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# The Role of Local Convexity in Lipschitz Maps

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*Dedicated to Professor Nigel Kalton in Memoriam.*

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Lipschitz maps between metric spaces (in particular, between Banach spaces) are abundant and afford a great deal of flexibility: they can be glued, pasted, and truncated without impairing the Lipschitz property. When their target space is the real line, they also can be extended to the whole space without increasing the Lipschitz contstant. However, if we drop the local convexity from the spaces, Lipschitz maps can behave in a completely different way and, in fact, we need not take for granted even their existence; for instance, the Lipschitz-dual of $L_p$ for $0 < p < 1$ is trivial, that is to say, there are no nonzero Lipschitz functions $f : L_p \to \mathbb{R}$ with $f(0) = 0$. In this note we emphasize the role of local convexity in some properties of Lipschitz maps by showing that local convexity is a necessary condition for these properties to hold, whence they cannot be translated to the nonlocally convex setting.

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## 1. Introduction

Let $X$ and $Y$ be real quasi-Banach spaces. A (possibly nonlinear) map $f : X \to Y$ is said to be *$L$-Lipschitz*, $L \geq 0$, if it satisfies an estimate

$$\|f(x) - f(y)\| \leq L\|x - y\|, \tag{1}$$

for every pair of points $x, y$ in $X$. The spaces $X$ and $Y$ are called *Lipschitz isomorphic* (also, *Lipschitz homeomorphic*) if there exists a Lipschitz bijection $f : X \to Y$ such that $f^{-1}$ is also Lipschitz. Here $\|\cdot\|$ denotes the quasi-norm on $X$ or $Y$.

Recall that a quasi-normed space $X$ is a locally bounded topological vector space. This is equivalent to saying that the topology on $X$ is induced by a quasi-norm, i.e., a map $\|\cdot\| : X \to [0, \infty)$ satisfying:

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(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ if $\alpha \in \mathbb{R}$, $x \in X$;
(iii) there is a constant $\kappa \geq 1$ so that for any $x$ and $y$ in $X$ we have

$$\|x + y\| \leq \kappa(\|x\| + \|y\|). \tag{2}$$

The least $\kappa$ in equation (2) is often referred to as the modulus of concavity of the quasi-norm. If it is possible to take $\kappa = 1$ we obtain a norm.

A quasi-norm $\|\cdot\|$ is called *p-norm* $(0 < p < 1)$ if it is *p-subadditive*, that is, if it satisfies

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \quad x, y \in X.$$

In this case the unit ball of $X$ is an absolutely $p$-convex set and $X$ is said to be *p-convex*. A $p$-subadditive quasi-norm $\|\cdot\|$ induces a metric topology on $X$. In fact, a metric can be defined by $d_p(x, y) = \|x - y\|^p$. The space $X$ is called a *quasi-Banach space* if $X$ is complete for this metric.

The purpose of this note is to encourage the study of the Lipschitz classfication of quasi-Banach spaces. For that, we first need to bring out to light the initial drawbacks we encounter in the lack of local convexity. Section 2 investigates the extension of real-valued Lipschitz maps defined on subsets of quasi-Banach spaces and shows that local convexity is a necessary and sufficient condition for McShane's theorem on extendability of Lipschitz maps to hold.

Lipschitz functions between Banach spaces are "smooth" in many cases, which makes differentiability an invaluable tool in their Lipschitz classification (see [8]). However, when one attempts to use differentiation to linearize Lipschitz maps between nonlocally convex quasi-Banach spaces, it does not take long to find spaces $X$ that are not suitable target spaces for studying differentiability properties of $X$-valued Lipschitz functions. Take, for instance, $X = L_p(0, 1)$ $(0 < p < 1)$ with the standard quasi-norm $\|f\|_p = (\int |f|^p)^{1/p}$, and consider the mapping

$$f : [0, 1] \to L_p(0, 1), \qquad t \to f(t) = \chi_{[0,t]}.$$

We have $\|f(t) - f(s)\|_p = |t - s|^{1/p}$, and so

$$\frac{\|f(t) - f(s)\|_p}{|t - s|} = |t - s|^{\frac{1}{p} - 1} \to 0 \ \text{ if } |t - s| \to 0.$$

That is, $f$ is a nonconstant Lipschitz function with zero derivative everywhere! This example of Rolewicz [15] (cf. [16, p. 198]) is just a particular case of a more general situation which occurss in quasi-Banach spaces with trivial dual. Kalton proved [9] that every quasi-Banach space $X$ such that $X^* = \{0\}$ admits a nonconstant Lipschitz function $f : [0, 1] \to X$ with $f'(t) = 0$ for all $t \in [0, 1]$. It seems only natural to ask: are there any quasi-Banach spaces $X$ such that the $X$-valued Lipschitz functions have nice differentiability properties? As it turns out, local convexity is not only a sufficient condition for some of these differentiability properties to work, but it is also necessary. We plan to study the differentiability of Lipschitz maps between quasi-Banach spaces with more detail in a later publication, but for the time being, in Section 3 we illustrate again the recurrent pattern of this paper by showing that a quasi-Banach space $X$ must be locally convex in case every differentiable Lipschitz function from $[0,1]$ into $X$ satisfies a mean value formula.

