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Locally Convex Quotient Cones

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We investigate suitable equivalence relations on locally convex cones which give rise to the definition of a locally convex quotient cone. In the special case of a locally convex vector space, this reduces to the known concept of quotient spaces. We recover most aspects of this case for our more general setting and provide a range of examples and applications for this approach, most of them new.

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1. Introduction

Endowed with suitable topologies, vector spaces yield rich and well-studied structures. They permit an extensive duality theory whose study gives valuable insight into the spaces themselves. However, some important mathematical settings, while close to the structure of vector spaces do not allow subtraction of their elements or multiplication by negative scalars. Examples are certain classes of functions that may take infinite values or are characterized through inequalities rather than equalities. They arise naturally in integration and in potential theory. Likewise, families of convex subsets of vector spaces which are of interest in various contexts, do not form vector spaces. If the cancellation law fails, domains of this type can not be embedded into larger vector spaces in order to apply results and techniques from classical functional analysis. The theory of locally convex cones, as developed in [3] and [4], uses order theoretical concepts to introduce a topological structure on ordered cones. In Section 2 of this paper we shall review some of the main concepts of this approach. In Section 3 we introduce convex equivalent relations, locally convex quotient cones, and some of their main properties. Section 4 contains an application to boundedness components.

2. Locally Convex Cones

A *cone* is a set $\mathcal{P}$ endowed with an addition $(a,b) \mapsto a+b$ and a scalar multiplication $(\alpha, a) \mapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. The

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cancellation law, stating that $a + c = b + c$ implies $a = b$, is not required in general. It holds if and only if the cone $\mathcal{P}$ can be embedded into a real vector space.

An *ordered cone* $\mathcal{P}$ carries a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. Anti-symmetry is however not required. Equality on $\mathcal{P}$ is obviously such an order.

The theory of locally convex cones as developed in [3] uses order theoretical concepts to introduce a quasiuniform topological structure on an ordered cone. In a first approach, the resulting topological neighborhoods themselves will be considered to be elements of the cone. In this vein, a *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, that is a subset of positive elements which is directed downward, closed for addition and multiplication by scalars $\alpha > 0$. The elements v of $\mathcal{V}$ define *upper* resp. *lower neighborhoods* for the elements of $\mathcal{P}$ by

$$v(a) = \{\, b \in \mathcal{P} \mid b \leq a + v \,\} \quad \text{resp. } (a)v = \{\, b \in \mathcal{P} \mid a \leq b + v \,\},$$

creating the *upper* resp. *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology* generated by the neighborhoods $v^s(a) = v(a) \cap (a)v$. All elements of $\mathcal{P}$ are supposed to be *bounded below*, that is for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda \geq 0$. They need however not be bounded above. An element $a \in \mathcal{P}$ is called *bounded (above)* if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $a \leq \lambda v$. The presence of unbounded elements represents the main difference between locally convex cones and locally convex vector spaces and accounts for much of the richness and subtlety of this setting.

Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the abstract neighborhood system $\mathcal{V}$. Every locally convex ordered topological vector space is a locally convex cone in this sense, as it may be canonically embedded into a full locally convex cone (see Examples 2.1(c) below and I.2.7 in [3]). The subsets $\{(a, b) \mid a, b \in \mathcal{P} \; a \leq b + v\}$ of $\mathcal{P}^2$, for all $v \in \mathcal{V}$ form a *convex quasiuniform structure* on $\mathcal{P}$. Conversely, it is shown in Chapter I.5.2 of [3] how a convex quasiuniform structure on a cone can be used to construct a full locally convex cone which contains the given one as a subcone and induces the given quasiuniform structure. This yields a second, equivalent approach to locally convex cones. We shall recall a few examples. Many more can be found in [4].

Examples 2.1. (a) On the extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we consider the usual order and algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. Endowed with the neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point. It is finer than the usual topology of $\overline{\mathbb{R}}$ where the intervals $[a, +\infty]$ for $a \in \mathbb{R}$ are the neighborhoods of $+\infty$.

(b) For the subcone $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} \mid a \geq 0\}$ of $\overline{\mathbb{R}}$ we may also consider the singleton neighborhood system $\mathcal{V} = \{0\}$. The elements of $\overline{\mathbb{R}}_+$ are obviously bounded below even with respect to the neighborhood $v = 0$, hence $\overline{\mathbb{R}}_+$ is a full locally convex cone. For $a \in \overline{\mathbb{R}}$ the intervals $(-\infty, a]$ and $[a, +\infty]$ are the only upper and lower neighborhoods, respectively. The symmetric topology is the discrete topology on $\overline{\mathbb{R}}_+$.

(c) Let $(E, \mathcal{V}, \leq)$ be a locally convex ordered topological vector space, where $\mathcal{V}$ is a basis of closed, convex, balanced and order convex neighborhoods of the origin in E. Recall that equality is an order relation, hence this example will cover locally convex spaces in general. In order to interpret E as a locally convex cone we shall embed it into a larger full cone. This is done in a canonical way: Let $\mathcal{P}$ be the cone of all non-empty convex subsets of E, endowed with the usual addition and multiplication of sets by non-negative scalars, that is $\alpha A = \{\alpha a \mid a \in A\}$ and $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$ for $A, B \in \mathcal{P}$ and $\alpha \geq 0$. We define the order on $\mathcal{P}$ by

$$A \leq B \quad \text{if } A \subset B + E_-,$$

where $E_- = \{x \in E \mid x \leq 0\}$ is the negative cone in E. The requirements for an ordered cone are easily checked. The neighborhood system in $\mathcal{P}$ is given by the neighborhood basis $\mathcal{V} \subset \mathcal{P}$. We observe that for every $A \in \mathcal{P}$ and $V \in \mathcal{V}$ there is $\rho > 0$ such that $\rho V \cap A \neq \emptyset$. This yields $0 \in A + \rho V$. Therefore $\{0\} \leq A + \rho V$, and every element $A \in \mathcal{P}$ is indeed bounded below. Thus $(\mathcal{P}, \mathcal{V})$ is a full locally convex cone. Via the embedding $x \mapsto \{x\} : E \to \mathcal{P}$ the space E itself is a subcone of $\mathcal{P}$. This embedding preserves the order structure of E, and on its image the symmetric topology of $\mathcal{P}$ coincides with the given vector space topology of E. Thus E is indeed a locally convex cone, but not a full cone. Other subcones of $\mathcal{P}$ that merit further investigation are those of all closed, closed and bounded, or compact convex sets in $\mathcal{P}$, respectively. Details on these and further related examples may be found in [3] and [4].

