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The Biduality Problem and M-Ideals in Weighted Spaces of Holomorphic Functions

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Given a weight v on an open subset U of $\mathbf{C}^n$, $\mathcal{H}_v(U)$ (resp. $\mathcal{H}_{v_o}(U)$) denotes the Banach space of holomorphic functions f on U such that vf is bounded on U (resp. converges to 0 on the boundary of U). We show that $\mathcal{H}_v(U)$ is canonically isometrically isomorphic to the bidual of $\mathcal{H}_{v_o}(U)$ if and only if $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ and the associated weights $\tilde{v}_o$ and $\tilde{v}$ coincide.

1. Introduction

Let U be an open subset of $\mathbf{C}^n$. A weight v on U is a bounded, strictly positive continuous real valued function on U. We use $\mathcal{H}_v(U)$ to denote the space of all holomorphic functions f on U which have the property that $\|f\|_v := \sup_{z\in U} v(z)|f(z)| < \infty$. Endowed with $\|\cdot\|_v$, $\mathcal{H}_v(U)$ becomes a Banach space. A separable subspace of $\mathcal{H}_v(U)$ is got by considering all f in $\mathcal{H}_v(U)$ with the property that $|f(z)|v(z)$ converges to 0 as z converges to the boundary of U, i.e. given $\epsilon > 0$ there is a compact subset K of U such that $v(z)|f(z)| < \epsilon$ for z in $U \setminus K$. This subspace is denoted by $\mathcal{H}_{v_o}(U)$. Thus $\mathcal{H}_v(U)$ may be regarded as all holomorphic functions on U which satisfy a growth condition of order $O(1/v(z))$ while $\mathcal{H}_{v_o}(U)$ are those functions with a growth rate of order $o(1/v(z))$.

Bonet and Wolf [9] show that $\mathcal{H}_{v_o}(U)$ is almost isometrically isomorphic to a subspace of c_o and observe that the proof can be adapted to show that $\mathcal{H}_v(U)$ is almost isometrically isomorphic to a subspace of ℓ_∞. A weight v on a balanced domain U is said to be radial if $v(z) = v(\lambda z)$ whenever $|\lambda| = 1$. Recently, Lusky, [19], shows that when v is a non-increasing radial weight on the unit disc, Δ, which vanishes on the boundary, we can be more specific about Bonet and Wolf's result. He shows that in this case $\mathcal{H}_{v_o}(\Delta)$ is either isomorphic to c_o or $\left(\sum_n H_n\right)_o$ while $\mathcal{H}_v(\Delta)$ is isomorphic to ℓ_∞ or $\mathcal{H}^\infty(\Delta)$. Here H_n denotes the span of the functions $\{1, z, z^2, \ldots, z^n\}$ on the unit circle endowed with the L_∞-norm. In [10, Corollary 17] the authors prove that for radial weights which vanish on the boundary $\mathcal{H}_{v_o}(U)$ is never isometrically isomorphic to c_o.

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These results lead us naturally to the question *'Is $\mathcal{H}_v(U)$ isometrically isomorphic, in a canonical way, to the bidual of $\mathcal{H}_{v_o}(U)$?'* This problem is known as the Biduality Problem. In next section we will explain more precisely what we mean by being isometrically isomorphic in a *canonical way.*

The Biduality Problem has been considered for as long as spaces of weighted holomorphic functions have been studied. The first reference to the Biduality Problem seems to be in a 1968 paper of Clunie and Köveri (see [14]). Rubel and Shields [22] show that $\mathcal{H}_v(\Delta)$ is isometrically isomorphic to the bidual of $\mathcal{H}_{v_o}(\Delta)$ in the case where v is a decreasing radial weight on the unit disc Δ, which vanishes on the boundary of Δ. A positive solution was also given by Anderson and Duncan [3] in the case of radial weights v on the complex plane which satisfy the condition $\lim_{r\to\infty}\frac{v(r)}{v(tr)}=0$ for all t in $(0,1)$. This later condition ensures that $\mathcal{H}_{v_o}(\mathbf{C})$ contains all polynomials. Some examples of non-radial weights where $\mathcal{H}_{v_o}(\mathbf{C})''=\mathcal{H}_v(\mathbf{C})$ are also given in [3]. In 1993 Bierstedt and Summers [8] showed that a necessary and sufficient condition for a positive solution to the Biduality Problem is that the closed unit ball of $\mathcal{H}_{v_o}(U)$ is dense in the closed unit ball of $\mathcal{H}_v(U)$ for the compact-open topology. This allowed them to recover both the results of Rubel and Shields [22], along with those of Anderson and Duncan [3]. Positive results on the Biduality Problem for the inductive limits $\mathcal{H}V_0(G)$ and $\mathcal{H}V(G)$ with G open in $\mathbb{C}^n$ and $V=(v_n)_n$ a sequence of weights on G can be found in [6]. In [7] the following result appears: A weight v gives a positive solution to the Biduality Problem if and only if its associated weights, $\tilde{v}$ and $\tilde{v}_o$, coincide (see Section 2 for the definition of associated weights). However, as later pointed out by the authors, the argument for one direction of the equivalence is incorrect and the question as to whether the identity $\tilde{v}=\tilde{v}_o$ implies a positive solution to the Biduality Problem remains open.

