mathcal{D}(x)$ such that the slice $S(x, f, s(t))$ has norm diameter less than t. In this case we show that X is already uniformly convex and $\delta(t)$ is of the order of square of $s(t)$.

2. The modulus U_S and U_I

U_S and U_I are the main object of studies in this paper. But first we show $\tilde{U}(x,t)$ defines nothing new but $\delta(x,t)$.

Proposition 2.1. *Let X be a Banach space. For all $x \in S_X$ and $t \in (0,2)$, $\tilde{U}(x,t) = \delta(x,t)$.*

Proof. It is easy to see that $\tilde{U}(x,t) \geq \delta(x,t)$. To see the other inequality, let $\delta(x,t) < s$ for some s. By definition of $\delta(x,t)$, there exists $y \in S_X,\ \|x-y\| \geq t$ such that $1 - \frac{1}{2}\|x+y\| < s$. We choose $f \in S_{X^*}$ such that $f(x-y) = \|x-y\|$. Hence $f(x-y) \geq t$ and $\tilde{U}(x,t) < s$ as well. This shows $\tilde{U}(x,t) \leq \delta(x,t)$. □

Remark 2.2. Proposition 2.1 provides us with another geometric interpretation of LUR points. Namely, $x \in S_X$ is a LUR point if and only if x is uniformly LUR point along all hyperplanes.

Recall that $x \in S_X$ is called a wLUR point if $(y_n) \subseteq B_X$, $\frac{1}{2}\|x + y_n\| \to 1$ implies $y_n \xrightarrow{w} x$. If we drop the uniformity assumption in Proposition 2.1, we get wLUR point.

Proposition 2.3. *Let X be a Banach space. Then $x \in S_X$ is a wLUR point if and only if for all $f \in S_{X^*}$ and all $t \in (0, 2)$, $U(x, f, t) > 0$.*

Proof. Let $x \in S_X$. If for some $f \in S_X$, $t > 0$ one has $U(x, f, t) = 0$ then there exists $(y_n) \subseteq B_X$ such that $f(x - y_n) > t$ but $1 - \frac{1}{2}\|x + y_n\| \to 0$. Hence x is not a wLUR point. Conversely, if x is not a wLUR point there exists $(y_n) \subseteq B_X$ and $f \in S_X$ such that $\frac{1}{2}\|x + y_n\| \to 1$ but (passing to a subsequence if necessary) $f(x - y_n) > t$ for some $t > 0$. Hence we have $U(x, f, t) = 0$. □

It is easy to see that in ℓ_1, if we denote (e_n) to be the standard unit vector basis, then $U_S(e_n, t) = t/2$ but $U_I(e_n, t) = 0$ for all $0 < t \leq 1$. Thus, locally, U_S and U_I are different. However, we show that $U_S(t) = U_I(t)$ for all $t \in (0, 2)$ and $U_I(t) > 0$ (equivalently $U_S(t) > 0$) for all $t \in (0, 2)$ characterizes U-convex spaces. In the process of proving it, we also establish some more properties of these moduli, which are of independent interest.

From the definition of $U_S(x, t)$ and $U_I(x, t)$ we have the following lemma.

Lemma 2.4. *Let X be a Banach space. If $x \in S_X$ is a smooth point then $U_S(x, t) = U_I(x, t)$ for all t.*

The proof of the following lemma, again, is a straightforward consequence of definition of U-convexity and the remark after [4, Definition 2.2].

Lemma 2.5. *X is U-convex space if and only if $U_I(t) > 0$ for all $t \in (0, 2)$. Also if X is a U-convex space, then given t, $U_I(t)$ is the largest constant > 0 such that for all $x, y \in S_X$, $\frac{1}{2}\|x + y\| > 1 - U_I(t) \Rightarrow \frac{1}{2}\|f + g\| > 1 - t$ for all $f \in \mathcal{D}(x)$, $g \in \mathcal{D}(y)$.*

Let us denote by U_I^* the corresponding U_I for X^*. The next result shows $U_I(t) = U_I^*(t)$ for all t. In this quantitative version, it is strengthening of the fact that U-convexity is self dual, that is, X is U-convex if and only if X^* is U-convex.

