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Corrigendum to "Optimality Conditions Using Approximations for Nonsmooth Vector Optimization Problems under General Inequality Constraints"

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Theorems 4.1 and 4.7 of our previous paper "Optimality conditions using approximations for nonsmooth vector optimization problems under general inequality constraints" [Journal of Convex Analysis 16 (2009) 169–186] are incorrect and need be changed. Hence, illustrative examples 4.2 and 4.8 should be modified.

Theorem 4.1. *Assume that C is polyhedral and $z^* \in K^*$ with $\langle z^*, g(x_0)\rangle = 0$. Impose further that $(f'(x_0), B_f(x_0))$ and $(g'(x_0), B_g(x_0))$ are asymptotically p-compact second-order approximations of f and g, respectively, at x_0 with norm-bounded $B_g(x_0)$.*

If $x_0 \in \mathrm{LWE}(f, g)$ then, for any $v \in T(G(z^*), x_0)$, there exists $y^* \in B$, where B is finite and $\mathrm{cone}(\mathrm{co}\, B) = C^*$, such that $\langle y^*, f'(x_0)v\rangle + \langle z^*, g'(x_0)v\rangle \geq 0$. If, furthermore, $y^* \circ f'(x_0) + z^* \circ g'(x_0) = 0$, we have either $M \in \mathrm{p}-\mathrm{cl}\, B_f(x_0)$ and $N \in \mathrm{p}-\mathrm{cl}\, B_g(x_0)$ such that

$$\langle y^*, M(v, v)\rangle + \langle z^*, N(v, v)\rangle \geq 0,$$

or $M \in \mathrm{p}- B_f(x_0)_\infty \setminus \{0\}$ such that

$$\langle y^*, M(v, v)\rangle \geq 0.$$

Proof. Fix $v \in T(G(z^*), x_0)$. There exists $(t_n, v_n) \to (0^+, v)$ such that $x_n := x_0 + t_n v_n \in G(z^*)$ for all n. As $x_0 \in \mathrm{LWE}(f, g)$ and C is polyhedral, there exists $y^* \in B$ such that (using a subsequence if necessary), for all n,

$$\langle y^*, f(x_n) - f(x_0)\rangle \geq 0.$$

Hence

$$\langle y^*, f(x_n) - f(x_0)\rangle + \langle z^*, g(x_n) - g(x_0)\rangle \geq 0.$$

Dividing this inequality by t_n and passing to limit one has

$$\langle y^*, f'(x_0)v\rangle + \langle z^*, g'(x_0)v\rangle \geq 0.$$

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If $y^* \circ f'(x_0) + z^* \circ g'(x_0) = 0$, then $(0, y^* \circ B_f(x_0) + z^* \circ B_g(x_0))$ is a second-order approximation of $L(., y^*, z^*) := \langle y^*, f(.)\rangle + \langle z^*, g(.)\rangle$ at x_0. Therefore, $M_n \in B_f(x_0)$ and $N_n \in B_g(x_0)$ exist such that, for large n,

$$L(x_0 + t_n v_n, y^*, z^*) - L(x_0, y^*, z^*) = t_n^2\langle y^*, M_n(v_n, v_n)\rangle + t_n^2\langle z^*, N_n(v_n, v_n)\rangle + o(t_n^2).$$

On the other hand,

$$L(x_0 + t_n v_n, y^*, z^*) - L(x_0, y^*, z^*) = \langle y^*, f(x_0 + t_n v_n) - f(x_0)\rangle \geq 0.$$

Consequently, for large n,

$$\langle y^*, M_n(v_n, v_n)\rangle + \langle z^*, N_n(v_n, v_n)\rangle + \frac{o(t_n^2)}{t_n^2} \geq 0.$$

The remaining part is unchanged. □

Example 4.2. Let $X = \mathbb{R}^2,\ Y = Z = \mathbb{R},\ C = \mathbb{R}_+,\ B = \{1\},\ K = \{0\},\ x_0 = (0, 0)$ and

$$f(x, y) = -\frac{2}{3}|x|^{\frac{3}{2}} + \frac{1}{2}y^2,$$
$$g(x, y) = x^2 - y.$$

Then $f'(x_0) = (0, 0)$, $g'(x_0) = (0, -1)$,

$$B_f(x_0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mid \alpha < -1 \right\},$$

$$\operatorname{cl} B_f(x_0) = \left\{ \begin{pmatrix} \beta & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mid \beta \leq -1 \right\},$$

$$B_f(x_0)_\infty = \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} \mid \gamma \leq 0 \right\},$$

$$B_g(x_0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Choose $z^* = 0 \in K^* = \mathbb{R}$ and $v = (1, 0) \in T(G(z^*), x_0) = \mathbb{R} \times \{0\}$. Then, for any $y^* \in B$, i.e. $y^* = 1$, we have $y^* \circ f'(x_0) + z^* \circ g'(x_0) = 0$ and

$$\langle y^*, M(v, v)\rangle + \langle z^*, N(v, v)\rangle = \alpha \leq -1 < 0$$

for all $M \in \operatorname{cl} B_f(x_0)$ and all $N \in \operatorname{cl} B_g(x_0)$, and

$$\langle y^*, M(v, v)\rangle = \gamma < 0$$

for all $M \in B_f(x_0)_\infty \setminus \{0\}$. Therefore, following Theorem 4.1, x_0 is not a local weakly efficient solution of problem (P).