In this paper we establish the forementioned implication in the M-ideal context. More precisely, we show that when v is a weight on an open subset U of $\mathbf{C}^n$ then $\mathcal{H}_v(U)$ is canonically the bidual of $\mathcal{H}_{v_o}(U)$ if and only if $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ and we have equality of the associated weights $\tilde{v}_o$ and $\tilde{v}$. Moreover, we prove that the condition on the coincidence of the associated weights can be replaced by a geometric condition on $\mathcal{H}_{v_o}(U)$ concerning the v-boundary of U. The v-boundary was introduced in [10] by the authors when studying the extreme points of the closed unit ball of the dual $\mathcal{H}_{v_o}(U)'$. In [10] a long list of weights whose v-boundary coincides with the set U was provided. Among them, most of the standard weights appear. As a consequence, for all these weights we prove that the Biduality Problem turns out to be equivalent to the problem of knowing whether $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$.

2. Preliminaries

Let us start this section by detailing what we mean by $\mathcal{H}_v(U)$ being isometrically isomorphic to the bidual of $\mathcal{H}_{v_o}(U)$ in a *canonical way.* Let U be an open subset of $\mathbf{C}^n$ and v be a weight on U. We let $\mathcal{G}_v(U)$ denote the space of all linear functionals on $\mathcal{H}_v(U)$ whose restriction to the closed unit ball of $\mathcal{H}_v(U)$ is continuous for the compact open topology, τ_o. We endow $\mathcal{G}_v(U)$ with the topology inherited from $\mathcal{H}_v(U)'$. Denote by $\tilde{R}$ the restriction mapping $\tilde{R}\colon \mathcal{G}_v(U)\to\mathcal{H}_{v_o}(U)'$ given by $\tilde{R}(\phi)=\phi|_{\mathcal{H}_{v_o}(U)}$. Denote by $\tilde{\Phi}\colon \mathcal{H}_v(U) \to \mathcal{G}_v(U)'$ the evaluation mapping $\tilde{\Phi}(f)(F) = F(f)$ for $f \in \mathcal{H}_v(U)$ and $F \in \mathcal{G}_v(U)$. By [8, Theorem 1.1], $\tilde{\Phi}$ is always an isomorphism, while $\tilde{R}$ is an isometric isomorphism if and only if the closed unit ball of $\mathcal{H}_{v_o}(U)$ is dense in the closed unit ball of $\mathcal{H}_v(U)$ for the compact open topology. Moreover, $\tilde{R}$ is a topological isomorphism if and only if a multiple of the closed unit ball of $\mathcal{H}_{v_o}(U)$ is dense in the closed unit ball of $\mathcal{H}_v(U)$ for the compact open topology. When we say that $\mathcal{H}_{v_o}(U)''$ is canonically isometrically (topologically) isomorphic to $\mathcal{H}_v(U)$, we will mean that the mapping $\tilde{\Phi}^{-1} \circ \tilde{R}^t$ is an isometric (a topological) isomorphism from $\mathcal{H}_{v_o}(U)''$ onto $\mathcal{H}_v(U)$.

It is shown in [10, 13] that the extreme points of $B_{\mathcal{H}_{v_o}(U)'}$ are contained in the set $\{\lambda v(z)\delta_z : z \in U, |\lambda| = 1\}$. Further, $v(z)\delta_z$ is an extreme point of $B_{\mathcal{H}_{v_o}(U)'}$ if and only if $\lambda v(z)\delta_z$ is an extreme point of $B_{\mathcal{H}_{v_o}(U)'}$ for every λ in $\mathbf{C}$ of modulus 1. Thus there is a distinguished subset of U, which we denote by $\mathcal{B}_v(U)$, such that the extreme points of $B_{\mathcal{H}_{v_o}(U)'}$ are given by $\{\lambda v(z)\delta_z : z \in \mathcal{B}_v(U), |\lambda| = 1\}$. We call $\mathcal{B}_v(U)$ the v-boundary of U. We say that $\mathcal{B}_v(U)$ is a determining set for $\mathcal{H}_v(U)$ if $f = g$ whenever f, $g \in \mathcal{H}_v(U)$ are such that $f(x) = g(x)$ for all $x \in \mathcal{B}_v(U)$.