Theorem 2.6. *Let X be such that $U_I(t) > 0$ for all $t \in (0, 2)$. Then for all $t \in (0, 2)$, $U_I(t) = U_I^*(t)$.*

Proof. The proof essentially follows the same argument as in [4]. Let $U_I(t) > 0$ and $f, g \in S_{X^*}$ be such that $\frac{\|f+g\|}{2} > 1 - U_I(t)$. Since U-convex spaces are super-reflexive, there exists $z \in S_X$ such that $(\frac{f+g}{2})(z) = \frac{\|f+g\|}{2}$. Then $f(z) > 1 - 2U_I(t)$ and $g(z) > 1 - 2U_I(t)$. Now if $x \in D(f), y \in D(y)$, $\frac{\|x+z\|}{2} > 1 - U_I(t)$, $\frac{\|y+z\|}{2} > 1 - U_I(t)$. By definition of $U_I(t)$, for all $h \in D(z), h(x) > 1 - t, h(y) > 1 - t$, hence $\frac{\|x+y\|}{2} > 1 - t$.

By the second part of Lemma 2.5 we conclude $U_I^*(t) \geq U_I(t)$. By duality we have $U_I^*(t) = U_I(t)$. □

In the next proposition we show continuity of $U_S(x,t)$ and $U_I(x,t)$ with respect to t.

Proposition 2.7. *Let X be a Banach space and $x \in S_X$. Then $U_S(x,\cdot)$ and $U_I(x,\cdot)$ are continuous function in t for $t \in (0,2)$.*

Proof. Let $x \in S_X$ and $t \in (0,2)$. We first assume $t \neq 1$. Let $s \in (0,2)$ be such that $1-s$ and $1-t$ have the same sign.

Assume $s > t$. Then $U_S(x,s) \geq U_S(x,t)$. For any $f \in \mathcal{D}(x)$, we have $S(x,f,s)^c \subseteq S(x,f,t)^c$ and hence it follows that $U_S(x,s) \geq U_S(x,t)$. Given $\varepsilon > 0$, we find $f \in \mathcal{D}(x)$ such that $U_s(x,s) < U(x,f,s) + \varepsilon/2$. Also we choose $y \in S(x,f,t)^c$ such that $U_S(x,t) + \varepsilon/2 \geq 1 - \frac{1}{2}\|x+y\|$. Then $f(\frac{1-s}{1-t}y) \leq 1-s$ and hence $\frac{(1-s)}{1-t}y \in S(x,f,s)^c$. Thus $U_S(x,s) < 1 - \frac{1}{2}\|x + \frac{(1-s)}{1-t}y\| + \varepsilon/2$. From triangle inequality, it follows that $1 - \frac{1}{2}\|x+y\| \geq 1 - \frac{1}{2}\|x + \frac{(1-s)}{1-t}y\| - \frac{1}{2}|\frac{s-t}{1-t}|$. Hence we have $U_S(x,s) - U_S(x,t) < \frac{1}{2}|\frac{s-t}{1-t}| + \varepsilon$.

If $s < t$ we repeat the above argument with the roles of s and t interchanged.

This shows the continuity of $U_S(x,\cdot)$ at t with $t \neq 1$.

For $t = 1$ we make little modification to the above argument. Since $U_S(x,1) \geq U(x,f,1)$ for any $f \in \mathcal{D}(x)$ as above and given $\varepsilon > 0$ we can choose $y \in S(x,f,1)^c$ such that $U_S(x,1) + \varepsilon/2 > 1 - \frac{1}{2}\|x+y\|$. By continuity of the norm we can further choose $y' \in S(x,f,1)^c$ such that $f(y') \leq -\varepsilon/2$ and $U_S(x,1) + \varepsilon > 1 - \frac{1}{2}\|x+y'\|$. Observe $y' \in S(x,f,1+\varepsilon/2)^c$ and hence if we choose $s \leq 1 + \varepsilon/2$ then $S(x,f,s)^c$. The rest of the argument follows.