Theorem 4.7. *Let C be polyhedral and $z^* \in K^*$ with $\langle z^*, g(x_0)\rangle = 0$. Assume that $(A_f(x_0), B_f(x_0))$ and $(A_g(x_0), B_g(x_0))$ are asymptotically p-compact second-order approximations of f and g, respectively, at x_0, with $A_f(x_0), A_g(x_0)$ and $B_g(x_0)$ being norm bounded. If $x_0 \in \mathrm{LWE}(f, g)$ then, for all $v \in T(G(z^*), x_0)$,*

(a) *for all* $w \in T^2(G(z^*), x_0, v)$, *there exist* $y^* \in B$ *(where* B *is finite and* $\text{cone}(\text{co}(B)) = C^*$*),* $P \in \text{p} - \text{cl}\, A_f(x_0)$ *and* $Q \in \text{p} - \text{cl}\, A_g(x_0)$ *such that* $\langle y^*, Pv\rangle + \langle z^*, Qv\rangle \geq 0$. *If, in addition,* $v \in P(x_0, y^*, z^*)$, *then either there are* $P \in \text{p} - \text{cl}\, A_f(x_0)$, $Q \in \text{p} - \text{cl}\, A_g(x_0)$, $M \in \text{p} - \text{cl}\, B_f(x_0)$ *and* $N \in \text{p} - \text{cl}\, B_g(x_0)$ *so that*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + 2\langle y^*, M(v, v)\rangle + 2\langle z^*, N(v, v)\rangle \geq 0$$

or $M \in \text{p}{-}B_f(x_0)_\infty \setminus \{0\}$ *exists with*

$$\langle y^*, M(v, v)\rangle \geq 0;$$

(b) *for all* $w \in T''(G(z^*), x_0, v)$, *there exist* $y^* \in B$, $P \in \text{p} - \text{cl}\, A_f(x_0)$ *and* $Q \in \text{p} - \text{cl}\, A_g(x_0)$ *such that* $\langle y^*, Pv\rangle + \langle z^*, Qv\rangle \geq 0$. *If, in addition,* $v \in P(x_0, y^*, z^*)$, *then either* $P \in \text{p} - \text{cl}\, A_f(x_0)$, $Q \in \text{p} - \text{cl}\, A_g(x_0)$ *and* $M \in \text{p}{-}B_f(x_0)_\infty$ *exist such that*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + \langle y^*, M(v, v)\rangle \geq 0$$

or, for some $M \in \text{p}{-}B_f(x_0)_\infty \setminus \{0\}$,

$$\langle y^*, M(v, v)\rangle \geq 0.$$

Proof. (a) Fix $v \in T(G(z^*), x_0)$ and $w \in T^2(G(z^*), x_0, v)$. There exist $t_n \to 0^+$, and $w_n \to w$ such that, for all n,

$$x_n := x_0 + t_n v + \frac{1}{2}t_n^2 w_n \in G(z^*).$$

As for Theorem 4.1, there exists $y^* \in B$ such that, for all n,

$$L(x_n, y^*, z^*) - L(x_0, y^*, z^*) = \langle y^*, f(x_n) - f(x_0)\rangle \geq 0.$$

On the other hand, there are $P'_n \in A_f(x_0)$ and $Q'_n \in A_g(x_0)$ such that, for large n

$$\begin{aligned} &L(x_n, y^*, z^*) - L(x_0, y^*, z^*) \\ &= t_n \left\langle y^*, P'_n \left(v + \frac{1}{2}t_n w_n\right)\right\rangle + t_n \left\langle z^*, Q'_n \left(v + \frac{1}{2}t_n w_n\right)\right\rangle + o(t_n). \end{aligned}$$

Hence,

$$\left\langle y^*, P'_n \left(v + \frac{1}{2}t_n w_n\right)\right\rangle + \left\langle z^*, Q'_n \left(v + \frac{1}{2}t_n w_n\right)\right\rangle + \frac{o(t_n)}{t_n} \geq 0.$$

By the assumed boundedness we can assume the existence of $P' \in \text{p} - \text{cl}\, A_f(x_0)$ and $Q' \in \text{p} - \text{cl}\, A_g(x_0)$ such that $P'_n \xrightarrow{\text{p}} P'$ and $Q'_n \xrightarrow{\text{p}} Q'$ and then passing to limit we obtain

$$\langle y^*, P'v\rangle + \langle z^*, Q'v\rangle \geq 0.$$

If $v \in P(x_0, y^*, z^*)$, from the definition of the Lagrangian we have $P_n \in A_f(x_0)$, $Q_n \in A_g(x_0)$, $M_n \in B_f(x_0)$ and $N_n \in B_g(x_0)$ such that, for large n,

$$\begin{aligned} &\langle y^*, P_n w_n\rangle + \langle z^*, Q_n w_n\rangle + 2\left\langle y^*, M_n \left(v + \frac{1}{2}t_n w_n, v + \frac{1}{2}t_n w_n\right)\right\rangle \\ &+ 2\left\langle z^*, N_n \left(v + \frac{1}{2}t_n w_n, v + \frac{1}{2}t_n w_n\right)\right\rangle + \frac{o(t_n^2)}{\frac{1}{2}t_n^2} \geq 0. \end{aligned}$$

The rest of (a) is unchanged.