Weights with the property that $\mathcal{B}_v(U) = U$ are said to be *complete* and are studied in [11]. In [10] it was shown that given $\alpha > 0$, $\beta \geq 1$, each of the following weights on the open unit ball of $\mathbf{C}^n$ is complete:

(a) $v_{\alpha,\beta}(z) = (1 - \|z\|^\beta)^\alpha$.
(b) $w_{\alpha,\beta}(z) = e^{\frac{-\alpha}{1-\|z\|^\beta}}$.
(c) $v(z) = (\log(2 - \|z\|))^\alpha$.
(d) $v(z) = (1 - \log(1 - \|z\|))^{-\alpha}$.
(e) $v(z) = \cos\left(\frac{\pi}{2}\|z\|\right)$.
(f) $v(z) = \cos^{-1}\|z\|$.
(g) Finite products of the examples in (a) to (f).

See [10], [11] and [12] for a more detailed discussion on the properties of the v-boundary.

In 1972 Alfsen and Effros, [2], introduced the concept of an M-ideal. Their motivation was to recapture much of the structure provided in a C*-algebra setting by two-sided ideals in a general Banach space setting. A year later the concept of M-ideal arose in the work of Ando, [4] under the name of splitting convex set. Given a Banach space E let B_E denote its closed unit ball. A subspace J of a Banach space E is said to be an M-ideal in E if $E' = J' \oplus_1 J^\perp$. Necessary and sufficient conditions for a subspace J of a Banach space E to be a M-ideal were found by Alfsen and Effros, [2], Alfsen, [1] and Behrends, [5]. Perhaps the most useful and the most famous is the 3-ball property. A subspace J of a Banach space E is said to satisfy the (restricted) 3-ball property if for all y_1, y_2, y_3 in B_J, all x in B_E and all $\epsilon > 0$ there is y in J such that

$$\|x + y_i - y\| \leq 1 + \epsilon$$

for $i = 1, 2, 3$.

To understand the connection between the Biduality Problem and the M-ideals we need to understand the concepts of the associated weights $\tilde{v}_o$ and $\tilde{v}$.

Given a weight $v\colon U \to \mathbf{R}$ we define $w\colon U \to \mathbf{R}$ by $w(z) = 1/v(z)$. The closed unit ball of $\mathcal{H}_{v_o}(U)$ is $\{f \in \mathcal{H}_{v_o}(U) : |f(z)| \le w(z), \text{ for all } z \in U\}$ whereas the closed unit ball of $\mathcal{H}_v(U)$ is $\{f \in \mathcal{H}_v(U) : |f(z)| \le w(z), \text{ for all } z \in U\}$. We define $\tilde{w}_o\colon U \to \mathbf{R}$ by

$$\tilde{w}_o(z) = \sup\{|f(z)| : f \in B_{\mathcal{H}_{v_o}(U)}\}$$

and $\tilde{w}\colon U \to \mathbf{R}$ by

$$\tilde{w}(z) = \sup\{|f(z)| : f \in B_{\mathcal{H}_v(U)}\}.$$

Let $\tilde{v}_o(z) = 1/\tilde{w}_o(z)$ and $\tilde{v}(z) = 1/\tilde{w}(z)$. Then $\tilde{v}_o$ and $\tilde{v}$ are weights on U which satisfy $0 < v \le \tilde{v} \le \tilde{v}_o$. Furthermore, Hadamard's Three Circles Theorem implies that $\log \tilde{w}_o$ and $\log \tilde{w}$ are convex functions of $\log |z|$. When v is a continuous decreasing radial weight on the unit disc [21, Theorem 2.6.6] tells us that $\log \tilde{w}_o$ and $\log \tilde{w}$ are subharmonic.

3. M-ideals and the Biduality Problem

The main aim of this paper is to prove the following equivalences:

Theorem 3.1. *Let U be an open subset of $\mathbf{C}^n$ and $v\colon U \to \mathbf{R}$ be a weight. Then the following are equivalent:*

(a) $\mathcal{H}_{v_o}(U)''$ *is canonically isometrically isomorphic to* $\mathcal{H}_v(U)$,
(b) $\mathcal{H}_{v_o}(U)''$ *is canonically topologically isomorphic to* $\mathcal{H}_v(U)$,
(c) $\mathcal{H}_{v_o}(U)$ *is an M-ideal in* $\mathcal{H}_v(U)$ *and* $\tilde{v}_o = \tilde{v}$ *on* U,
(d) $\mathcal{H}_{v_o}(U)$ *is an M-ideal in* $\mathcal{H}_v(U)$ *and* $\mathcal{B}_v(U)$ *is a determining set for* $\mathcal{H}_v(U)$.