To show continuity for $U_I(x,\cdot)$ we indicate the modifications necessary in the above argument for $U_S(x,t)$. Again let $t \neq 1$ and $s > t$ be such that $1-s$ and $1-t$ have the same sign. Note that $U_I(x,s) \geq U_I(x,t)$. For any $f \in \mathcal{D}(x)$, $U_I(x,s) < U(x,f,s)$. Given $\varepsilon > 0$, we choose a $f \in \mathcal{D}(x)$ such that $U_I(x,t) + \frac{\varepsilon}{4} \geq U_I(x,f,t)$. We can further choose $y \in S(x,f,t)^c$ such that $U_I(x,f,t) + \frac{\varepsilon}{4} \geq 1 - \frac{1}{2}\|x+y\|$. The rest of the argument is same as in the case of $U_S(x,t)$. □

Our next result shows that to determine $U_S(t)$ and $U_I(t)$ it is enough to consider a dense set of S_X.

Theorem 2.8. *Let X be a Banach space and $D \subseteq S_X$ a norm dense set. Then*

$$U_S(t) = \inf_{x \in D} U_S(x,t); \qquad U_I(t) = \inf_{x \in D} U_I(x,t).$$

Proof. We first show it for U_S and later indicate the modifications to be done for U_I.

Let us denote $\inf_{x\in D} U_S(x,t)$ by $U'(t)$. Clearly we have $U'(t) \geq U_S(t)$. Now if $U'(t) > U_S(t)$ for some t, then there exists $\varepsilon > 0$ and $x \in S_X \backslash D$ such that $U'(t) > U_S(x,t) + 3\varepsilon$. Let $(x_n) \subseteq D$ be such that $x_n \to x$ in norm. Thus we have $U_S(x_n,t) > U_S(x,t) + 3\varepsilon$.

For each n we choose $f_n \in \mathcal{D}(x_n)$ such that $U_S(x_n,t) < U(x_n,f_n,t) + \varepsilon$. Thus we have $U(x_n,f_n,t) > U_S(x,t) + 2\varepsilon$ for all n.

Let f be a weak*-cluster point of (f_n). Then $f(x) = 1$.

Let $s > t$. Then $U_S(x, s) \geq U(x, f, s)$ and we choose $y \in S(x, f, s)^c$ such that $U_S(x, s) + \varepsilon > 1 - \frac{1}{2}\|x + y\|$. Then for n large enough we have $U_S(x_n, s) + \varepsilon > 1 - \frac{1}{2}\|x_n + y\|$. Since f is a weak*-cluster point of (f_n) and $s > t$ there exists n such that $y \in S(x_n, f_n, t)^c$ as well. Thus $1 - \frac{1}{2}\|x + y\| \geq U(x_n, f_n, t)$. Combining this with the other inequality we have $U_S(x, s) - U_S(x, t) > \varepsilon$. But then this is true for all $s > t$ which contradicts the continuity of $U_S(x, \cdot)$ at t. Thus we have $U'(t) = U_S(t)$ for all t.

To prove the result for U_I, if $\inf_{x \in D} U_I(x, t) > U_I(t)$ for some t then, as before we can get $x \in S_X \setminus D$, $x_n \in D, x_n \to x$ in norm such that $U_I(x_n, t) > U_I(x, t) + 3\varepsilon$. This implies for all $f_n \in \mathcal{D}(x_n)$, $U(x_n, f_n, t) > U_I(x, t) + 3\varepsilon$. Once again let $s > t$ and we choose $f \in \mathcal{D}(x)$ such that $U_I(x, s) + \varepsilon > U(x, f, s)$. Since $x_n \to x$ in norm, a standard argument shows that we have f is a weak*-cluster point of some (f_n), $f_n \in \mathcal{D}(x_n)$. The rest of argument is same as before. □

In [1] it was shown that if $U_S(t) > 0$ for all t then X is U-convex and by our Lemma 2.5 it follows that $U_I(t) > 0$ for all t is equivalent to X being U-convex. As a corollary to the Theorem 2.8, we show that in this case $U_S(t) = U_I(t)$.

