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Comparing Various Methods of Resolving Differential Inclusions

M. A. Sychev*
Laboratory of Differential Equations and Related Problems in Analysis,
Sobolev Institute for Mathematics, Koptuyg Avenue 4, Novosibirsk 630090, Russia
masychev@math.nsc.ru

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In this paper we compare all existing methods of resolving the homogeneous differential inclusion problem. We emphasize that there is an elementary approach to construct solutions as limit of strongly convergent sequences of approximate solutions. We discuss which role plays an universal functional which measures maximal oscillations produced by admissible for the problem functions at a given one. We suggest a constructive way to generate stable solutions of the inclusions. Finally we prove higher regularity of solutions of the inclusions.

Keywords: Functional analysis, calculus of variations, PDE, differential inclusions, sequences obtained by almost maximal perturbations, stable solutions, higher regularity

1. PDE and Differential Inclusions

At the International Congress of Mathematicians in Zürich (1994) Šverák suggested to consider a problem of differential inclusions

$$Du \in K,\ u \in W^{1,1}(\Omega;\mathbf{R}^m),\ u\big|_{\partial\Omega} = f \quad (\Omega \subset \mathbf{R}^n) \tag{1}$$

as a new direction in PDE theory, [57]. He mentioned that there are at least two important PDE problems that can be reduced to the situation (1): 2×2 elliptic problems

$$\operatorname{div} L(Du) = 0 \tag{2}$$

since divergent-free fields are rotated gradients and, therefore, the problem can be rewritten as a differential inclusion for the case $m \times n = 4 \times 2$ and $m \times 2$ hyperbolic problems satisfying the entropy conditions.

Earlier theory of differential inclusions was developed basically for ODE, see [12], [1], [19, 20], [10].

Certain facts and tools to study (1) were obtained in context of studying variational models for solid-solid phase transitions, as this was suggested by Ball & James in [6, 7]. Šverák systematically exposed how that work is related to general problems (1)

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and communicated certain nontrivial results already obtained, see also [8]. Parallely the multi-dimensional case of differential inclusions was considered in Italian School, see very interesting papers by Cellina [13], Cellina & Perrotta [15], Bressan & Flores [11], see also a nice introduction to this subject by Cellina [14].

At the next International Congress (Berlin, 1998) an example of a pathological elliptic problem (2) was addressed, [36]. In this example the equation (2) is the Euler-Lagrange equation for a multiple integral with strictly quasiconvex integrand in the sense of Evans [27], however there exists a weak Lipschitz solution with everywhere oscillating gradients, contrary to the situation with strong local minimizers, [27], [32]. Later the result of Müller and Šverák was improved by Szekelyhidi [58] who showed that the same situation turned out to be possible for polyconvex L, which is the most studied case in elliptic nonconvex problems. One more example of similar irregular solutions was obtained for parabolic problems, [34]. Dacorogna & Marcellini used (1)-setting to study a.e. solutions of Hamilton-Jacobi equations and systems, [23, 24].

More recently differential inclusions were again involved

1) to study regularity of solutions of planar linear elliptic systems with measurable coefficients where higher integrability result is valid for gradients of weak solutions, i.e. of those which already have appropriately high exponent of integrability (though still below the energetic one), and to study very weak solutions with worse integrability of the gradients that is given by the exponent. Both the bounds of integrability for weak and very weak solutions were confirmed by examples, see Astala & Faraco & Szekelyhidi [3];

2) to construct pathological examples of weak solutions of the Euler equations describing the motion of an ideal incompressible fluid with pressure and velocity compactly supported. The examples by De Lellis & Szekelyhidi [22] are sharper than those earlier obtained by Scheffer [53] and by Shnirelman [54, 55].

Therefore the differential inclusion approach turned out to be enough productive one for studies PDE, at least in the sense of constructing pathological examples in various classes of problems. Since so a systematic development of solvability theory for problems (1) is required.