To accomplish our task, we first need to prove several results which are of independent interest.

We consider the subset $\mathcal{F}_v(U)$ of $\mathcal{G}_v(U)$ given by

$$\mathcal{F}_v(U) = \{\phi \in \mathcal{H}_v(U)' : \phi|_{B_{\mathcal{H}_v(U)}} \text{ is } \tau_o\text{-continuous and } \|\phi\| = \|\phi|_{\mathcal{H}_{v_o}(U)}\|\}.$$

When $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$, for each ψ in $\mathcal{H}_{v_o}(U)'$ there is a unique $\tilde{\psi}$ in $\mathcal{H}_v(U)'$ with $\tilde{\psi}|_{\mathcal{H}_{v_o}(U)} = \psi$ and $\|\tilde{\psi}\| = \|\psi\|$. Hence $\mathcal{F}_v(U)$ is a vector subspace of $\mathcal{G}_v(U)$. Again, we endow $\mathcal{F}_v(U)$ with the topology induced from $\mathcal{H}_v(U)'$.

Propositions 3.2 and 3.3 are motivated by [8] and [20]. The key to the proof of these propositions is the fact that since $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ the restriction mapping R from $\mathcal{F}_v(U)$ onto $\mathcal{H}_{v_o}(U)'$, $R(\phi) = \phi|_{\mathcal{H}_{v_o}(U)}$, is an isometry and, in particular, injective.

Proposition 3.2. *Let U be an open subset of $\mathbf{C}^n$ and $v\colon U \to \mathbf{R}$ be a weight. If $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ then $\mathcal{F}_v(U)$ is isometrically isomorphic to $\mathcal{H}_{v_o}(U)'$.*

Proof. Consider the restriction mapping $R\colon \mathcal{F}_v(U) \to \mathcal{H}_{v_o}(U)'$, $R(\phi) = \phi|_{\mathcal{H}_{v_o}(U)}$. By definition of $\mathcal{F}_v(U)$, R is an isometry and in particular injective. Given ϕ in $\mathcal{H}_{v_o}(U)'$, by the Riesz Representation Theorem, there exists a regular Borel measure μ on U

with $\|\phi\| = \|\mu\|$ such that

$$\phi(f) = \int_U v(z)f(z)\,d\mu(z)$$

for all f in $\mathcal{H}_{v_o}(U)$. Define $\tilde{\phi}$ on $\mathcal{H}_v(U)$ by

$$\tilde{\phi}(f) = \int_U v(z)f(z)\,d\mu(z).$$

Then

$$\|\tilde{\phi}\| = \sup_{\|f\|_v \leq 1, f \in \mathcal{H}_v(U)} \left| \int_U v(z)f(z)\,d\mu(z) \right| \geq \|\phi\| = \|\mu\|.$$

Conversely, given f in $B_{\mathcal{H}_v(U)}$

$$|\tilde{\phi}(f)| = \left| \int_U v(z)f(z)\,d\mu(z) \right| \leq \sup_{z \in U} v(z)|f(z)|\|\mu\|.$$

Thus $\|\tilde{\phi}\| \leq \|\mu\|$. Hence $\|\tilde{\phi}\| = \|\mu\| = \|\phi\|$. The inner regularity of μ means that $\tilde{\phi}|_{B_{\mathcal{H}_v(U)}}$ is τ_o-continuous. Hence R is an isometric isomorphism of $\mathcal{F}_v(U)$ onto $\mathcal{H}_{v_o}(U)'$. □

Let us now see that when $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ we can replace $\mathcal{G}_v(U)$ with $\mathcal{F}_v(U)$ in the construction of the predual of $\mathcal{H}_v(U)$.

Proposition 3.3. *Let U be an open subset of $\mathbf{C}^n$ and $v\colon U \to \mathbf{R}$ be a weight. If $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ and $\tilde{v}_o = \tilde{v}$ on U then $\mathcal{H}_v(U)$ is isometrically isomorphic to $\mathcal{F}_v(U)'$.*

Proof. Consider the evaluation map $e\colon \mathcal{H}_v(U) \to \mathcal{F}_v(U)'$ given by $e(f) = e_f$ where $e_f(\phi) = \phi(f)$ for f in $\mathcal{H}_v(U)$ and ϕ in $\mathcal{F}_v(U)$. As

$$|e_f(\phi) - e_g(\phi)| = |\phi(f) - \phi(g)| = |\phi(f-g)| \leq \|\phi\|\|f-g\|_v$$

it follows that $\|e_f - e_g\| \leq \|f-g\|_v$ and therefore e is continuous.