(b) For any $v \in T(G(z^*), x_0)$ and $w \in T''(G(z^*), x_0, v)$, there are $(t_n, r_n) \to (0^+, 0^+)$ and $w_n \to w$ such that $\frac{t_n}{r_n} \to 0^+$ and, for all n,

$$x_n := x_0 + t_n v + \frac{1}{2} t_n r_n w_n \in G(z^*).$$

As in part (a), there is $y^* \in B$ such that, for all n, $L(x_n, y^*, z^*) - L(x_0, y^*, z^*) \geq 0$. Then, there are $P'_n \in A_f(x_0)$ and $Q'_n \in A_g(x_0)$ such that, for large n,

$$\left\langle y^*, P'_n \left(v + \frac{1}{2} r_n w_n\right)\right\rangle + \left\langle z^*, Q'_n \left(v + \frac{1}{2} r_n w_n\right)\right\rangle + \frac{o(t_n)}{t_n} \geq 0.$$

By the assumed boundedness we have $P'_n \xrightarrow{\text{p}} P' \in \text{p} - \text{cl}\, A_f(x_0)$ and $Q'_n \xrightarrow{\text{p}} Q' \in \text{p} - \text{cl}\, A_g(x_0)$. Passing the above inequality to limit we obtain

$$\langle y^*, P'v\rangle + \langle z^*, Q'v\rangle \geq 0.$$

If $v \in P(x_0, y^*, z^*)$ there exists $P_n \in A_f(x_0)$, $Q_n \in A_g(x_0)$, $M_n \in B_f(x_0)$ and $N_n \in B_g(x_0)$ such that

$$\begin{aligned}
&\left(\frac{2}{t_n r_n}\right)(L(x_n, y^*, z^*) - L(x_0, y^*, z^*)) \\
&= \langle y^*, P_n w_n\rangle + \langle z^*, Q_n w_n\rangle + \left\langle y^*, \left(\frac{2t_n}{r_n}\right) M_n \left(v + \frac{1}{2} r_n w_n, v + \frac{1}{2} r_n w_n\right)\right\rangle \\
&\quad + \left\langle z^*, \left(\frac{2t_n}{r_n}\right) N_n \left(v + \frac{1}{2} r_n w_n, v + \frac{1}{2} r_n w_n\right)\right\rangle + \frac{2o(t_n^2)}{t_n r_n} \geq 0.
\end{aligned}$$

The remaining part is unchanged. □

Example 4.8. Let $X = Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $C = \mathbb{R}^2_+$, $B = \{y_1^* = (1, 0),\ y_2^* = (0, 1)\}$, $K = \{0\}$, $x_0 = (0, 0)$, $f(x, y) = (-y, -x + |y|)$ and $g(x, y) = -x^3 + y^2$. Then we have the following approximations

$$A_f(x_0) = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & \pm 1 \end{pmatrix} \right\}, \qquad B_f(x_0) = \{0\},$$

$$A_g(x_0) = \{0\}, \qquad B_g(x_0) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let $z^* = 0 \in K^*$. Then

$$G(z^*) = \{(x, y) \in \mathbb{R}^2 \mid -x^3 + y^2 = 0\},$$

$$T(G(z^*), x_0) = \mathbb{R}_+ \times \{0\}.$$

Choosing $v = (1, 0) \in T(G(z^*), x_0)$, we have

$$T^2(G(z^*), x_0, v) = \emptyset, \qquad T''(G(z^*), x_0, v) = \mathbb{R}^2.$$

Now let $w = (0,1) \in T''(G(z^*), x_0, v)$. Then for $y_1^* = (1,0) \in B$, we have $P \in \operatorname{cl} A_f(x_0)$ and $Q \in \operatorname{cl} A_g(x_0)$ such that

$$\langle y_1^*, Pv \rangle + \langle z^*, Qv \rangle \geq 0$$

and

$$v \in P(x_0, y_1^*, z^*) = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_2 = 0\}.$$

For all $P \in \operatorname{cl} A_f(x_0)$, all $Q \in \operatorname{cl} A_g(x_0)$ and all $M \in B_f(x_0)_\infty$, one has

$$\langle y_1^*, Pw \rangle + \langle z^*, Qw \rangle + \langle y_1^*, M(v,v) \rangle = -1 < 0.$$

For $y_2^* = (0,1) \in B$, we have $v \notin P(x_0, y_2^*, z^*)$.

Taking into account Theorem 4.7 one sees that $x_0 \notin \mathrm{LWE}(f,g)$.