The methods employed in the uniform classification of Banach spaces rely heavily on the theory of Lipschitz maps because, thanks to the convexity, uniformly homeomorphic Banach spaces are Lipschitz isomorphic for large distances. This, in combination with the good properties of ultraproducts of Banach spaces and the duality theory, opens up the connecting door between the uniform and Lipschitz classifications in Banach spaces [8]. In contrast, the study of the uniform classication of quasi-Banach spaces encounters (from a very early stage) major disadvantages. The problem is that $p$-Banach spaces for $p < 1$ need not be metrically convex with the metric induced by the $p$-norm and so uniformly continuous maps defined on them satisfy instead Lipschitz conditions for large distances of order $> 1$, which does not lead anywhere. In Section 4 the Lipschitz condition for large distances, called (LLD)-condition for short, is introduced and studied for non-convex spaces. This class of nonlinear homeomorphism between quasi-Banach spaces bypasses the need of metric convexity before taking ultraproducts.

The rest of the paper gathers positive results concerning extensions of Lipschitz maps on quasi-Banach spaces in certain scenarios. Section 5 gives that if two quasi-Banach spaces are Lipschitz isomorphic, then their respective Banach envelopes must be Lipschitz isomorphic as well; Section 6 provides a splitting result of exact sequences of quasi-Banach spaces which is useful for linearizing Lipschitz liftings of quotient maps between those spaces.

The reader will find the background of these problems in the context of Banach spaces in the excellent book by Benyamini and Lindenstrauss [4] and the informative survey by Kalton [11].

## 2. McShane's theorem equates with local convexity

It is well-known that the Hahn-Banach theorem does not hold in a quasi-Banach space $X$ unless $X$ is locally convex [12]. Metric spaces also have their own version of the Hahn-Banach theorem for real-valued Lipschitz maps defined on them. Indeed, the classical theorem of McShane [13] states that real-valued Lipschitz functions defined on metric spaces (in particular, in Banach spaces) can always be extended to the whole space with preservation of the Lipschitz norm. In fact, if $f : A \to \mathbb{R}$ is an $L$-Lipschitz function on a subset $A$ of a Banach space $X$, the formula

$$F(x) = \inf \{f(a) + L\|x - a\| : a \in A\}, \quad x \in X,$$

defines an $L$-Lipschitz extension of $f$ to the whole of $X$. Of course, Lipschitz extensions need not exist in general when we replace the target real line by an arbitrary Banach space. We next show that in this respect, the Lipschitz structure of a quasi-Banach space exhibits some degree of consistency with the linear structure, and that, roughly speaking, once we have a real-valued Lipschitz map defined on a quasi-Banach space then we get lots of linear functionals. Then we use this to deduce that local convexity is not only a sufficient condition for McShane's theorem to hold but it is also necessary.

Suppose that $X$ is a quasi-normed space. The *Lipschitz dual of* $X$, denoted $\mathrm{Lip}_0(X)$, is the vector space of all real-valued Lipschitz functions $f$ defined on $X$ such that $f(0)=0$, endowed with the *Lipschitz norm*,

$$\|f\|_{\mathrm{Lip}} = \sup\left\{\frac{|f(x)-f(y)|}{\|x-y\|_X} : x, y \in X, x \neq y\right\}.$$

It can be readily checked that $(\mathrm{Lip}_0(X), \|\cdot\|_{\mathrm{Lip}})$ is a Banach space.

The $L_p$ spaces for $0<p<1$ have trivial dual [6] and also satisfy that $\mathrm{Lip}_0(L_p)=\{0\}$ [1, Proposition 2.8]. This is not a coincidence, and the two facts are related, as the next result shows.

**Proposition 2.1.** *Let $X$ be a quasi-Banach space. Put*

$$|||x||| = \sup\left\{|f(x)| : f \in \mathrm{Lip}_0(X), \|f\|_{\mathrm{Lip}} \leq 1\right\}, \quad x \in X. \tag{3}$$

(*i*) *The mapping $x \to |||x|||$ defines a semi-norm on $X$;*
(*ii*) *If $\mathrm{Lip}_0(X)$ is nontrivial then $X^*$ is nontrivial;*
(*iii*) *If $X$ has a separating Lipschitz dual, then $|||\cdot|||$ is a norm on $X$ and $X$ has a separating (linear) dual.*

**Proof.** (*i*) Let us see that $|||\cdot|||$ satisfies the triangle law and leave the other straightforward details to the reader. Fix $x, y \in X$. For any $f \in \mathrm{Lip}_0(X)$ with $\|f\|_{\mathrm{Lip}} \leq 1$, the map

$$g : X \to \mathbb{R}, \qquad u \to g(u) = f(u+y) - f(y)$$

verifies $g \in \mathrm{Lip}_0(X)$ and $\|g\|_{\mathrm{Lip}} \leq 1$. Hence by definition we have $g(x) \leq |||x|||$, i.e., $f(x+y) \leq |||x||| + f(y)$, which implies $|f(x+y)| \leq |||x||| + |f(y)|$. If we sup over all $f \in \mathrm{Lip}_0(X)$ with $\|f\|_{\mathrm{Lip}} \leq 1$ we obtain $|||x+y||| \leq |||x||| + |||y|||$.