We claim that the restriction of e to the closed unit ball of $\mathcal{H}_v(U)$ is continuous when $B_{\mathcal{H}_v(U)}$ is endowed with the compact open topology, τ_o, and $\mathcal{F}_v(U)'$ is given the $\sigma(\mathcal{F}_v(U)', \mathcal{F}_v(U))$-topology. To see this let $(f_i)_{i \in I}$ and f belong to the closed unit ball of $\mathcal{H}_v(U)$ with f_i converging to f in the compact-open topology. Since each ϕ in $\mathcal{F}_v(U)$ is τ_o-continuous when restricted to the unit ball of $\mathcal{H}_v(U)$ we have that $e_{f_i}(\phi) = \phi(f_i)$ converges to $e_f(\phi) = \phi(f)$ for all ϕ in $\mathcal{F}_v(U)$ hence our claim is proven.

To see that e is injective, suppose that f, g belong to $\mathcal{H}_v(U)$ with $e_f = e_g$. For each x in U, the function $\delta_x\colon \mathcal{H}_v(U) \to \mathbf{C}$, defined by $\delta_x(h) = h(x)$, is τ_o-continuous and

$$\|\delta_x\| = \frac{1}{\tilde{v}(x)} = \frac{1}{\tilde{v}_o(x)} = \|\delta_x|_{\mathcal{H}_{v_o}(U)}\|.$$

Hence each δ_x belongs to $\mathcal{F}_v(U)$ and $f(x) = e_f(\delta_x) = e_g(\delta_x) = g(x)$ for all x in U. Thus $f = g$ and e is injective.

As $B_{\mathcal{H}_v(U)}$ is τ_o-compact and e is linear $\tau_o - \sigma(\mathcal{F}_v(U)', \mathcal{F}_v(U))$ continuous when restricted to $B_{\mathcal{H}_v(U)}$, $e(B_{\mathcal{H}_v(U)})$ is $\sigma(\mathcal{F}_v(U)', \mathcal{F}_v(U))$-compact and convex. By the Bipolar Theorem

$$e(B_{\mathcal{H}_v(U)}) = \overline{\Gamma}^{\sigma(\mathcal{F}_v(U)', \mathcal{F}_v(U))}(e(B_{\mathcal{H}_v(U)})) = e(B_{\mathcal{H}_v(U)})^{\circ\circ}$$

with respect to the duality between $\mathcal{F}_v(U)'$ and $\mathcal{F}_v(U)$. Since

$$e(B_{\mathcal{H}_v(U)})^{\circ} = \{\phi \in \mathcal{F}_v(U) : |\phi(f)| \leq 1 \text{ for all } f \in B_{\mathcal{H}_v(U)}\} = B_{\mathcal{F}_v(U)},$$

$e(B_{\mathcal{H}_v(U)}) = e(B_{\mathcal{H}_v(U)})^{\circ\circ}$ is the closed unit ball of $\mathcal{F}_v(U)'$ and therefore e is surjective. This proves that e is an isometric isomorphism of $\mathcal{H}_v(U)$ onto $\mathcal{F}_v(U)'$. □

We can replace the condition that $\tilde{v}_o = \tilde{v}$ on U with the condition that the v-boundary is a determining set for $\mathcal{H}_v(U)$. Using [11, Proposition 24] we obtain the following proposition. (The requirement in the statement of [11, Proposition 24] that v tends to 0 on the boundary of U may be removed.)

Proposition 3.4. *Let U be an open subset of $\mathbf{C}^n$ and $v\colon U \to \mathbf{R}$ be a weight. Suppose that $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ and $\mathcal{B}_v(U)$ is a determining set for $\mathcal{H}_v(U)$. Then $\mathcal{H}_v(U)$ is isometrically isomorphic to $\mathcal{F}_v(U)'$.*

Let us investigate the 'converse' of Proposition 3.3. Since, however, we do not know that $\mathcal{F}_v(U)$ is necessarily a vector space in general we will work with $\mathcal{G}_v(U)$ rather than $\mathcal{F}_v(U)$.