(d) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, X a set and let $\mathcal{F}(X, \mathcal{P})$ be the cone of all $\mathcal{P}$-valued functions on X, endowed with the pointwise operations and order. If $\bar{\mathcal{P}}$ is a full cone containing both $\mathcal{P}$ and $\mathcal{V}$, then we may identify the elements $v \in \mathcal{V}$ with the constant functions $x \mapsto v$ for all $x \in X$, hence $\mathcal{V}$ is a subset and a neighborhood system for $\mathcal{F}(X, \bar{\mathcal{P}})$. A function $f \in \mathcal{F}(X, \bar{\mathcal{P}})$ is uniformly bounded below, if for every $v \in \mathcal{V}$ there is $\rho \geq 0$ such that $0 \leq f + \rho v$. These functions form a full locally convex cone $\big(\mathcal{F}_b(X, \bar{\mathcal{P}}), \mathcal{V}\big)$, carrying the topology of uniform convergence. As a subcone, $\big(\mathcal{F}_b(X, \mathcal{P}), \mathcal{V}\big)$ is a locally convex cone. Alternatively, a more general neighborhood system $\mathcal{V}_{\mathcal{Y}}$ for $\mathcal{F}(X, \mathcal{P})$ may be created using a suitable family $\mathcal{Y}$ of subsets Y of X, directed downward with respect to set inclusion, and the neighborhoods v_Y for $v \in \mathcal{V}$ and $Y \in \mathcal{Y}$, defined for functions $f, g \in \mathcal{F}(X, \mathcal{P})$ as $f \leq g + v_Y$ if $f(x) \leq g(x) + v$ for all $x \in Y$. In this case we consider the subcone $\mathcal{F}_{b_{\mathcal{Y}}}(X, \mathcal{P})$ of all functions in $\mathcal{F}(X, \mathcal{P})$ that are uniformly bounded below on the sets in $\mathcal{Y}$. Together with the neighborhood system $\mathcal{V}_{\mathcal{Y}}$, it forms a locally convex cone. $\big(\mathcal{F}_{b_{\mathcal{Y}}}(X, \mathcal{P}), \mathcal{V}_{\mathcal{Y}}\big)$ carries the topology of uniform convergence on the sets in $\mathcal{Y}$.

(e) For $x \in \overline{\mathbb{R}}$ denote $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$. For $1 \leq p \leq +\infty$ and a sequence $(x_i)_{i \in \mathbb{N}}$ in $\overline{\mathbb{R}}$ let $\|(x_i)\|_p$ denote the usual l^p norm, that is $\|(x_i)\|_p = \big(\sum_{i=1}^{\infty} |x_i|^p\big)^{(1/p)} \in \overline{\mathbb{R}}$ for $p < +\infty$ and $\|(x_i)\|_\infty = \sup\{|x_i| \mid i \in \mathbb{N}\} \in \overline{\mathbb{R}}$. Now let $\mathcal{C}^p$ be the cone of all sequences $(x_i)_{i \in \mathbb{N}}$ in $\overline{\mathbb{R}}$ such that $\|(x_i^-)\|_p \leq +\infty$. We use the pointwise order in $\mathcal{C}^p$ and the neighborhood system $\mathcal{V}_p = \{\rho v_p \mid \rho > 0\}$, where

$$(x_i)_{i\in\mathbb{N}} \le (y_i)_{i\in\mathbb{N}} + \rho v_p$$

means that $\|(x_i - y_i)^+\|_p \le \rho$. (In this expression the l^p norm is evaluated only over the indexes $i \in \mathbb{N}$ for which $y_i < +\infty$.) It can be easily verified that $(\mathcal{C}^p, \mathcal{V}_p)$ is a locally convex cone. In fact $(\mathcal{C}^p, \mathcal{V}_p)$ can be embedded into a full cone following a procedure analogous to that in 2.1(c). The case for $p = +\infty$ is of course already covered by Part (d).

For cones $\mathcal{P}$ and $\mathcal{Q}$ a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a+b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ holds for all $a, b \in \mathcal{P}$ and $\alpha \ge 0$. If both $\mathcal{P}$ and $\mathcal{Q}$ are ordered, then T is called *monotone,* if $a \le b$ implies $T(a) \le T(b)$. If both $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $w \in \mathcal{W}$ one can find $v \in \mathcal{V}$ such that $T(a) \le T(b) + w$ whenever $a \le b + v$ for $a, b \in \mathcal{P}$. It is immediate from the definition that uniform continuity implies and combines continuity for the operator $T : \mathcal{P} \to \mathcal{Q}$ with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$, respectively. Continuous operators are monotone, even though in a slightly modified sense (Lemma II.1.4 in [3]).

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathcal{V})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all *polars* v° of neighborhoods $v \in \mathcal{V}$, where $\mu \in v^\circ$ means that $\mu(a) \le \mu(b) + 1$ whenever $a \le b + v$ for $a, b \in \mathcal{P}$. Continuity implies that a linear functional μ is monotone, and for a full cone $\mathcal{P}$ it requires just that $\mu(v) \le 1$ holds for some $v \in \mathcal{V}$ in addition. We endow $\mathcal{P}^*$ with the topology $w(\mathcal{P}^*, \mathcal{P})$ of pointwise convergence on the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$ with its usual topology. As in locally convex topological vector spaces, the polar v° of a neighborhood $v \in \mathcal{V}$ is seen to be $w(\mathcal{P}^*, \mathcal{P})$-compact and convex ([3], Theorem II.2.4). A variety of Hahn-Banach type extension and separation theorems for linear functionals is available and can be found in [3], [5] and [4]. These are essential for the development of a powerful duality theory for locally convex cones.