(*ii*) By hypothesis there exist $f_0 \in \mathrm{Lip}_0(X)$ and $x_0 \in X$ so that $f_0(x_0)=1$. Thus in particular $|||x_0||| \neq 0$. Let $X_0$ be the 1-dimensional linear subspace of $X$ generated by $x_0$, i.e., $X_0 = \{rx_0 : r \in \mathbb{R}\}$. Let us consider the linear functional on $X_0$ given by

$$x_0^{\#} : X_0 \to \mathbb{R}, \qquad rx_0 \to r.$$

Since $x_0^{\#}(x) \leq |||x|||$ for all $x \in X_0$, the (algebraic) Hahn-Banach theorem yields a linear map $x_0^* : X \to \mathbb{R}$ such that $x_0^*|_{X_0} = x_0^{\#}$ and $x_0^*(x) \leq |||x|||$ for all $x \in X$. In particular, $x_0^*$ is continuous on $X$.

(*iii*) If $0 \neq x \in X$, by assumption there exists $f \in \mathrm{Lip}_0(X)$ so that $f(x) \neq f(0) = 0$, which implies $|||x||| \neq 0$. This way $(X, |||\cdot|||)$ is a normed space, whence given any two different vectors in $X$, we can pick a $|||\cdot|||$-continuous linear functional $x^* : X \to \mathbb{R}$ that separates them. Since $|f(x)| \leq \|x\|$ for every $f \in \mathrm{Lip}_0(X)$ with $\|f\|_{\mathrm{Lip}} \leq 1$ and $x \in X$, we have

$$|||x||| \leq \|x\|, \quad x \in X,$$

and so $x^*$ is also $\|\cdot\|$-continuous. □

**Definition 2.2.** If $X$ is a quasi-Banach space with a separating Lipschitz dual, the norm given in equation (3) will be referred to as the *Lipschitz envelope norm* of $X$.

**Remark 2.3.** (a) Let $X$ be a quasi-Banach space with a separating (linear) dual $X^*$. Then the *Banach envelope norm* of $X$, defined as the gauge functional of the convex hull of the closed unit ball of $X$ [12],

$$\|x\|_c = \inf\left\{\lambda > 0 : \lambda^{-1}x \in \operatorname{co}(B_X)\right\}, \quad x \in X,$$

and which is equivalently given by

$$\|x\|_c = \sup_{x^* \in B_{X^*}} |x^*(x)|,$$

coincides with the Lipschitz envelope norm of $X$.

(b) Let us notice that there are quasi-Banach spaces $X$ which have nontrivial dual but do not have enough linear functionals to separate the points of $X$. Take for instance $X = L_p \oplus \mathbb{R}$ for $0 < p < 1$.

**Proposition 2.4.** *Suppose $X$ is a quasi-Banach space such that whenever $f : E \to \mathbb{R}$ is an $L$-Lipschitz map defined on a subset $E$ of $X$, $f$ can be extended to an $L'$-Lipschitz map $F : X \to \mathbb{R}$ with $L' \geq L$. Then $X$ is locally convex.*

**Proof.** Given $0 \neq x \in X$ of norm $\|x\| = 1$, consider the subspace $E = \{rx : r \in \mathbb{R}\}$ of $X$ and the linear functional on $E$ given by

$$f : E \to \mathbb{R}, \qquad rx \to r.$$

By hypothesis there is a $L'$-Lipschitz function $F : X \to \mathbb{R}$ so that $F|_E = f$. Thus, going back to the definition in (3),

$$|||x||| \geq \frac{F(x)}{L'} = \frac{f(x)}{L'} = \frac{1}{L'}.$$

We deduce that $|||\cdot|||$ is now a norm and, furthermore,

$$\|x\| \leq L'|||x|||, \quad x \in X.$$

On the other hand, we have $|||x||| \leq \|x\|$ for $x \in X$. The last two inequalities show that the quasi-Banach space $X$ can be equipped with an equivalent norm, i.e., $X$ is locally convex. □

## 3. The mean value formula equates with local convexity

Differentiable functions on Banach spaces satisfy the following result (cf. [4, p. 84]):

**Proposition 3.1.** *Let $X$ and $Y$ be Banach spaces. Assume that $f : X \to Y$ is Gâteaux differentiable on the interval $I = \{x_0 + t(y_0 - x_0) : t \in [0,1]\}$ connecting $x_0$ with $y_0$ in $X$. Then*

$$\|f(y_0) - f(x_0)\| \leq \sup_{x \in I} \|f'(x)\| \|y_0 - x_0\|.$$

The fact that the mean value formula characterizes local convexity was explicitly established in [10, Theorem 3.1]. Here we give a slightly improved version of this result.

**Proposition 3.2.** *Let $X$ be a quasi-Banach space with a separating dual. Suppose that for some constant $C \geq 1$ we have that every differentiable Lipschitz function $F : [0,1] \to X$ satisfies a Mean Value formula*

$$\|F(y) - F(x)\| \leq C \sup_{t\in[0,1]} \|F'(x + t(y-x))\| |y - x|, \quad \forall\, x, y \in [0,1].$$

*Then $X$ is locally convex.*

**Proof.** For $n \in \mathbb{N}$ arbitrary, let $\{x_i\}_{i=1}^n$ be any vectors contained in the closed unit ball $B_X$ of $X$, and let $\{a_i\}_{i=1}^n$ be nonnegative scalars with $\sum_{i=1}^n a_i = 1$. We will show that $\| \sum_{i=1}^n a_i x_i \| \leq K$ for some constant $K$, which will imply that the origin has a convex neighborhood and so $X$ will be normable.