Proposition 3.5. *Let U be an open subset of $\mathbf{C}^n$ and $v\colon U \to \mathbf{R}$ be a weight. If $\tilde{R}\colon \mathcal{G}_v(U) \to \mathcal{H}_{v_o}(U)'$ is a topological isomorphism then $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$.*

Proof. We first observe that [8, Remarks 1] implies that we can find $\lambda > 0$ so that $B_{\mathcal{H}_v(U)}$ is contained in the τ_o closure of $\lambda B_{\mathcal{H}_{v_o}(U)}$. We adapt the proof outlined in [23]. See also [18]. We assume without loss of generality that $v(z) \leq 1$ on U. Let f belong to $B_{\mathcal{H}_v(U)}$, g_1, g_2, g_3 belong to $B_{\mathcal{H}_{v_o}(U)}$ and $\epsilon > 0$. Let k be the smallest positive integer with $k \geq \max\{2, (\lambda+1)/\epsilon\}$. We begin by choosing a compact subset K_1 of U such that $v(z)|g_i(z)| \leq \epsilon$ for $i = 1, 2, 3$ and all z in $U \setminus K_1$. We can find a function f_1 in $\lambda B_{\mathcal{H}_{v_o}(U)}$ such that $\sup_{z\in K_1} |f(z) - f_1(z)| \leq \epsilon$. As f_1 belongs to $\mathcal{H}_{v_o}(U)$ we can find a compact subset K_2 of U with $K_2 \supset K_1$ such that $v(z)|f_1(z)| \leq \epsilon$ for all z in $U \setminus K_2$. Next, we choose f_2 in $\lambda B_{\mathcal{H}_{v_o}(U)}$ such that $\sup_{z\in K_2} |f(z) - f_2(z)| \leq \epsilon$. We continue this process, to obtain an increasing chain of k compact sets $K_1 \subset K_2 \subset \cdots \subset K_k$ and functions $(f_j)_{j=1}^k$ in $\lambda B_{\mathcal{H}_{v_o}(U)}$ such that $v(z)|f_l(z)| \leq \epsilon$ for all z in $U \setminus K_{l+1}$, $l = 1, \ldots, k-1$, and $\sup_{z\in K_l} |f(z) - f_l(z)| \leq \epsilon$, $l = 1, \ldots, k$. Then, for $1 \leq l \leq k$, we have that $v(z)|f(z) - f_l(z)| \leq 1 + \epsilon$ for all z in U with the possible exception of those z in $K_{l+1} \setminus K_l$. Let $h = \frac{1}{k}\sum_{j=1}^k f_j$ and consider z in U. If $z \in K_1$ then, since $|f(z) - f_l(z)| \leq \epsilon$ for $1 \leq l \leq k$, we have

$$v(z)|f(z) - g_i(z) - h(z)| \leq 1 + \epsilon$$

for $i = 1, 2, 3$. For z in $K_{l+1} \setminus K_l$ we have $v(z)|g_i(z)| \leq \epsilon$ for $i = 1, 2, 3$ and $v(z)|f(z) - f_j(z)| \leq 1+\epsilon$ for all $1 \leq j \leq k$ with the possible exception of l. As $\|f - f_l\|_v \leq \lambda + 1$, for z in $K_{l+1} \setminus K_l$ and $i = 1, 2, 3$ we have

$$\begin{aligned}
&v(z)|f(z) - g_i(z) - h(z)| \\
&\leq \sum_{\substack{j=1 \\ j\neq l}}^{k} \frac{1}{k} v(z)|f(z) - f_j(z)| + \frac{1}{k} v(z)|f(z) - f_l(z)| + v(z)|g_i(z)| \\
&\leq \frac{k-1}{k}(1+\epsilon) + \frac{1}{k}(\lambda + 1) + \epsilon \\
&\leq 1 + 3\epsilon,
\end{aligned}$$

Thus, as $\mathcal{H}_{v_o}(U)$, $\mathcal{H}_v(U)$ satisfies the 3-ball property, $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$. □

Now we are ready to prove our main theorem:

Proof of Theorem 3.1. Let us prove that $(b) \Rightarrow (a)$. Since, by [8, Theorem 1.1 (a)], $\tilde{\Phi}\colon \mathcal{H}_v(U) \to \mathcal{G}_v(U)'$ is always an isometric isomorphism it follows that $\mathcal{H}_{v_o}(U)''$ is canonically topologically isomorphic to $\mathcal{H}_v(U)$ if and only if $\tilde{R}\colon \mathcal{G}_v(U) \to \mathcal{H}_{v_o}(U)'$ is a topological isomorphism. It now follows from Proposition 3.5 that (b) implies that $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$. The proof of Proposition 3.2 gives us that R is an isometric isomorphism of $\mathcal{F}_v(U)$ onto $\mathcal{H}_{v_o}(U)'$. As both R and $\tilde{R}$ are the restriction mappings and $\mathcal{F}_v(U) \subseteq \mathcal{G}_v(U)$ we see that $\mathcal{G}_v(U) = \mathcal{F}_v(U)$ and $\tilde{R} = R$ proving that $\tilde{R}$ is also an isometric isomorphism of $\mathcal{G}_v(U)$ onto $\mathcal{H}_{v_o}(U)'$. Therefore (a) holds.