Examples 2.2. Revisiting the preceding Examples 2.1, we observe that the dual cone $\overline{\mathbb{R}}^*$ of $\overline{\mathbb{R}}$ (see 2.1(a)) consists of all positive reals (via the usual multiplication), and the singular functional $\bar{0}$ such that $\bar{0}(a) = 0$ for all $a \in \mathbb{R}$ and $\bar{0}(+\infty) = +\infty$. Likewise, in 2.1(b), the continuous linear functionals on $\overline{\mathbb{R}}_+$, endowed with the neighborhood system $\mathcal{V} = \{0\}$, are the positive reals together with $\bar{0}$, but further include the element $+\infty$, acting as $+\infty(0) = 0$ and $+\infty(a) = +\infty$ for all $0 \ne a \in \overline{\mathbb{R}}_+$. This functional is obviously contained in the polar of the neighborhood $0 \in \mathcal{V}$. In 2.1(c) and (d) on the other hand, due to the generality of the settings, a complete description for the respective dual cones is not immediately available. We may, however, identify some of their elements: In 2.1(c), let μ be a continuous monotone linear function on the locally convex ordered topological vector space $(E, \le)$. Then the mapping

$$A \mapsto \sup\{\mu(a) \mid a \in A\} : \mathrm{Conv}(E) \to \overline{\mathbb{R}}$$

is seen to be an element of $\mathrm{Conv}(E)^*$. In 2.1(d), if $\mu \in \mathcal{P}^*$ and if $x \in Y$ for some $Y \in \mathcal{Y}$, then the mapping $\mu_x : \mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P}) \to \overline{\mathbb{R}}$ such that

$$\mu_x(f) = \mu\big(f(x)\big) \text{ for all } f \in \mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P})$$

is a continuous linear functional on $\mathcal{F}_{b_\mathcal{Y}}(X,\mathcal{P})$; more precisely: If $\mu \in v^\circ$ for $v \in \mathcal{V}$ and $x \in Y$ for $Y \in \mathcal{Y}$, then $\mu_x \in v_Y^\circ$. In 2.1(e) for $p < +\infty$ the dual cone of $\mathcal{C}^p$ consists of all sequences $(y_i)_{i\in\mathbb{N}}$ such that $y_i \geq 0$ for all $i \in \mathbb{N}$ and $\|(y_i)\|_q < +\infty$, where q is the conjugate index of p.

We also consider a (topological and linear) closure of the given order on a locally convex cone, called the weak preorder $\preceq$ which is defined as follows (see I.3 in [4]): We set $a \preceq b+v$ for $a,b \in \mathcal{P}$ and $v \in \mathcal{V}$ if for every $\varepsilon > 0$ there is $1 \leq \gamma \leq 1+\varepsilon$ such that $a \leq \gamma b + (1+\varepsilon)v$, and set $a \preceq b$ if $a \preceq b+v$ for all $v \in \mathcal{V}$. This order is clearly weaker than the given order, that is $a \leq b$ or $a \leq b+v$ implies $a \preceq b$ or $a \preceq b+v$. Importantly, the weak preorder on a locally convex cone is entirely determined by its dual cone $\mathcal{P}^*$, that is $a \preceq b$ holds if and only if $\mu(a) \leq \mu(b)$ for all $\mu \in \mathcal{P}^*$, and $a \preceq b+v$ if and only $\mu(a) \leq \mu(b)+1$ for all v° (Corollaries I.4.31 and I.4.34 in [4]).

Corresponding to the weak preorder, the *upper, lower and symmetric relative topologies* on a locally convex cone $(\mathcal{P},\mathcal{V})$ are generated by the neighborhoods $v_\varepsilon(a)$, $(a)v_\varepsilon$ and $v_\varepsilon^s(a) = v_\varepsilon(a) \cap (a)_\varepsilon$, respectively, for $a \in \mathcal{P}$, $v \in \mathcal{V}$ and $\varepsilon > 0$, where

$$v_\varepsilon(a) = \{b \in \mathcal{P} \mid b \leq \gamma a + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1+\varepsilon\},$$
$$(a)v_\varepsilon = \{b \in \mathcal{P} \mid a \leq \gamma b + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1+\varepsilon\}.$$

The relative topologies are generally coarser than the given upper, lower and symmetric topologies, but locally coincide with them at bounded elements of $\mathcal{P}$ (Proposition I.4.2(iv) in [4]). The symmetric relative topology is known to be Hausdorff if and only if the weak preorder on $\mathcal{P}$ is antisymmetric (Proposition I.4.8 in [4]).

3. Locally Convex Quotient Cones

We consider an equivalence relation $\sim$ on a locally convex cone $(\mathcal{P},\mathcal{V})$ which is compatible with the algebraic operations in $\mathcal{P}$, that is $a+c \sim b+c$ and $\alpha a \sim \alpha b$ whenever $a \sim b$ for $a,b,c \in \mathcal{P}$ and $\alpha \geq 0$. By $\tilde{a}$ we denote the equivalence class of an element $a \in \mathcal{P}$. Since $a \sim a'$ and $b \sim b'$ implies that $a+b \sim a'+b$ and $b+a' \sim b'+a'$, thus $a+b \sim a'+b'$, the operations $\tilde{a}+\tilde{b} = \widetilde{a+b}$ and $\alpha\tilde{a} = \widetilde{\alpha a}$ are well-defined for $a,b \in \mathcal{P}$ and $\alpha \geq 0$, and

$$\widetilde{\mathcal{P}} = \{\tilde{a} \mid a \in \mathcal{P}\}$$

becomes a cone with these operations. It is fairly obvious how to assign a suitable order $\lesssim$ and a locally convex cone topology to $\widetilde{\mathcal{P}}$. The order $\lesssim$ is the strongest order on $\widetilde{\mathcal{P}}$ which is compatible with the algebraic operations of $\widetilde{\mathcal{P}}$ and guarantees that the canonical projection $\Pi : \mathcal{P} \to \widetilde{\mathcal{P}}$, that is $\Pi(a) = \tilde{a}$ for $a \in \mathcal{P}$, is monotone. The weakest order relating all elements of $\widetilde{\mathcal{P}}$ obviously satisfies this requirement, and the intersection over any family of such order relations is again of this type. Therefore such a strongest order exists on $\widetilde{\mathcal{P}}$. Likewise, there is a finest convex quasiuniform structure on $\widetilde{\mathcal{P}}$ which guarantees that the projection $\Pi : \mathcal{P} \to \widetilde{\mathcal{P}}$ is a continuous linear operator. This can be expressed explicitly, although not very elegantly as follows:

Definition 3.1. For $a,b \in \mathcal{P}$ and $v \in \mathcal{V} \cup \{0\}$ we set

$$\tilde{a} \lesssim \tilde{b} + \tilde{v}$$

if there are $c_1, d_1, \ldots, c_n, d_n \in \mathcal{P}$ and $0 \leq \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\lambda_1 + \ldots + \lambda_n \leq 1$ and $c_1 \sim a$, $d_n \sim b$, as well as $c_i \leq d_i + \lambda_i v$ for all $i = 1, \ldots, n$, and $d_i \sim c_{i+1}$ for all $i = 1, \ldots, n-1$.

As required, this order is reflexive, transitive and compatible with the algebraic operations. The subsets $\{(\tilde{a}, \tilde{b}) \mid \tilde{a} \lesssim \tilde{b} + \tilde{v}\} \subset \widetilde{\mathcal{P}}^2$ for all $v \in \mathcal{V}$ describe a convex quasiuniform structure on $\widetilde{\mathcal{P}}$. Then according to I.5.4 in [3] there exists a full cone $\widetilde{\mathcal{P}} \oplus \widetilde{\mathcal{V}}$ whose neighborhoods yield the same quasiuniform structure on $\widetilde{\mathcal{P}}$. The neighborhoods $\tilde{v} \in \widetilde{\mathcal{V}}$ from above form a basis for $\widetilde{\mathcal{V}}$ in the following sense: For every $w \in \widetilde{\mathcal{V}}$ there is $v \in \mathcal{V}$ such that $\tilde{a} \lesssim \tilde{b} + \tilde{v}$ for $\tilde{a}, \tilde{b} \in \widetilde{\mathcal{P}}$ implies that $\tilde{a} \lesssim \tilde{b} + w$. The locally convex cone $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{V}})$ is called *the locally convex quotient cone of* $(\mathcal{P}, \mathcal{V})$ *over* $\sim$.

Proposition 3.2. *Let* $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{V}})$ *be the locally convex quotient cone of* $(\mathcal{P}, \mathcal{V})$ *over* $\sim$ *. The projection* $\Pi : \mathcal{P} \to \widetilde{\mathcal{P}}$ *is a monotone and continuous linear operator.*

Proposition 3.3. *Let* $(\mathcal{P}, \mathcal{V})$ *and* $(\mathcal{Q}, \mathcal{W})$ *be locally convex cones and* $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{V}})$ *the quotient of* $(\mathcal{P}, \mathcal{V})$ *over* $\sim$ *. If* $T : \mathcal{P} \to \mathcal{Q}$ *is a continuous linear operator such that* $T(a) = T(b)$ *whenever* $a \sim b$*, then the operator* $\widetilde{T} : \widetilde{\mathcal{P}} \to \mathcal{Q}$ *such that* $\widetilde{T}(\tilde{a}) = T(a)$ *is also linear and continuous.*

Proof. The continuity of the operator $\Pi : \mathcal{P} \to \widetilde{\mathcal{P}}$ is obvious, since $a \leq b + v$ implies $\tilde{a} \lesssim \tilde{b} + \tilde{v}$. For 3.3 let $T : \mathcal{P} \to \mathcal{Q}$ be a continuous linear operator such that $T(a) = T(b)$ whenever $a \sim b$ for $a, b \in \mathcal{P}$. Given $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $a \leq b + v$ implies $T(a) \leq T(b) + w$. Let $\tilde{a}, \tilde{b} \in \widetilde{\mathcal{P}}$ such that $\tilde{a} \leq \tilde{b} + \tilde{v}$, and let $c_1, d_1, \ldots, c_n, d_n \in \mathcal{P}$ and $0 \leq \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be as in the definition of the neighborhood $\tilde{v}$. Then $T(c_1) = T(a)$, $T(d_n) = T(b)$, as well as $T(d_i) = T(c_{i+1})$ for all $i = 1, \ldots, n-1$, and $T(c_i) \leq T(d_i) + \lambda_i w$ for all $i = 1, \ldots, n$. This yields $T(a) \leq T(b) + \sum_{i=1}^{n-1} \lambda_i w \leq T(b) + w$, hence $\widetilde{T}(\tilde{a}) \leq \widetilde{T}(\tilde{b}) + w$ as well. □

Corollary 3.4. *Let* $(\mathcal{P}, \mathcal{V})$ *and* $(\mathcal{Q}, \mathcal{W})$ *be locally convex cones and let* $T : \mathcal{P} \to \mathcal{Q}$ *be a continuous linear operator. If an equivalence relation is defined on* $\mathcal{P}$ *by* $a \sim b$ *if* $T(a) = T(b)$*, then the operator* $\widetilde{T} : \widetilde{\mathcal{P}} \to \mathcal{Q}$ *such that* $\widetilde{T}(\tilde{a}) = T(a)$ *is one-to-one, linear and continuous.*

Let $\mathcal{P}^*_\sim$ denote the subcone of $\mathcal{P}^*$ consisting of all linear functionals $\mu \in \mathcal{P}^*$ such $\mu(a) = \mu(b)$ whenever $a \sim b$ for $a, b \in \mathcal{P}$. According to Proposition 3.3 then every $\mu \in \mathcal{P}^*_\sim$ corresponds to a linear functional $\tilde{\mu} \in \widetilde{\mathcal{P}}^*$. Conversely, for every $\tilde{\mu} \in \widetilde{\mathcal{P}}^*$, the functional $\tilde{\mu} \circ \Pi : \mathcal{P} \to \overline{\mathbb{R}}$ is contained in $\mathcal{P}^*_\sim$. This yields:

Proposition 3.5. *Let* $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{V}})$ *be the locally convex quotient cone of* $(\mathcal{P}, \mathcal{V})$ *over* $\sim$ *. The dual cone* $\widetilde{\mathcal{P}}^*$ *of* $\widetilde{\mathcal{P}}$ *can be identified with* $\mathcal{P}^*_\sim$*. The polar* $\tilde{v}^\circ$ *of a neighborhood* $\tilde{v} \in \widetilde{\mathcal{V}}$ *is given by* $v^\circ \cap \mathcal{P}^*_\sim$*.*

Let us denote the weak preorder on $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{V}})$ by $\precsim$. Then $\tilde{a} \precsim \tilde{b} + \tilde{v}$ holds for $\tilde{a}, \tilde{b} \in \widetilde{\mathcal{P}}$ and $\tilde{v} \in \widetilde{\mathcal{V}}$ if and only if for every $\varepsilon > 0$ there is $1 \leq \gamma \leq 1 + \varepsilon$ such that $\tilde{a} \lesssim \gamma \tilde{b} + (1 + \varepsilon)\tilde{v}$, and $a \precsim b$ if $\tilde{a} \precsim \tilde{b} + \tilde{v}$ for all $\tilde{v} \in \widetilde{\mathcal{V}}$. According to Corollaries I.4.43 and I.4.31 in [4] and Proposition 3.5 this is equivalent to $\mu(a) \leq \mu(b) + 1$ for all $\mu \in v^\circ \cap \mathcal{P}^*_\sim$, and $\mu(a) \leq \mu(b)$ for all $\mu \in \mathcal{P}^*_\sim$, respectively.

If Ω is a subset of the dual cone $\mathcal{P}^*$ of $\mathcal{P}$, then $a \overset{\Omega}{\sim} b$ if $\mu(a) = \mu(b)$ for all $\mu \in \Omega$ defines a compatible equivalence relation on $\mathcal{P}$. According to Proposition 3.5, Ω is contained in the dual $\widetilde{\mathcal{P}}^*$ of the quotient cone $\widetilde{\mathcal{P}}$ in this case.

Proposition 3.6. *Let $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{V}})$ be the locally convex quotient cone of $(\mathcal{P}, \mathcal{V})$ over $\sim$, and let $\Omega = \mathcal{P}^*_\sim \subset \mathcal{P}^*$. The symmetric relative topology is Hausdorff for $\widetilde{\mathcal{P}}$ if and only if the equivalence relations $\sim$ and $\overset{\Omega}{\sim}$ coincide on $\mathcal{P}$.*

Proof. According to Proposition I.4.8 in [4] the symmetric relative topology is Hausdorff if and only if the weak preorder is antisymmetric. On the other hand we have $\tilde{a} \precsim \tilde{b}$ for $\tilde{a}, \tilde{b} \in \widetilde{\mathcal{P}}$ if and only if $\mu(a) \leq \mu(b)$ for all $\mu \in \Omega = \mathcal{P}^*_\sim$. Thus $a \overset{\Omega}{\sim} b$ is equivalent to $\tilde{a} \precsim \tilde{b}$ and $\tilde{b} \precsim \tilde{a}$. The latter therefore yields $\tilde{a} = \tilde{b}$ if and only if the equivalence relations $\sim$ and $\overset{\Omega}{\sim}$ coincide on $\mathcal{P}$. □

Remarks and Examples 3.7. (a) If the order relation $\leq$ is indeed the equality for elements of $\mathcal{P}$, then a brief inspection of Definition 3.1 yields that $\lesssim$ is the equality for elements of $\widetilde{\mathcal{P}}$. Relations involving neighborhoods can however not necessarily be simplified, even in this case.

(b) If $\sim$ is defined on $\mathcal{P}$ as $a \sim b$ for $a, b \in \mathcal{P}$ if $a \leq b$ and $b \leq a$, then $\widetilde{\mathcal{P}}$ is the *antisymmetric reflection* of $\mathcal{P}$, that is $\tilde{a} \lesssim \tilde{b}$ if $a \leq b$, and the order $\lesssim$ is antisymmetric on $\widetilde{\mathcal{P}}$.

(c) If $\sim$ is defined on $\mathcal{P}$ as $a \sim b$ for $a, b \in \mathcal{P}$ if $a \leq b + v$ and $b \leq a + v$ for all $v \in \mathcal{V}$, then $\widetilde{\mathcal{P}}$ is the *separate reflection* of $\mathcal{P}$. Indeed, $\tilde{a} \leq \tilde{b} + \tilde{v}$ and $\tilde{b} \leq \tilde{a} + \tilde{v}$ for $\tilde{a}, \tilde{b} \in \widetilde{\mathcal{P}}$ and all $\tilde{v} \in \widetilde{\mathcal{V}}$ implies that $a \leq b + v$ and $b \leq a + v$ for all $v \in \mathcal{V}$, hence $\tilde{a} = \tilde{b}$. Thus for $\tilde{a} \neq \tilde{b}$ there is $\tilde{v} \in \widetilde{\mathcal{V}}$ such that $\tilde{b}$ is not contained in the symmetric neighborhood $\tilde{v}^s(\tilde{a})$. With $\tilde{u} = (1/2)\tilde{v} \in \widetilde{\mathcal{V}}$ then $\tilde{u}^s(\tilde{a}) \cap \tilde{u}^s(\tilde{b}) = \emptyset$, and the symmetric topology of $\widetilde{\mathcal{P}}$ is seen to be Hausdorff. A similar construction can be carried out using the weak preorder $\preceq$ and the corresponding relative topologies as elaborated in I.3 of [4].

(d) If $(\mathcal{P}, \mathcal{V})$ is indeed a locally convex ordered topological vector space (see Example 2.1(c)), then $M = \{m \in \mathcal{P} \mid m \sim 0\}$ is a subspace of $\mathcal{P}$, and $a \sim b$ holds for $a, b \in \mathcal{P}$ if and only if $a - b \in M$. Thus $\widetilde{\mathcal{P}}$ is the usual quotient space $\mathcal{P}/M$ (see I.2 in [8]). We have $\tilde{a} \lesssim \tilde{b}$ for $\tilde{a}, \tilde{b} \in \mathcal{P}/M$ if $a \leq b + m$ for some $m \in M$, and $\tilde{a} \lesssim \tilde{b} + V$ for $V \in \mathcal{V}$ if $a \leq b + m + v$ for some $m \in M$ and $v \in V$.