Corollary 2.9. *Let X be a U-convex space. Then $U_S(t) = U_I(t)$ for all $t \in (0, 2)$.*

Proof. If X is U-convex then as mentioned in the introduction, X is a super reflexive space and hence, in particular, an Asplund space. Thus there exists a dense set $D \subseteq S_X$ consisting of Fréchet smooth points of X. But for $x \in D$ we have $U_S(x, t) = U_I(x, t)$ for all t. The conclusion follows from Theorem 2.8. □

We have mentioned in the introduction that if X is uniformly smooth then X is U-convex. In the next section we will show (with a quantitative estimate) that if $x \in S_X$ is a Fréchet smooth point, then $U_I(x, t) > 0$. But x just being a smooth point does not give that $U_S(x, t) > 0$ for all t. Before we give an example we prove the following proposition. Recall that if in X we have an equivalent norm such that X^{**} is LUR, then X is reflexive. The proposition below shows this is true even if $U_S^{**}(x^{**}, t) > 0$ for all t and all $x^{**} \in S_{X^{**}}$.

Proposition 2.10. *Let X be a Banach space and $x^{**} \in S_{X^{**}}$ such that $U_S^{**}(x^{**}, t) > 0$ for all $t < 1$. Then x^{**} attains its norm over X^*. In particular, if for a Banach space X $U_S^{**}(x^{**}, t) > 0$ for all t and all $x^{**} \in S_{X^{**}}$ then X is reflexive.*

Proof. Let $U_S^{**}(x^{**}, t) > 0$ for all $t < 1$. We choose a net $(x_\alpha) \subseteq B_X$ which weak* converges to x^{**}. By weak*-lower semi-continuity of the norm we have $\frac{1}{2}\|x_\alpha + x^{**}\| \to 1$. But then by the remark following Definition 1.1, there exists $F \in S_{X^{***}}, F(x^{**}) = 1$ such that $F(x_\alpha) \to 1$. Let $f = F|_X$. And thus $f(x_\alpha) \to 1$ and $\|f\| = 1$. But (x_α) weak* converges to x^{**} and hence $x^{**}(f) = 1$. □

The following example shows that if x is a smooth point $U_S(x, t)$ need not be positive for all t.

Example 2.11. In [7] it was shown that the James' space J, which is quasi-reflexive, admits a renorming such that J^{***} is strictly convex, that is, in particular J^{**} is smooth. Since, J is not reflexive, by Proposition 2.10, there exists $x^{**} \in J^{**}$ for which $U_S(x^{**}, t) = 0$ for some $t < 1$.

However, if $U_I(x, t)$ is large for some t then x is a smooth point.

Proposition 2.12. *Suppose $x \in S_X$ is such that $U_I(x, t) \geq \frac{1}{2}$ for some $t \in (0, 1)$. Then x is a smooth point.*

Proof. Let $x \in S_X$ such that $U_I(x, t) > \frac{1}{2}$ for some $t \in (0, 1)$ and $f, g \in \mathcal{D}(x)$. We claim for any $y \in \ker f$, $g(y) < 1 - 2U_I(x, t)$. Thus if $U_I(x, t) \geq \frac{1}{2}$ then $\ker f \subseteq \ker g$ and since $f(x) = g(x) = 1$ we have $f = g$. This will show x is a smooth point.

To see the claim, assume on the contrary that $g(y) \geq 1 - 2U_I(x, t)$. Then $\frac{1}{2}\|x + y\| \geq \frac{1}{2}g(x + y) \geq 1 - U_I(x, t)$. Thus, by definition of $U_I(x, t)$, we have $f(y) > 1 - t$ which is a contradiction. □

Our next result shows relation between U_S, U_I to U_S^{**}, U_I^{**}. For $x \in S_X$, by $U_S^{**}(x, t)$ and $U_I^{**}(x, t)$ we denote the corresponding moduli calculated for x in X^{**}.

Proposition 2.13. *For any Banach space X and $x \in S_X$ $t \in (0, 2)$ we have*

$$U_S(x, t) \geq U_S^{**}(x, t) \geq U_I(x, t) \geq U_I^{**}(x, t).$$

Proof. The first inequality is easy to see. The second and the third follows from the fact that any $f \in S_X$, can be, canonically identified as $\hat{f}$ in the third dual of X and by the weak*-density of B_X in $B_{X^{**}}$, we have $U(x, f, t) = U^{**}(x, \hat{f}, t)$. □

It follows from the Proposition 2.13 that if $x \in S_X$ is a smooth point of X^{**} (so called very smooth point) then equality holds throughout.

Question 2.14.

(a) Suppose $U_S(x, t) > 0$ and $U_S(x, t) = U_I(x, t)$ for all $t \in (0, 2)$. Is x a smooth point?