Proposition 3.5 tells us that (a) implies the first assertion in both (c) and (d). Also, if (a) holds then it follows from [7, Theorem 1.13] that $\tilde{v} = \tilde{v}_o$ on U. In addition when (a) holds, $\mathcal{H}_{v_o}(U)'$ is the closed linear span of $\{v(z)\delta_z : z \in U\}$ and hence is a separable dual space. It therefore has the Radon-Nikodým property. Thus, by [15, 6.20a], the extreme points of the unit ball of $\mathcal{H}_{v_o}(U)'$ are a determining set of $\mathcal{H}_v(U)$. However, this set can, by definition, be identified with $\mathcal{B}_v(U)$ and then (a) implies both (c) and (d).

Next suppose that (c) or (d) holds. By the proof of Propositions 3.3 and 3.4 the mapping $e\colon \mathcal{H}_v(U) \to \mathcal{F}_v(U)'$ is an isometric isomorphism. Since $\tilde{\Phi}\colon \mathcal{H}_v(U) \to \mathcal{G}_v(U)'$ is always an isometric isomorphism we have that $\tilde{\Phi} \circ e^{-1}$ is an isometric isomorphism from $\mathcal{F}_v(U)'$ onto $\mathcal{G}_v(U)'$. By Proposition 3.2 $\mathcal{F}_v(U)$ is isometrically isomorphic to $\mathcal{H}_{v_o}(U)'$. As $\mathcal{H}_{v_o}(U)'$ is a separable dual space it has the Radon-Nikodým property. Hence $\mathcal{F}_v(U)$ has the Radon-Nikodým property. Applying [16, Theorem 10] we see that $\mathcal{F}_v(U)$ is the unique isometric predual of its dual. Thus $(\tilde{\Phi} \circ e^{-1})^t$ is an isometric isomorphism from $i_{\mathcal{G}_v(U)}(\mathcal{G}_v(U))$ onto $i_{\mathcal{F}_v(U)}(\mathcal{F}_v(U))$, where i_X is the canonical inclusion of a Banach space X into its bidual. For f in $\mathcal{H}_v(U)$ and ϕ in $\mathcal{G}_v(U)$ we have

$$\begin{aligned}
(\tilde{\Phi} \circ e^{-1})^t(i_{\mathcal{G}_v(U)}(\phi))(e(f)) &= (i_{\mathcal{G}_v(U)}(\phi) \circ \tilde{\Phi} \circ e^{-1})(e(f)) = i_{\mathcal{G}_v(U)}(\phi)(\tilde{\Phi}(f)) \\
&= \tilde{\Phi}(f)(\phi) = \phi(f).
\end{aligned}$$

Hence $\mathcal{G}_v(U) = \mathcal{F}_v(U)$, $\tilde{R} = R$ and $\tilde{\Phi} = e$.

By Proposition 3.2 it follows that $\tilde{R}$ is an isometric isomorphism of $\mathcal{G}_v(U)$ onto $\mathcal{H}_{v_o}(U)'$. Applying Proposition 3.3 in the case of (c) or Proposition 3.4 in the case of (d) we have that $\tilde{\Phi}$ is an isometric isomorphism of $\mathcal{H}_v(U)$ onto $\mathcal{F}_v(U)'$. This gives us that $\mathcal{H}_v(U)$ is isometrically isomorphic in a canonical way to $\mathcal{H}_{v_o}(U)''$ which is (a). □

The largest and most important context in which M-ideals are studied is the situation where a Banach space is an M-ideal in its bidual. In [23] Dirk Werner proved that when v is a radial weight on the open unit disc, Δ, which converges to 0 on the boundary of Δ then $\mathcal{H}_{v_o}(\Delta)$ is an M-ideal in $\mathcal{H}_v(\Delta)$. Our next result shows that, for arbitrary weights v, $\mathcal{H}_{v_o}(U)$ is always an M-ideal in its bidual. Therefore, Werner's example follows from the fact that $\mathcal{H}_{v_o}(\Delta)''$ is canonically isometrically isomorphic to $\mathcal{H}_v(\Delta)$ whenever v is a radial weight on the open unit disc Δ (see [22]).