(e) As in 2.1(c) let $\mathcal{P}$ be the cone of all non-empty convex subsets of a locally convex ordered topological vector space $(E, \mathcal{V}, \leq)$ endowed with the canonical order and neighborhood system from 2.1(c) and consider the following equivalence relation: We set $A \sim B$ for $A, B \in \mathcal{P}$ if $\bar{A} = \bar{B}$, that is if the topological closures of A and B coincide. Then $\tilde{A} \lesssim \tilde{B}$ if $A \subset \bar{B} + E_-$ and $\tilde{A} \lesssim \tilde{B} + \tilde{V}$ if $A \subset B + V + E_-$ for $\tilde{A}, \tilde{B} \in \widetilde{\mathcal{P}}$ and $V \in \mathcal{V}$. Thus $\widetilde{\mathcal{P}}$ corresponds to the cone of all non-empty closed convex subsets of E endowed with the given scalar multiplication, the modified addition $A \oplus B = \overline{A + B}$ for closed convex subsets E, and the set inclusion as order.

(f) Let X be a set and let $\mathcal{Y}$ be a family of subsets of X, directed downward with respect to set inclusion. Let $\mathcal{P} = \mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})$ be the cone of all $\overline{\mathbb{R}}$-valued functions on X that are bounded below on the sets in $\mathcal{Y}$ (see Example 2.1(d)). The neighborhood system $\mathcal{V}_\mathcal{Y}$ is generated by the neighborhoods ε_Y for $\varepsilon > 0$ and $Y \in \mathcal{Y}$ such that $f \leq g + \varepsilon_Y$ for $f, g \in \mathcal{P}$ if $f(y) \leq g(y) + \varepsilon$ for all $y \in Y$. For $f \in \mathcal{P}$ we set $I_f = \{x \in X \mid f(x) = +\infty\}$ and define a compatible equivalence relation on $\mathcal{P}$ by $f \sim g$ for $f, g \in \mathcal{P}$ if $I_f = I_g$. The order on the quotient cone $\widetilde{\mathcal{P}}$ is given by $\tilde{f} \lesssim \tilde{g}$ if $I_f \subset I_g$ for $\tilde{f}, \tilde{g} \in \widetilde{\mathcal{P}}$ and $\tilde{f} \lesssim \tilde{g} + \tilde{\varepsilon}_Y$ for $\varepsilon_Y \in \mathcal{V}_\mathcal{Y}$ if $I_f \cap Y \subset I_g$. We have $I_{f+g} = I_f \cup I_g$ and $I_{\alpha f} = I_f$ for $\alpha > 0$. Thus $\widetilde{\mathcal{P}}$ corresponds to $\mathfrak{P}(X)$, the family of all subsets of X, endowed with the operations $A \oplus B = A \cup B$ and $\alpha A = A$ for $A, B \in \mathfrak{P}$ and $\alpha > 0$, and $0 \cdot A = \emptyset$. The order on $\mathfrak{P}$ is the set inclusion, and the neighborhoods are given by $A \leq B + \varepsilon_Y$ for $\varepsilon_Y \in \mathcal{V}_\mathcal{Y}$ if $A \cap Y \subset B$.

4. Factorization over Boundedness Components

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. We define an equivalence relation $\sim$ on $\mathcal{P}$ by $a \sim b$ for $a, b \in \mathcal{P}$ if for every neighborhood $v \in \mathcal{V}$ there are constants $\alpha, \beta, \rho > 0$ such that $a \leq \beta b + \rho v$ and $b \leq \alpha a + \rho v$. This relation is compatible with the algebraic operations of $\mathcal{P}$. Indeed, suppose that $a \sim b$. This obviously implies that $\delta a \sim \gamma b$ for all $\delta, \gamma > 0$. Moreover, for $c \in \mathcal{P}$ and $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$, $0 \leq b + \lambda v$ and $0 \leq c + \lambda v$. Now for $a \leq \beta b + \rho v$ set $\gamma = \max\{\beta, 1\}$. Then

$$\begin{aligned} a + c \leq \beta b + c + \rho v &\leq \beta b + c + \rho v + (\gamma - \beta)(b + \lambda v) + (\gamma - 1)(c + \lambda v) \\ &= \gamma(b + c) + \delta v, \end{aligned}$$

where $\delta = \rho + \lambda(2\gamma - \beta - 1)$. Similarly, for $b \leq \alpha a + \rho v$ and $\gamma = \max\{\alpha, 1\}$ one argues that $b + c \leq \gamma(a + c) + \delta v$. Hence $a + c \sim b + c$. The equivalence classes $\tilde{a}$ of this relation are called the *boundedness components* of $\mathcal{P}$ and are in the sequel denoted as $\mathcal{B}^s(a)$. We shall list some of the properties of boundedness components which can be found in Section I.4 of [4].

Proposition 4.1. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, and let $a, b, c \in \mathcal{P}$.*

(*a*) *If $b, c \in \mathcal{B}^s(a)$ and $\alpha > 0$, then $\alpha b, b + c \in \mathcal{B}^s(a)$.*

(*b*) *$b \in \mathcal{B}^s(a)$ if and only if for every $\mu \in \mathcal{P}^*$ we have $\mu(a) + \infty$ if and only if $\mu(b) = +\infty$.*

We shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ has *uniform boundedness components* (see 4.23 in [4]) if there is a particular neighborhood $v_0 \in \mathcal{V}$ such that the neighborhood subsystem $\mathcal{V}_0 = \{\lambda v_0 \mid \lambda > 0\}$ generates the same boundedness components in $\mathcal{P}$ as the full system $\mathcal{V}$. Ordered locally convex topological vector spaces are obviously of this type since they have only a single boundedness component. We cite the following from Section I.4 of [4]. (For notions concerning connectedness see Ch. 8 in [9].)