(b) Suppose $U_S(x, t) > 0$ and equality holds in Proposition 2.13. Is x a smooth point in X^{**}?

The next result is often helpful in calculation of U_S or U_I.

Proposition 2.15. *Let X be a Banach space. Let $x, x_1, x_2 \in S_X$ be such that $x = \lambda x_1 + (1 - \lambda)x_2$ for some $\lambda \in (0, 1)$. Then $U_S(x, t) \geq \lambda U_I(x_1, t) + (1 - \lambda)U_I(x_2, t)$. In particular if x is a smooth point then we can replace $U_S(x, t)$ by $U_I(x, t)$.*

Proof. Let $x = \lambda x_1 + (1 - \lambda)x_2$ for some $\lambda \in (0, 1)$ and $x, x_1, x_2 \in S_X$. We first observe that if $f \in \mathcal{D}(x)$ then $f \in \mathcal{D}(x_1)$, and $f \in \mathcal{D}(x_2)$. Also for any $y \in B_X$, we have $\frac{1}{2}\|x + y\| \leq \lambda\frac{1}{2}\|x_1 + y\| + (1 - \lambda)\frac{1}{2}\|x_2 + y\|$. Hence for any $f \in \mathcal{D}(x)$ we have, $U(x, f, t) \geq \lambda(1 - \frac{1}{2}\|x_1 + y\|) + (1 - \lambda)(1 - \frac{1}{2}\|x + y\|) \geq \lambda U_I(x_1, t) + (1 - \lambda)U_I(x_2, t)$. □

We illustrate the use of Proposition 2.15 to calculate $U_I(t)$ for ℓ_p-spaces, $1 < p < \infty$.

Note that every point S_{ℓ_p}, $1 < p < \infty$ is smooth point and hence we have $U_S(x,t) = U_I(x,t)$ for all t and all $x \in S_{\ell_p}$.

Let (e_n) be unit vector basis of ℓ_p. It is easy to see that for all n, $U_I(e_n,t)$ (which is equal to $U_S(e_n,t)$) satisfies $U_I(e_n,t) \geq 1 - [(1-\frac{t}{2})^p + 1 - (\frac{1}{2} - \frac{t}{2})^p]^{\frac{1}{p}}$. Now by Theorem 2.8 to calculate $U_I(t)$ for ℓ_p, it is enough to calculate $\inf_{x \in D} U_I(x,t)$ for a dense set $D \subseteq S_{\ell_p}$. We consider D to be set of finitely supported points in S_{ℓ_p}.

Let $x = (x_1, x_2, \cdots x_n, 0, 0, \cdots) \in S_{\ell_p}$. Let $\|x\|_1$ denote the ℓ_1 norm of x. We write $x = \|x\|_1 \sum_{i=1}^n \frac{|x_i|}{\|x\|_1} sgn x_i e_i$. Thus by Proposition 2.15 we have $U_I(x,t) \geq \|x\|_1 \frac{|x_i|}{\|x\|_1} U_I(e_i,t) = \|x\|_1 U_I(e_i,t)$ and since $\|x\|_1 \geq \|x\|_p = 1$ we have $U_I(x,t) \geq U_I(e_i,t) \geq 1 - [(1-\frac{t}{2})^p + 1 - (\frac{1}{2} - \frac{t}{2})^p]^{\frac{1}{p}}$.

However, by Proposition 2.6 we have $U_I^*(t) = U_I(t)$ and the same calculation as above shows that $U_I^*(t) \geq 1 - [(1-\frac{t}{2})^q + 1 - (\frac{1}{2} - \frac{t}{2})^q]^{\frac{1}{q}}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Hence we have for ℓ_p spaces,

$$U_I(t) \geq \max\left\{1 - \left[\left(1-\frac{t}{2}\right)^p + 1 - \left(\frac{1}{2} - \frac{t}{2}\right)^p\right]^{\frac{1}{p}}, 1 - \left[\left(1-\frac{t}{2}\right)^q + 1 - \left(\frac{1}{2} - \frac{t}{2}\right)^q\right]^{\frac{1}{q}}\right\}.$$

3. Relation to $\delta(x,t)$

In this section we establish relationship with U_I and U_S to modulus of convexity δ. We prove all our results locally, but also indicate that interesting conclusions can be drawn from uniformizations of the local results.