Combining the main result in [9] and the fact that a Banach space $(1+\epsilon)$-isomorphic to a subspace of c_0 must be an M-ideal in its bidual we obtain the following result.

Proposition 3.6. *Let U be an open subset of $\mathbf{C}^n$ and $v\colon U \to \mathbf{R}$ be a weight. Then $\mathcal{H}_{v_o}(U)$ is an M-ideal in its bidual.*

Proof. Let y_1, y_2, y_3 belong to $B_{\mathcal{H}_{v_o}(U)}$, x belong to $B_{\mathcal{H}_{v_o}(U)''}$ and $\epsilon > 0$. It follows from [9, Theorem 1] that we can find a subspace X_ϵ of c_o and an isomorphism $T\colon \mathcal{H}_{v_o}(U) \to X_\epsilon$ such that $\|T\| = 1$ and $\|T^{-1}\| \le 1+\epsilon$. Then $T(y_1), T(y_2), T(y_3)$ belong to the unit ball of X_ϵ and $T''(x)$ belongs to the unit ball of X_ϵ''. It follows from [18, Theorem III.1.6 (a)] that X_ϵ is an M-ideal in its bidual. Hence we can find y in X_ϵ'' such that $\|T''(x) + T(y_i) - y\| \le 1+\epsilon$ for $i = 1, 2, 3$ and therefore $\|x + y_i - T^{-1}(y)\| \le (1+\epsilon)^2$ for $i = 1, 2, 3$ proving that $\mathcal{H}_{v_o}(U)$ is an M-ideal in its bidual. □

As a consequence, a positive solution to the Biduality Problem would yield that $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ for any arbitrary weight v. Let us see that in most cases, the problem of knowing when $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ is indeed equivalent to the Biduality Problem. This follows from the fact that the condition of $\mathcal{B}_v(U)$ being a determining set for $\mathcal{H}_v(U)$ is trivially satisfied when v is complete.

Corollary 3.7. *Let U be an open subset of $\mathbf{C}^n$ and $v\colon U \to \mathbf{R}$ be a weight. If v is complete then $\mathcal{H}_v(U)$ is isometrically isomorphic in a canonical way to the bidual of $\mathcal{H}_{v_o}(U)$ if and only if $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$.*

We finish the paper with the following application that completes [12, Theorem 8].

Proposition 3.8. *Let U be an open subset of $\mathbf{C}^n$ and $v\colon U \to \mathbf{R}$ be a weight which converges to 0 on the boundary of U such that $\mathcal{H}_{v_o}(U)''$ is isometrically isomorphic in a canonical way to $\mathcal{H}_v(U)$. Let $T\colon \mathcal{H}_v(U) \to \mathcal{H}_v(U)$ be an isometric isomorphism, then there is a homeomorphism $\phi\colon \mathcal{B}_v(U) \to \mathcal{B}_v(U)$ and $h_\phi \in \mathcal{H}_{v_o}(U)$ such that*

$$T(f)(z) = h_\phi(z) f \circ \phi(z),$$

for all $f \in \mathcal{H}_v(U)$, $z \in \mathcal{B}_v(U)$. Moreover, if $\overset{\circ}{\mathcal{B}}_v(U)$ is non-empty then $\phi : \overset{\circ}{\mathcal{B}}_v(U) \to$

$\overset{\circ}{\mathcal{B}}_v(U)$ *is a biholomorphic mapping.*

Proof. By [17, Proposition 4.2] every isometry of $\mathcal{H}_v(U)$ is the bitranspose of an isometry of $\mathcal{H}_{v_o}(U)$ and hence the transpose of an isometry of $\mathcal{H}_{v_o}(U)'$. By the proof of [12, Theorem 1] all isometries, T^*, of $\mathcal{H}_{v_o}(U)'$ have the form

$$T^*(v(z)\delta_z) = \alpha(z)v(\phi(z))\delta_{\phi(z)}$$

for $z \in \mathcal{B}_v(U)$ where ϕ is an homeomorphism of $\mathcal{B}_v(U)$. From this observation we obtain that every isometry of $\mathcal{H}_v(U)$ has the form $T(f)(z) = h_\phi(z) f\circ\phi(z)$ for $h_\phi\colon U \to \mathbf{C}$. Finally when we take $f \equiv 1$, as T maps $\mathcal{H}_{v_o}(U)$ onto $\mathcal{H}_{v_o}(U)$ we see that $h_\phi \in \mathcal{H}_{v_o}(U)$. From the proof of [12, Theorem 1] it follows that ϕ is an automorphism of $\overset{\circ}{\mathcal{B}}_v(U)$ when $\overset{\circ}{\mathcal{B}}_v(U)$ is non-empty. □

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