Proposition 4.2. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. With respect to the symmetric relative topology on $\mathcal{P}$ the following hold:*

(*a*) *The boundedness components are connected and coincide with the connectedness components of* $\mathcal{P}$.
(*b*) *The boundedness components are closed in* $\mathcal{P}$.
(*c*) *If* $\mathcal{P}$ *has uniform boundedness components, then* $\mathcal{P}$ *is locally connected and the boundedness components are also open in* $\mathcal{P}$.

Now let us consider the quotient cone $\widetilde{\mathcal{P}} = \{\mathcal{B}^s(a) \mid a \in \mathcal{P}\}$ together with its neighborhood system $\widetilde{\mathcal{V}}$ and the quotient order $\lesssim$. According to Definition 3.1 we have $\mathcal{B}^s(a) \lesssim \mathcal{B}^s(b) + \tilde{v}$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V} \cup \{0\}$ if there are $c_1, d_1, \ldots, c_n, d_n \in \mathcal{P}$ and $0 \leq \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\lambda_1 + \ldots + \lambda_n \leq 1$ and $c_1 \sim a$, $d_n \sim b$, as well as $c_i \leq d_i + \lambda_i v$ for all $i = 1, \ldots, n$, and $d_i \sim c_{i+1}$ for all $i = 1, \ldots, n-1$. The induced weak preorder on $\widetilde{\mathcal{P}}$, on the other hand, is given by $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b) + \tilde{v}$ if for every $\varepsilon > 0$ there is $1 \leq \gamma \leq 1 + \varepsilon$ such that $\mathcal{B}^s(a) \lesssim \gamma \mathcal{B}^s(b) + (1+\varepsilon)\tilde{v}$. Thus $\mathcal{B}^s(a) \lesssim \mathcal{B}^s(b) + \tilde{v}$ implies that $a \leq \beta b + \rho v$ for some $\beta, \rho > 0$. Conversely, if $a \leq \beta b + \rho v$ holds for $a, b \in \mathcal{P}$, $v \in \mathcal{V}$ and $\beta, \rho > 0$, then we set $c = (1/\rho)a$ and $d = (\beta/\rho)b$. Thus $a \sim c$, $d \sim b$ and $c \leq d + v$. This yields $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b) + \tilde{v}$. Furthermore, $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b)$ holds if and only if $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b) + \tilde{v}$ for every $v \in \mathcal{V}$. Thus $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b)$ and $\mathcal{B}^s(b) \precsim \mathcal{B}^s(a)$ implies that $a \sim b$, hence $\mathcal{B}^s(a) = \mathcal{B}^s(b)$. The weak preorder is therefore antisymmetric, hence the symmetric relative topology on $\widetilde{\mathcal{P}}$ is Hausdorff. Moreover, if $\mathcal{P}$ has uniform boundedness components generated by the neighborhood $v_0 \in \mathcal{V}$, then $\mathcal{B}^s(b) \in (\widetilde{v}_0)^s_\varepsilon\big(\mathcal{B}^s(a)\big)$ for $a, b \in \mathcal{P}$ and $\varepsilon > 0$ implies that $a \leq \beta b + \rho v_0$ and $b \leq \alpha a + \rho v_0$ for some $\alpha, \beta, \rho > 0$. Therefore b is contained in the boundedness component of a generated by the single neighborhood v_0 which in this case coincides with $\mathcal{B}^s(a)$. Hence $\mathcal{B}^s(b) = \mathcal{B}^s(a)$ and $(\widetilde{v}_0)^s_\varepsilon\big(\mathcal{B}^s(a)\big) \subset \big\{\mathcal{B}^s(a)\big\}$. We summarize:

Lemma 4.3. *Let* $(\mathcal{P}, \mathcal{V})$ *be a locally convex cone,* $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{V}})$ *the quotient cone of its boundedness components. The symmetric relative topology on* $\widetilde{\mathcal{P}}$ *is Hausdorff, and indeed discrete if* $\mathcal{P}$ *has uniform boundedness components. The weak preorder* $\precsim$ *on* $\widetilde{\mathcal{P}}$ *is antisymmetric and given by* $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b) + \tilde{v}$ *for* $a, b \in \mathcal{P}$ *and* $v \in \mathcal{V}$ *if there are* $\beta, \rho > 0$ *such that* $a \leq \beta b + \rho v$.

For the algebraic operations in $\widetilde{\mathcal{P}}$ we observe:

Lemma 4.4. *Let* $(\mathcal{P}, \mathcal{V})$ *be a locally convex cone, and let* $a, b \in \mathcal{P}$.

(*a*) $\alpha \mathcal{B}^s(a) = \mathcal{B}^s(a)$ *for all* $\alpha > 0$.
(*b*) $\mathcal{B}^s(a), \mathcal{B}^s(b) \precsim \mathcal{B}^s(a) + \mathcal{B}^s(b)$.
(*c*) $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b)$ *if and only if* $\mathcal{B}^s(a) + \mathcal{B}^s(b) = \mathcal{B}^s(b)$.

Proof. Part (*a*) follows from Proposition 4.1(*a*), which yields $\alpha a \in \mathcal{B}^s(a)$ for all $\alpha > 0$. Part (*b*) is evident, since for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq b + \lambda v$, hence $a \leq a + (b + \lambda v) = (a+b) + \lambda v$. This yields $\mathcal{B}^s(a) \precsim \mathcal{B}^s(a+b) + \tilde{v}$ for every $v \in \mathcal{V}$, hence $\mathcal{B}^s(a) \precsim \mathcal{B}^s(a+b) = \mathcal{B}^s(a) + \mathcal{B}^s(b)$. For (*c*) suppose that $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b)$. Then for every $v \in \mathcal{V}$ there are $\beta, \rho > 0$ such that $a \leq \beta b + \rho v$. Thus $a + b \leq (\beta+1)b + \rho v$. This yields $\mathcal{B}^s(a+b) \precsim \mathcal{B}^s(b)$. Since $\mathcal{B}^s(b) \precsim \mathcal{B}^s(a+b)$ by Part (*b*), we conclude that $\mathcal{B}^s(a+b) = \mathcal{B}^s(b)$ since $\precsim$ is antisymmetric on $\widetilde{\mathcal{P}}$. Conversely, if $\mathcal{B}^s(a) + \mathcal{B}^s(b) = \mathcal{B}^s(b)$, then for every $v \in \mathcal{V}$ there are $\beta, \rho, \lambda > 0$ such that $a + b \leq \beta b + \rho v$ and $0 \leq b + \lambda v$. Thus $a \leq a + (b + \lambda v) \leq \beta b + (\rho + \lambda)v$. This shows $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b)$ as claimed. □