Let $f : [0,1] \to X$ be the piecewise linear function given by

$$f(t) = \begin{cases} tx_1 & \text{if } 0 \leq t \leq a_1, \\ a_1x_1 + (t - a_1)x_2 & \text{if } a_1 \leq t \leq a_1 + a_2, \\ a_1x_1 + a_2x_2 + (t - (a_1 + a_2))x_3 & \text{if } a_1 + a_2 \leq t \leq a_1 + a_2 + a_3, \\ \cdots & \\ \sum_{i=1}^{n-1} a_ix_i + (t - \sum_{i=1}^{n-1} a_i)x_n & \text{if } \sum_{i=1}^{n-1} a_i \leq t \leq 1. \end{cases}$$

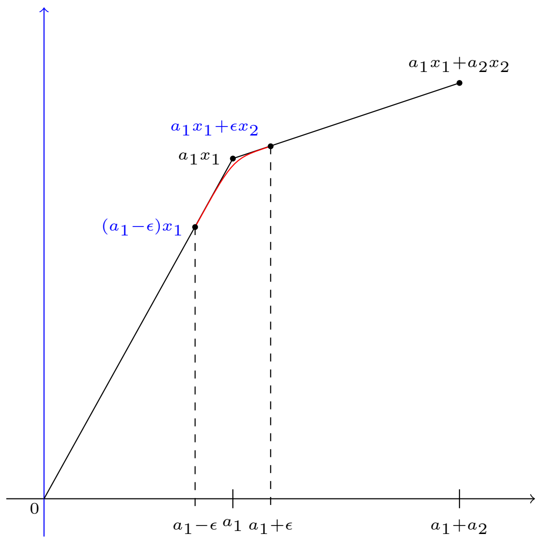
It is clear that $f$ is Lipschitz on $[0,1]$ and differentiable at every $t \in [0,1]$ except for the $n-1$ points of the set $\mathcal{V}_f = \{a_1, a_1 + a_2, \cdots, a_1 + a_2 + \cdots + a_{n-1}\}$. Its derivative is

$$f'(t) = \begin{cases} x_1 & \text{if } 0 \leq t < a_1, \\ x_2 & \text{if } a_1 < t < a_1 + a_2, \\ x_3 & \text{if } a_1 + a_2 < t < a_1 + a_2 + a_3, \\ \cdots & \\ x_n & \text{if } \sum_{i=1}^{n-1} a_i < t \leq 1. \end{cases}$$

Now we are going to smooth out $f$ in a small neighborhood of each point of $\mathcal{V}_f$. Take $\epsilon > 0$ sufficiently small and for the sake of simplicity let us restrict our attention to the interval $[0, a_1 + a_2]$. Put

$$F|_{[0,a_1+a_2]}(t) = \begin{cases} tx_1 & \text{if } 0 \leq t \leq a_1 - \epsilon, \\ \varphi_1(t) & \text{if } a_1 - \epsilon \leq t \leq a_1 + \epsilon, \\ a_1x_1 + (t - a_1)x_2 & \text{if } a_1 + \epsilon \leq t \leq a_1 + a_2, \end{cases}$$

where $\varphi_1(t)$ is a second degree interpolation polynomial on $[a_1 - \epsilon, a_1 + \epsilon]$ with $\varphi_1(a_1 \mp \epsilon) = f(a_1 \mp \epsilon)$, $\varphi_1'(a_1 - \epsilon) = x_1$, and $\varphi_1'(a_1 + \epsilon) = x_2$. Pictorially,




We quickly obtain the estimate $\sup\{\|F'(t)\| : t \in [0, a_1+a_2]\} \leq \kappa$, the modulus of concavity of the quasi-norm. Repeating this argument on each interval $[\sum_{i=1}^{k-1} a_i, \sum_{i=1}^{k} a_i]$ for $1 \leq k \leq n$ we obtain a Lipschitz map $F : [0,1] \to X$ that is differentiable at every $t \in [0,1]$. Hence using the hypothesis,

$$\|F(1) - F(0)\| \leq C \sup_{t\in[0,1]} \|F'(t)\|,$$

that is,

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq \kappa C,$$

which finishes the proof. □

## 4. The Lipschitz for large distance condition in nonlocally convex spaces

Let us start this section by making a definition.

**Definition 4.1.** A map $f$ from a metric space $(X,d)$ into a metric space $(Y,\rho)$ is said to verify a *Lipschitz condition for large distances (or (LLD)-condition) of order* $\alpha \geq 1$ if for every $\delta > 0$ there is a constant $K = K(\delta)$ such that

$$\rho(f(x), f(y)) \leq K\, d(x,y)^{\alpha},$$

for any $x, y \in X$ with $d(x,y) > \delta$.

The well-known *Corson-Klee lemma* guarantees that if a map $f$ is uniformly continuous and the space $(X,d)$ is metrically convex then $f$ is Lipschitz of order 1 for large distances [5]. But if $X$ is a $p$-Banach space for $p < 1$, then the metric space $(X, d_p)$, where $d_p = \|x-y\|^p$, is not ussualy metrically convex and so uniformly continuous maps on $X$ satisfy weaker (LLD)-conditions. Let us see this with an easy example.

**Example 4.2.** Let $0 < p < 1$.

(i) If $f : (\ell_p, d_p) \to (\ell_1, d_1)$ is uniformly continuous then for each $\delta > 0$ there is a constant $K = K(\delta)$ such that
$$d_1(f(x), f(y)) \leq K\, d_p(x, y)^{1/p}$$
whenever $x, y \in \ell_p$ with $d_p(x, y) > \delta^p$.