We introduce the following quantity which is crucial for our results in this section.

Let $x \in S_X, f \in S_{X^*}, t \in (0,2)$. Define

$$s(x,f,t) = \inf_{y \in \ker f:\ \|y\| \geq t/4} \|x+y\| - 1.$$

Lemma 3.1. *Let $x \in S_X$ and $f \in S_{X^*}$ be such that $f(x) > 0$. If $s(x,f,t) > 0$ then* $\operatorname{diam} S(x,f,f(x)s(x,f,t)) < t$.

Proof. We first note that $s(x,f,t) \leq t/4$. Let $z \in B_X$ be such that $f(z) > f(x)(1 - s(x,f,t))$. We put $y = z - \frac{f(z)}{f(x)}x$. Then $y \in \ker f$ and

$$\|x+y\| - 1 \leq \left\|z + x - \frac{f(z)}{f(x)}x\right\| - 1 \leq \left|1 - \frac{f(z)}{f(x)}\right| < s(x,f,t).$$

Hence, $\|y\| < t/4$. But then $\|z - x\| < t/4 + \left|1 - \frac{f(z)}{f(x)}\right| < t/4 + s(x,f,t) < t/4 + t/4 < t/2$. Thus we have $\operatorname{diam} S(x,f,f(x)s(x,f,t)) < t$. □

We now define the following two moduli.

Definition 3.2. Let $x \in S_X$. For $t \in (0,2)$ define

$$d(x,t) = \sup_{f \in S_{X^*}} s(x,f,t)$$

$$s(x,t) = \sup_{f \in \mathcal{D}(x)} s(x,f,t),$$

and their uniform versions,

$$d(t) = \inf_{x \in S_X} d(x,t); \qquad s(t) = \inf_{x \in S_X} s(x,t).$$

The following Proposition shows $d(x,t) > 0$ (respectively $s(x,t) > 0$ for all $t \in (0,2)$) characterizes denting points (respectively strongly exposed points) of B_X.

Proposition 3.3.

(*a*) *$x \in S_X$ is denting point of B_X if and only if $d(x,t) > 0$ for all $t \in (0,2)$.*

(*b*) *$x \in S_X$ is strongly exposed point of B_X if and only if $s(x,t) > 0$ for all $t \in (0,2)$. Furthermore, if $s(x,t) > 0$ then there exists $f \in \mathcal{D}(x)$ such that* $\operatorname{diam}\{y \in B_X : f(y) > 1 - s(x,t)\} < t$.

Proof. The proof of (a) and the first part of (b) follow directly from Lemma 3.1.

To show the second part let $s(x,t) > 0$. We choose a sequence (r_n), $r_n \in (0,1)$, $r_n \uparrow 1$ and $f_n \in \mathcal{D}(x)$ such that $s(x,f_n,t) > r_n s(x,t)$. Let f be a weak*-cluster point of (f_n). Then $f \in \mathcal{D}(x)$ as well. Let $y \in \{y \in B_X : f(y) > 1 - s(x,t)\}$. Then we can choose some n large such that $f_n(y) > 1 - r_n s(x,t)$ and hence $f_n(y) > 1 - s(x,f_n,t)$. Applying Lemma 3.1 we have the desired conclusion. □

Remark 3.4. (a) Starting with $s(x,f,t)$ one could also consider $\tilde{d}(x,t) = \inf_{f \in S_{X^*}} s(x,f,t)$ and $\tilde{s}(x,t) = \inf_{f \in \mathcal{D}(x)} s(x,f,t)$. While it is easy to check that $\tilde{d}(x,t) \le -t/4$, but if $\tilde{s}(x,t) > 0$ for all $t \in (0,2)$ then we have every $f \in \mathcal{D}(x)$ is an Fréchet smooth. However, the condition $\tilde{s}(x,t) > 0$ for all $t \in (0,2)$ is apparently stronger than every $f \in \mathcal{D}(x)$ is Fréchet smooth and we do not know if it already implies x is a LUR point.