Lemma 4.4(a) implies in particular that the linear functionals $\tilde{\mu} \in \widetilde{\mathcal{P}}$ can only take the values 0 and $+\infty$. For a linear functional $\mu \in \mathcal{P}^*$ set $\tilde{\mu}\left(\mathcal{B}^s(a)\right) = +\infty$ if $\mu(a) = +\infty$, and $\tilde{\mu}\left(\mathcal{B}^s(a)\right) = 0$ else. Part (b) of Proposition 4.1 implies that on a given boundedness component, the values of μ are either all finite or identically $+\infty$. Thus $\tilde{\mu} \in \widetilde{\mathcal{P}}^* = \mathcal{P}^*_{\sim}$. Moreover, since $\tilde{\mu} \circ \Pi \in \mathcal{P}^*_{\sim}$ and $\widetilde{\tilde{\mu} \circ \Pi} = \tilde{\mu}$ for every $\tilde{\mu} \in \widetilde{\mathcal{P}}^*$, the mapping $\mu \mapsto \tilde{\mu} : \mathcal{P}^* \to \widetilde{\mathcal{P}}^*$ is linear and onto.

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *quasi-linear* if there is a neighborhood $v_0 \in \mathcal{V}$ such that $a \leq b + v_0$ holds for $a, b \in \mathcal{P}$ and if and only if $a \leq b + s$ for some bounded element $s \in \mathcal{P}$ such that $s \leq v_0$. Obviously, every ordered locally convex topological vector space is quasi-linear in this sense, likewise every full locally convex cone whose neighborhood system $\mathcal{V} \subset \mathcal{P}$ is generated by a single neighborhood $v_0 \in \mathcal{P}$. A quasi-linear locally convex cone $(\mathcal{P}, \mathcal{V})$ has uniform boundedness components, since $a \leq b + v_0$ for $a, b \in \mathcal{P}$ implies that $a \leq b + \lambda v$ with some $\lambda \geq 0$ for every other neighborhood $v \in \mathcal{V}$. Moreover, if $\mathcal{P}$ is quasi-linear, then the quotient order $\lesssim$ and the weak preorder $\precsim$ coincide on $\widetilde{\mathcal{P}}$. Indeed, if $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b) + \tilde{v}_0$ for the particular neighborhood $v_0 \in \mathcal{V}$, then there are $\beta, \rho > 0$ such that $a \leq \beta b + \rho v$, that is $a \leq d$ with $d = \beta b + \rho s \in \mathcal{P}$ for some bounded element $s \leq v_0$. We shall verify that $b \sim d$. Indeed, given $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that both $0 \leq s + \lambda v$ and $s \leq \lambda v$. This shows $d \leq \beta b + (\rho\lambda)v$ as well as $b \leq b + \rho(s + \lambda u) \leq d + (\rho\lambda)v$. Thus $b \sim d$. We infer $\mathcal{B}^s(a) \lesssim \mathcal{B}^s(b)$ according to Definition 3.1. On the other hand, the statement $\mathcal{B}^s(a) \lesssim \mathcal{B}^s(b)$ is stronger than either of the statements $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b)$ or $\mathcal{B}^s(a) \lesssim \mathcal{B}^s(b) + v_0$, both of which are stronger than $\mathcal{B}^s(a) \precsim \mathcal{B}^s(b) + v_0$. Thus these four notions coincide coincide for a quasi-linear cone.

Example 4.5. Let us reconsider Example 3.7(e) (a special case of 2.1(d)), that is $\mathcal{P} = \mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})$, the cone of all bounded below (on the sets in $\mathcal{Y}$) $\overline{\mathbb{R}}$-valued functions on X with the neighborhood system $\mathcal{V}_\mathcal{Y}$ generated by the neighborhoods ε_Y for $\varepsilon > 0$ and $Y \in \mathcal{Y}$ (see 3.7(e)). For a function $f \in \mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})$ its boundedness component

$$\mathcal{B}^s(f) = \left\{ g \in \mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}}) \;\middle|\; \begin{array}{l} \text{for every } Y \in \mathcal{Y} \text{ there are } \alpha, \beta, \gamma, \delta > 0 \text{ such that} \\ \gamma f(x) - \delta \leq g(x) \leq \alpha f(x) + \beta \;\text{ for all } x \in Y \end{array} \right\}$$

is closed in the symmetric relative topology according to 4.2(b). (This can also be easily checked directly.) If the set X itself is contained in $\mathcal{Y}$, then the cone $(\mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}}), \mathcal{V})$ is quasi-linear. Indeed, for the neighborhood $v_0 = 1_X$ we have $f \leq g + 1_X$ if and only if $f \leq g + 1$, where 1 denotes the (bounded) constant function $x \mapsto 1$ in $\mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})$. Thus the order $\lesssim$ of the quotient cone $\widetilde{\mathcal{F}}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})$ coincides with its weak preorder $\precsim$ and is easy to describe (Lemma 4.3). Following 4.2(c) the boundedness components are also open with respect to the symmetric relative topology of $\mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})$ and the symmetric relative topology of $\widetilde{\mathcal{F}}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})$ is discrete in this case (4.3). If, for another special case, $\mathcal{Y}$ consists of all finite subsets of X, then the above description yields that two functions $f, g \in \mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})$ are contained in the same boundedness component if and only if for every $Y \in \mathcal{Y}$ they take the value $+\infty$ at exactly the same points of Y, that is if they take the value $+\infty$ at exactly the same points of $X_0 = \bigcup_{Y \in \mathcal{Y}} Y$. The symmetric relative topology is the topology of pointwise convergence on X_0 in this case. If X_0 is an infinite set, then the boundedness components are seen to be closed but not open in $\mathcal{F}_{b_{\mathcal{Y}}}(X, \overline{\mathbb{R}})$ in this topology.

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