(ii) If $g : (\ell_1, d_1) \to (\ell_p, d_p)$ is uniformly continuous then for each $\delta > 0$ there is a constant $L = L(\delta)$ such that
$$d_p(g(x), g(y)) \leq L\, d_1(x, y),$$
whenever $x, y \in \ell_1$ with $d_1(x, y) > \delta$.

Let us show (i). Given $\delta > 0$ there is a constant $C_\delta$ so that
$$\sup\{\|f(a) - f(b)\|_1 : a, b \in \ell_p, \|a - b\|_p \leq \delta\} \leq C_\delta. \tag{4}$$
If $x, y \in \ell_p$ are such that $\|x - y\|_p > \delta$ we pick the smallest $N \in \mathbb{N}$ such that $\|x - y\|_p / N < \delta$. Set
$$x_j = \left(1 - \frac{j}{N}\right) x + \frac{j}{N} y, \quad j = 0, 1, \dots, N.$$
Since
$$\|x_j - x_{j+1}\|_p = \left\| \frac{x}{N} - \frac{y}{N} \right\|_p = \frac{\|x - y\|_p}{N} < \delta,$$
by (4) and our choice of $N$ we have,
$$\|f(x) - f(y)\|_1 \leq \sum_{j=0}^{N-1} \|f(x_j) - f(x_{j+1})\|_1 \leq N C_\delta < \frac{2}{\delta} C_\delta \|x - y\|_p,$$
and the conclusion of (i) follows with $K_\delta = 2\delta^{-1} C_\delta$.

To obtain (ii), let
$$C_\delta = \sup\left\{\|g(a) - g(b)\|_p^p : a, b \in \ell_1, \|a - b\|_1 \leq \delta\right\}.$$
If $x, y \in \ell_1$ are such that $\|x - y\|_1 \geq \delta$ we can find (by the metric convexity of $\ell_1$) $n \leq 2\|x - y\|_1 / \delta$ points in $\ell_1$, $x = x_0, x_1, \dots, x_n = y$ with $\|x_i - x_{i+1}\|_1 \leq \delta$ for every $0 \leq i \leq n - 1$. Thus,
$$\|g(x) - g(y)\|_p^p \leq \sum_{j=0}^{n-1} \|g(x_j) - g(x_{j+1})\|_p^p \leq n C_\delta \leq L_\delta \|x - y\|_1,$$
where $L_\delta = 2\delta^{-1} C_\delta$.

**Definition 4.3.** We say that two quasi-Banach spaces $X$ and $Y$ are *Lipschitz homeomorphic for large distances*, (LLD)-homeomorphic for short, if there is a bijection $f : X \to Y$ so that for some constants $C$ and $c$ we have
$$\frac{1}{C}\|x - y\| \leq \|f(x) - f(y)\| \leq C\|x - y\|,$$
whenever $\|x - y\| \geq c$.

Of course, if $X$ and $Y$ are both Banach spaces and $f : X \to Y$ is a uniform homeomorphism, then $X$ and $Y$ are (LLD)-homeomorphic. However, this property does not follow when at least one of the two spaces is non-Banach.

**Proposition 4.4.** *Let $X$ be a $p$-normed space and $Y$ be a $q$-normed space with $0 < p < q \leq 1$. If $X$ (LLD)-embeds into $Y$ then $X$ is $q$-normable. In particular, if $X$ is (LLD)-homeomorphic to a subset of a normed space then $X$ is locally convex.*

The proof of the proposition will require the next lemma, which is part of the proof of the *Aoki-Rolewicz theorem* [3, 14]. This basic (yet fundamental) result states that every quasi-normed space $X$ is $p$-normable for some $p \leq 1$.

**Lemma 4.5 (Aoki-Rolewicz).** *Let $(X, \|\cdot\|)$ be a quasi-normed space and $0 < p \leq 1$. Suppose that there is a constant $C \geq 1$ so that for any $\{x_i\}_{i=1}^n$ in $X$ and any $n \in \mathbb{N}$ we have*

$$\left\|\sum_{j=1}^n x_j\right\| \leq C\left(\sum_{j=1}^n \|x_j\|^p\right)^{1/p}.$$

*Then $X$ is $p$-normable.*

**Proof of Proposition 4.4.** Suppose that $f : X \to Y$ is an (LLD)-embedding such that $f(0) = 0$ and for some constant $C$,

$$C^{-1}\|x - y\| \leq \|f(x) - f(y)\| \leq C\|x - y\|,$$

whenever $\|x - y\| \geq 1$.