(b) For $f \in S_{X^*}$ we can weak* version of the moduli d and s, namely, we take $d_*(f,t) = \sup_{x \in S_X} s(f,x,t)$ and if f attains its norm over X, $s_*(f,t) = \sup_{x \in S_X : f(x)=1} s(f,x,t)$. Again by Lemma 3.1 it follows that f is a weak*-denting (respectively weak*-strongly exposed) point of B_{X^*} if and only if $d_*(f,t) > 0$ (respectively $s_*(f,t) > 0$) for all $t \in (0,2)$. The corresponding estimate on diameter of the slices from Proposition 3.3 are also true.

(c) It is easy to see that $s(x,t) \ge 2\delta(x,t)$ for all $x \in S_X$ and $t \in (0,2)$.

We now state the main Theorem of this section.

Theorem 3.5. *Let X be a Banach space and $x \in S_X$. Then for all $t \in (0,2)$,*

(*a*) *There exists $f_0 \in S_{X^*}$ such that for all $f \in S_{X^*}$, $U(x,f,t) \ge \delta(x,t) \ge U(x,f_0,f_0(x)d(x,t))$.*

(*b*) *$U_I(x,t) \ge \delta(x,t) \ge U_I(x,s(x,t))$.*

Proof. The proof for (a) and (b) are similar. We prove it here for (b).

Fix t. That $U_I(x,t) \geq \delta(x,t)$ follows from definition.

For any $f \in \mathcal{D}(x)$, we have

$$\left\{y \in B_X : \left\|\frac{x+y}{2}\right\| > 1 - U_I(x, s(x,t))\right\} \subseteq \{y \in B_X : f(y) > 1 - s(x,t)\}.$$

If $s(x,t) > 0$, there exists $f \in \mathcal{D}(x)$ such that $\operatorname{diam}\{y \in B_X : f(y) > 1 - s(x,t)\} < t$. Hence if $\|x-y\| \geq t$ then $\|\frac{x+y}{2}\| \leq 1 - U_I(x, s(x,t))$. Thus $\delta(x,t) \geq U_I(x, s(x,t))$. $\square$

Recall from Proposition 2.3 that $x \in S_X$ is a wLUR point if and only if for all $f \in S_{X^*}$ and all $t \in (0,2)$, $U(x,f,t) > 0$. Thus Theorem 3.5(a) says that $x \in S_X$ is a LUR point if and only if it is wLUR point and denting point of B_X. Since wLUR point are already extreme point and by [3], a denting point is precisely point which is point of continuity for weak-norm topology and extreme point, we actually recover the fact that x is LUR point if and only if x is wLUR and is a point of continuity for weak-norm topology in B_X.

Combining with Proposition 3.3(b), Theorem 3.5(b) says $x \in S_X$ is a LUR point if and only if x is a strongly exposed point in B_X and $U_I(x,t) > 0$ for all $t \in (0,2)$. For the rest of this section we will be concerned with the implications of this result.

From Remark 3.4(a) and Theorem 3.5(b) it follows that for $f \in S_{X^*}$ is norm attaining functional then $\delta(f,t) > U_I^*(f, s_*(f,t))$.

If $f \in S_{X^*}$ and $x \in S_X$ are such that $f(x) = 1$ and f is a LUR point then x is a Fréchet smooth point. There is no characterization known so far how to get the converse and Theorem 3.5(b) provides us with one such. Note that if $x \in S_X$ is Fréchet smooth point then $f \in \mathcal{D}(x)$ is weak*-strongly exposed.

Corollary 3.6. *For a Banach space X let $x \in S_X$ and $f \in \mathcal{D}(x)$. Then the following are equivalent.*

(a) *f is a LUR point of B_{X^*}.*
(b) *x is Fréchet smooth and $U_I(f,t) > 0$ for all $t \in (0,2)$.*

As mentioned before Uniformly smooth spaces are U-convex. We now show that the local version of this is also true.

Proposition 3.7. *Let X be a Banach space and $x \in S_X$ a Fréchet smooth point and $f \in \mathcal{D}(x)$. Then $U_I(x,t) \geq \frac{1}{2}s_*(f, 4t/5)$.*

Proof. Let $y \in B_X$ be such that $\frac{1}{2}\|x+y\| > 1 - \frac{1}{2}s_*(f,t)$. We choose $g \in \mathcal{D}(\frac{x+y}{2})$. Then $g(x) > 1 - s_*(f,t)$, $g(y) > 1 - s_*(f,t)$. By Proposition 3.3(b) we have $\|f-g\| < t$. Hence $f(y) > -t + 1 - s_*(f,t) \geq 1 - 5t/4$. Thus we conclude $U_I(x, 5t/4) \geq \frac{1}{2}s_*(f,t)$ or equivalently, $U_I(x,t) \geq \frac{1}{2}s_*(f, 4t/5)$. $\square$

For the rest of the section we give an application of the uniform version of the inequality in Theorem 3.5(b).

First suppose $d(t) > 0$ for $t \in (0,2)$. Then it implies that all $x \in S_X$ is uniformly denting points. Given t one can define the quantity $D(X,t)$ which is the dentability index of X at t. We do not define it here and the interested reader can find the details in [5]. A standard argument shows that if $d(t) > 0$ then $D(X,t)$ is finite and hence if $d(t) > 0$ for all $t \in (0,2)$, we have dentability index of X is finite and in this case X is super reflexive. Recall that any super reflexive space has a uniformly convex renormimg.

If $s(t) > 0$ for all t then X is 'uniformly strongly exposed', meaning for all $x \in S_X$, given t there exists $f \in \mathcal{D}(x)$ such that the slice $S(x, f, s(t))$ has norm diameter less than t. This will give X is a super-reflexive space since $d(t) \geq s(t)$ for all t. However we show $s(t) > 0$ for all t implies $\delta(t) > 0$ and hence X is Uniformly convex and we also show that $\delta(t)$ is of the order of square of $s(t)$.

Theorem 3.8. *Let X be such that $s(t) > 0$ for all $t \in (0,2)$. Then $\delta(t) \geq \frac{1}{2}s(s(4t/5))$. In particular X a uniformly convex.*

Proof. If $s(t) > 0$ for all $t \in (0,2)$ then X is super reflexive and hence there exists a dense set $D \subseteq S_{X^*}$ such that every $f \in D$ is a Fréchet smooth point. Let $f \in D$ and choose $x \in \mathcal{D}(f)$. Since $s(x,t) > s(t) > 0$, we have x is a strongly exposed point of B_X. By Proposition 3.7 we have $U_I^*(f,t) \geq \frac{1}{2}s(4t/5)$. Thus $\inf_{f\in D} U_I^*(f,t) \geq \frac{1}{2}s(4t/5)$. Since D is dense in S_{X^*}, by Theorem 2.8 we have $U_I^*(t) > \frac{1}{2}s(4t/5)$ for all t. But by Theorem 2.6 $U_I(t) = U_I^*(t)$ and thus applying Theorem 3.5(b), we get $\delta(t) \geq \frac{1}{2}s(s(4t/5))$. □

Acknowledgements. We would like the thank the referee for a careful reading of the earlier version of the manuscript and suggesting necessary changes and corrections which helped us to prepare the current version.

References

[1] J. Gao: Normal Structure and modulus of U-convexity in Banach spaces, in: Function Spaces, Differential Operators and Nonlinear Analysis (Prague, 1995), J. Rákosník (ed.), Prometheus, Prague (1996) 195–199, MR1480939 (99m:46033).

[2] J. Gao, K.-S. Lau: On two classes of Banach spaces with uniform normal structure, Stud. Math. 99(1) (1991) 41–56, MR1120738 (92h:46017).

[3] B.-L. Lin, P.-K. Lin, S. Troyanski: Characterizations of denting points, Proc. Amer. Math. Soc. 102(3) (1988) 526–528, MR928972 (89e:46016).

[4] K.-S. Lau: Best approximation by closed sets in Banach spaces, J. Approx. Theory 23 (1978) 29–36, MR493114 (58#12150).

[5] G. Lancien: A survey on the Szlenk index and some of its applications, RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 100(1–2) (2006) 209–235, MR2267410 (2008b:46022).

[6] S. Saejung: On the modulus of U-convexity, Abstr. Appl. sadsadsadsad Anal. 2005(1) (2005) 59–66, MR2142156 (2005m:46030).

[7] M. A. Smith: Rotundity and smoothness in conjugate spaces, Proc. Amer. Math. Soc. 61(2) (1976) 232–234, MR435807 (55#8763).