Given any $\{x_i\}_{i=1}^n$ in $X$, pick an integer $N$ so that $N\|x_k\| > 1$ for all $1 \leq k \leq n$ and $N\|x_1 + x_2 + \cdots + x_n\| > 1$. Put $z_k = x_1 + \cdots + x_k$ for $1 \leq k \leq n$ and let $z_0 = 0$. Then,

$$\begin{aligned}
\left\|\sum_{i=1}^n x_i\right\|^q = \frac{1}{N^q}\left\|N\sum_{i=1}^n x_i\right\|^q &= \frac{1}{N^q}\|Nz_n - Nz_0\|^q \\
&\leq \frac{C^q}{N^q}\|f(Nz_n) - f(Nz_0)\|^q \\
&= \frac{C^q}{N^q}\left\|\sum_{k=1}^n f(Nz_k) - f(Nz_{k-1})\right\|^q \\
&\leq \frac{C^q}{N^q}\sum_{k=1}^n \|f(Nz_k) - f(Nz_{k-1})\|^q \\
&\leq \frac{C^{2q}}{N^q}\sum_{k=1}^n \|Nz_k - Nz_{k-1}\|^q \\
&= C^{2q}\sum_{k=1}^n \|x_k\|^q,
\end{aligned}$$

and so, by Lemma 4.5, $X$ is $q$-normable. □

**Corollary 4.6.** *Suppose $0 < p < q < \infty$. Then no function between the quasi-Banach spaces $\ell_p$ and $\ell_q$ [respectively, $L_p$ and $L_q$] is and (LLD)-embedding.*

**Proof.** If $\ell_p$ were (LLD)-homeomorphic to a subset of $\ell_q$ then $\ell_p$ would be $q$-normable, which is impossible (see [12, Lemma 2.7]). The corresponding statement for the spaces $L_p$ goes along the same lines since $L_p$ is not $q$-normable for any $p < q \leq 1$ [12, Theorem 2.2]. □

**Problem 4.7.** *Is the class of (LLD)-homeomorphic quasi-Banach spaces different from the class of Lipschitz isomorphic quasi-Banach spaces?*

**Proposition 4.8.** *Let $X$ be a $p$-normed space for some $0 < p < 1$ and let $Y = \{x \in X : \|x\| \leq 2^{-1/p}\}$. Suppose that a map $f : Y \to \mathbb{R}$ satisfies a local Lipschitz condition*

$$|f(x) - f(y)| \leq C \|x - y\|, \quad x, y \in Y, \tag{5}$$

*for some $C > 0$. Then $f$ can be extended to a function $F : X \to \mathbb{R}$ that satisfies an (LLD)-condition,*

$$|F(x) - F(y)| \leq C \|x - y\|$$

*for all $x$ and $y$ in $X$ with $\|x - y\| \geq 1$.*

**Proof.** Let $x, y \in Y$. We have,

$$\|x - y\|^p \leq \|x\|^p + \|y\|^p \leq 2.2^{-1} = 1,$$

which implies $\|x - y\| \leq \|x - y\|^p$. Therefore we obtain

$$|f(x) - f(y)| \leq C \|x - y\|^p, \quad x, y \in Y. \tag{6}$$

Let $\|f\|_{\mathrm{Lip}}$ be the least constant in (6). Taking into account that $X$ is a metric space with the distance $d_p(x, y) = \|x - y\|^p$, $f$ extends to the whole of $X$ via the function $F : X \to \mathbb{R}$ defined by

$$F(x) = \inf_{y \in Y} \left\{ f(y) + \|f\|_{\mathrm{Lip}} \|x - y\|^p \right\}.$$

Moreover,

$$|F(x) - F(y)| \leq \|f\|_{\mathrm{Lip}} \|x - y\|^p, \quad x, y \in X.$$

Notice that if $\|x - y\| \geq 1$ we have $\|x - y\|^p \leq \|x - y\|$, which yields

$$|F(x) - F(y)| \leq \|f\|_{\mathrm{Lip}} \|x - y\|^p \leq \|f\|_{\mathrm{Lip}} \|x - y\|.$$

□

## 5. Extending Lipschitz maps to the Banach envelope

Recall that the Banach envelope $\hat{X}$ of a quasi-Banach space $X$ with a separating dual is the "smallest" Banach space containing $X$ (see [12, p. 27]). The space $\hat{X}$ has the property that every continuous linear operator $T : X \to E$ mapping into a Banach space $E$ extends to $\hat{X}$ with preservation of norm. We next prove that the same is true for Banach space-valued Lipschitz functions defined on $X$.

**Proposition 5.1.** *Let $X$ be a quasi-Banach space with a separating dual.*

($i$) *Suppose that $Z$ is a Banach space and that $f : X \to Z$ is L-Lipschitz. Then there exists a unique Lipschitz extension of $f$ to the Banach envelope of $X$, $\tilde{f} : \hat{X} \to Z$ with preservation of norm. In particular, if $f \in \mathrm{Lip}_0(X)$ then there exists a unique $\tilde{f} \in \mathrm{Lip}_0(\hat{X})$ with $\|\tilde{f}\|_{\mathrm{Lip}} = \|f\|_{\mathrm{Lip}}$.*

($ii$) *If $Y$ is a quasi-Banach space Lipschitz isomorphic to $X$, then $Y$ has a separating dual.*

($iii$) *If $Y$ is a quasi-Banach space Lipschitz isomorphic to $X$ then the Banach envelopes $\hat{X}$ and $\hat{Y}$ are Lipschitz isomorphic.*

**Proof.** ($i$) Without loss of generality let us assume $\|f\|_{\mathrm{Lip}} = 1$. We will show

$$\|f(x) - f(x')\|_Z \leq \|x - x'\|_c, \quad x, x' \in X. \tag{7}$$

Once this is proved, the density of $X$ in $\hat{X}$ will yield the required extension and its uniqueness.

Let us observe that for $x \in X$,

$$\|x\|_c = \inf\left\{\sum_{i=1}^{n} \|y_i\|_X : (y_i)_{i=1}^n \subset X, \sum_{i=1}^{n} y_i = x, n \in \mathbb{N}\right\}. \tag{8}$$

Then given $x, x'$ in $X$, for any $\epsilon > 0$ there exist $y_1, \ldots, y_n$ in $X$ with $x - x' = \sum_{i=1}^n y_i$ and $\sum_{i=1}^n \|y_i\|_X < \|x - x'\|_c + \epsilon$. Let $x_0 = x'$ and for $1 \leq k \leq n$ put $x_k = x_0 + \sum_{i=1}^k y_i$. Then

$$\begin{aligned}\|f(x) - f(x')\|_Z &= \left\|\sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))\right\|_Z \\ &\leq \|f\|_{\mathrm{Lip}} \sum_{i=1}^{n} \|x_i - x_{i-1}\|_X \\ &= \sum_{i=1}^{n} \|y_i\|_X \\ &< \|x - x'\|_c + \epsilon.\end{aligned}$$

Since $\epsilon$ was arbitrary we obtain (7).

($ii$) By Proposition 2.1, the formula

$$|y|_L = \sup\{|f(x)| : f \in Lip_0(Y), \|f\|_{\mathrm{Lip}} \leq 1\}$$

defines a norm on $Y$ and by the Hahn-Banach theorem $(Y, |\cdot|_L)$ has a separating dual. Since $|\cdot|_L$ is dominated by the original quasi-norm $\|\cdot\|_Y$ on $Y$, it follows that $(Y, \|\cdot\|_Y)$ has a separating dual as well.

($iii$) Let $F : X \to Y$ be a bi-Lipschitz map (i.e., a bijection so that both $F$ and its inverse are Lipschitz) and call $j$ the inclusion of $Y$ into its Banach envelope $\hat{Y}$. Part ($i$) yields a unique Lipschitz extension $\tilde{F} : \hat{X} \to \hat{Y}$ with $\|\tilde{F}\|_{\mathrm{Lip}} = \|F\|_{\mathrm{Lip}}$.

The same argument with $Y \overset{F^{-1}}{\rightarrow} X \hookrightarrow \hat{X}$ yields a unique Lipschitz extension $\widetilde{F^{-1}}$ : $\hat{Y} \to \hat{X}$ with $\|\widetilde{F^{-1}}\|_{\mathrm{Lip}} = \|F^{-1}\|_{\mathrm{Lip}}$. Finally, one checks that $\widetilde{F^{-1}} = \tilde{F}^{-1}$. □

**Example 5.2.** Suppose $0 < p, q < 1$.

(i) The quasi-Banach spaces $L_q$ and $\ell_p$ are not Lipschitz isomorphic;
(ii) The quasi-Banach spaces $\ell_2(\ell_p^n)$ and $\ell_q$ are not Lipschitz isomorphic.

**Proof.** (i) $\ell_p$ and $L_q$ cannot be Lipscthiz isomorphic because $L_q^* = \{0\}$ [6] whereas $\ell_p^* = \ell_\infty$. In fact, this is an immediate consequence of Weston's result that $L_q$ and $\ell_p$ are not uniformly isomorphic [17].

(ii) If $\ell_2(\ell_p^n)$ and $\ell_q$ were Lipschitz isomorphic then their respective Banach envelopes $\ell_2(\ell_1^n)$ and $\ell_1$ would be Lipschitz isomorphic as well. But we know [8] that if a Banach space $X$ is Lipschitz isomorphic to $\ell_1$ and is a dual space, then $X$ is linearly isomorphic to $\ell_1$, which is not the case with $X = \ell_2(\ell_1^n)$. □

## 6. Lifting Lipschiz maps between quasi-Banach spaces

In nonlinear theory we are interested in knowing when the existence of a Lipschitz (or uniformly continuous) map of a certain kind (namely, embedding, embedding onto, quotient map) between quasi-Banach spaces ensures the existence of a linear map of the same kind.

Let $Y$ be a quasi-Banach space and $X$ be a closed subspace. Denote by $Z$ the quotient quasi-Banach space $Y/X$ and let $q : Y \to Z$ be the quotient mapping. A *lifting* (or *section*) of $q$ is a map (not necessarily linear or continuous) $f : Z \to Y$ such that $q \circ f = \mathrm{Id}_Z$. Clearly, $q$ induces a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{q} Z \longrightarrow 0.$$

Suppose we can find a Lipschitz lifting of $q$. Then the map

$$Y \to X \oplus Z, \quad x \to (x - f(q(x)), q(x)),$$

is a Lipschitz isomorphism (onto). The short exact sequence *splits* if $X$ is complemented in $Y$, or, equivalently, there is a bounded linear section of $q$, in which case $Y$ must be linearly isomorphic to $X \oplus Z$.

The following lifting theorem is a generalization of a result from [7] that was stated without a proof in [2].

**Theorem 6.1.** *Let*

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{q} Z \longrightarrow 0$$

*be a short exact sequence of quasi-Banach spaces such that $Z$ is separable, $X$ is a Banach space and there exists a Lipschitz map $f : Z \to Y$ such that $q \circ f = Id_Z$. Then the sequence splits, i.e., there exists a bounded linear operator $T : Z \to Y$ with $q \circ T = Id_X$.*

**Proof.** We follow the ideas of Theorem 3.1 of [7]. We can assume (by renorming) that $Y$ is $p$-normed for some $0 < p \leq 1$ and that $X$ is a subspace of $Y$ on which the given $p$-norm is a norm. Indeed, let $\|\cdot\|$ be the original $p$-norm on $Y$ and let $\|\cdot\|_X$ be an equivalent norm on $X$ such that $\|x\|_X \leq \|x\|$ for $x \in Y$. Put

$$\|y\|_Y = \inf\{(\|x\|^p + \|y - x\|^p)^{1/p} : \; x \in X\}, \quad y \in Y.$$

This defines a new $p$-norm on $Y$ so that $\|x\|_Y = \|x\|_X$ for $x \in X$.

Let $K$ be the Lipschitz constant of $f$. Let $(z_n)_{n=1}^\infty$ be a sequence in $Z$ which is linearly independent, has dense linear span and satisfies $\|z_n\|_Z < 2^{-n}$. Pick $y_n \in Y$ so that $q(y_n) = z_n$ and $\|y_n\|_Y < 2^{-n}$.

Let us define a map $G : \ell_\infty \to X$ by

$$G(t) = f\left(\sum_{n=1}^\infty t_n z_n\right) - \sum_{n=1}^\infty t_n y_n, \quad t = (t_n)_{n=1}^\infty.$$

The map $G : \ell_\infty \to X$ is weak$^*$-to-norm continuous on norm-bounded sets.

Let $(\xi_n)_{n=1}^\infty$ be a sequence of independent random variables each with uniform distribution on $[0,1]$. Let $(e_n)_{n=1}^\infty$ be the canonical basis of $c_0 \subset \ell_\infty$. Then the series $\sum_{n=1}^\infty \xi_n e_n$ converges weak$^*$ (everywhere) in $\ell_\infty$. We will define $x_n \in X$ by

$$x_n = \mathbb{E}G\left(e_n + \sum_{k \neq n} \xi_k e_k\right) - \mathbb{E}G\left(\sum_{k \neq n} \xi_k e_k\right).$$

Now define $F : \ell_\infty \to X$ by

$$F(t) = \mathbb{E}G\left(t + \sum_{k=1}^\infty \xi_k e_k\right), \quad t \in \ell_\infty.$$

Consider the restriction of $F$ to $[e_j]_{j=1}^n$. If we set

$$F_n(t) = \mathbb{E}G\left(t + \sum_{k=n+1}^\infty \xi_k e_k\right), \quad t \in [e_j]_{j=1}^n.$$

then $F_n$ is obviously continuous and

$$F(t) = \mathbb{E}F_n\left(t + \sum_{k=1}^n \xi_k e_k\right), \quad t \in [e_j]_{j=1}^n.$$

Thus $F$ is continuously differentiable as a map $F : [e_j]_{j=1}^n \to X$ and its derivative at the origin is given by

$$\nabla F(0)(t) = \sum_{k=1}^n t_k x_k.$$

For any $t \in [e_j]_{j=1}^n$ we have

$$G\left(t + \sum_{k=1}^{\infty} \xi_k e_k\right) - G\left(\sum_{k=1}^{\infty} \xi_k e_k\right) + \sum_{k=1}^{n} t_k y_k$$
$$= f\left(\sum_{k=1}^{n} t_k z_k + \sum_{k=1}^{\infty} \xi_k z_k\right) - f\left(\sum_{k=1}^{n} \xi_k z_k\right),$$

so that

$$\left\| G\left(t + \sum_{k=1}^{\infty} \xi_k e_k\right) - G\left(\sum_{k=1}^{\infty} \xi_k e_k\right) + \sum_{k=1}^{n} t_k y_k \right\| \le K \left\| \sum_{k=1}^{n} t_k z_k \right\|.$$

Since

$$\left\| G(t) - G(0) + \sum_{k=1}^{n} t_k y_k \right\| \le K \left\| \sum_{k=1}^{n} t_k z_k \right\|,$$

we deduce that

$$\left\| G\left(t + \sum_{k=1}^{\infty} \xi_k e_k\right) - G\left(\sum_{k=1}^{\infty} \xi_k e_k\right) - G(t) + G(0) \right\| \le 2^{\frac{1}{p}} K \left\| \sum_{k=1}^{n} t_k z_k \right\|.$$

Since $G$ takes values in $X$ this implies that

$$\| F(t) - F(0) - G(t) + G(0) \| \le 2^{\frac{1}{p}} K \left\| \sum_{k=1}^{n} t_k z_k \right\|$$

and so

$$\left\| F(t) - F(0) + \sum_{k=1}^{n} t_k y_k \right\| \le 3^{\frac{1}{p}} K \left\| \sum_{k=1}^{n} t_k z_k \right\|.$$

Hence, replacing $t$ by $\alpha t$ and letting $\alpha \to 0$,

$$\left\| \nabla F(0)(t) + \sum_{k=1}^{n} t_k y_k \right\| \le 3^{\frac{1}{p}} K \left\| \sum_{k=1}^{n} t_k z_k \right\|$$

i.e.,

$$\left\| \sum_{k=1}^{n} t_k (x_k + y_k) \right\| \le 3^{\frac{1}{p}} K \left\| \sum_{k=1}^{n} t_k z_k \right\|.$$

Now if we define a linear operator $T$ on the linear span of $(z_n)_{n=1}^{\infty}$ by $Tz_n = x_n + y_n$ then $T$ extends to a bounded operator $T : Z \to Y$ and $q \circ T = Id_Z$. $\square$

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