2. Elementary approach to solvability result. Renormalizations.

There were a number of methods suggested to obtain solvability of (1): an approach of "convex integration for Lipschitz functions" by Müller & Šverák [36–39], [33], [57], "martingale convergence" approach [§5, 42], "Baire category approach" by Dacorogna & Marcellini [23, 24] (originally "Baire category approach" was suggested by Cellina [12] to study ODE problems and was developed by De Blasi & Pianigiani, see e.g. [19, 20], when in multi-dimensional case it was first applied by Bressan & Flores [11]), "point of continuity" approach by Kirchheim [29, 30]. However a more elementary approach to study (1) turned out to be to apply classical theory of weak convergence developed long time ago in Functional Analysis and in the Calculus of Variations, as we suggested in [42, 43]. Though we probably were insufficiently explicit with exposition of this statement due to the way the theory of differential inclusions are continued to be commented by the experts, see [17, 18], [26], [30], [31], [§5, 42], and by those who follow them.

An optimal setting of the problem (1) was elaborated as follows:

Theorem 2.1 (Theorem 1.1 of [44, 45]). *Let U and K be bounded and compact subsets of $\mathbf{R}^{m\times n}$ respectively and assume that given $A \in U$ there exists a sequence of piece-wise affine functions $u_i \in l_A + W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ (l_A is an affine function with $Dl_A = A$) such that*

$$Du_i \in U \ a.e., \ \operatorname{dist}(Du_i, K) \to 0 \ \ in\ L^1. \tag{3}$$

Then for each piece-wise affine admissible function $f \in W^{1,\infty}(\Omega;\mathbf{R}^m)$, i.e. such that $Df \in U$ a.e., and for each $\epsilon > 0$ there exists a solution $u_\epsilon \in f + W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ of the problem (1) *with $||f - u_\epsilon||_{L^\infty} \le \epsilon$.*

Here we say that a function $f \in W^{1,1}(\Omega;\mathbf{R}^m)$ is **piece-wise affine** provided there exists a decomposition of Ω (we always assume that Ω is a bounded set with Lipschitz boundary) into open subsets Ω_j with $\operatorname{meas}(\partial\Omega_j) = 0$, $j \in \mathbf{N}$, with f is affine in each of these sets, and a set of zero measure.

The setting (3) of the problem (1) remained unchanged in last decade. An analogous setting for the nonhomogeneous case, i.e. when $K = K(x,u)$, $U = U(x,u)$, was suggested in [35]; however there is a case which does not fit this setting, see the interesting paper by Bertone & Cellina [5].

There was a statement by Dacorogna, see [17, 18], that Theorem 2.1 was first suggested by Dacorogna & Marcellini [24] and then proved in [44]. We find difficult to agree with this statement since the preprint version [45] of the paper [44] was published already in 1998. This preprint was handed to Dacorogna and to Marcellini at the conference in Pisa held in September 1998 and the results were discussed with them. In [24] a solvability result was proved with weakly extreme points of K instead of K, where the set of weakly extreme points could be in general larger than the original set, see [35, §5] for an example where K is countable and the set of weakly extreme points of K is continual.

The setting (3) suggests to consider approximate solutions of the problem (1) following the scheme

Definition 2.2 ([42, Def. 2.4], [43, Def. 1.1]). We say that a sequence of piece-wise affine functions $u_i \in W^{1,1}(\Omega;\mathbf{R}^m)$ is **obtained by perturbation** if for each element u_i of the sequence there exists an at most countable family of disjoint open subsets Ω_j^i of Ω, $j \in \mathbf{N}$, such that $\operatorname{meas}(\Omega \setminus \cup_j \Omega_j^i) = 0$ and for each $j \in \mathbf{N}$ we have: $\operatorname{meas}(\partial\Omega_j^i) = 0$, u_i is affine in Ω_j^i, $u_i = u_{i+k}$ on $\partial\Omega_j^i$ for all $k \in \mathbf{N}$.

In fact given a piece-wise affine function $u_i \in W^{1,1}(\Omega;\mathbf{R}^m)$ with $Du_i \in U$ a.e. we can decompose Ω into open sets Ω_j^i, $j \in \mathbf{N}$, and a set of zero measure with u_i affine in each Ω_j^i, $j \in \mathbf{N}$. Then we want to use (3) to define $u_{i+1} = u_i + \phi_j^i$ in Ω_j^i with

$\phi_j^i|_{\partial\Omega_j^i} = 0$, $j \in \mathbf{N}$, and with

$$||\operatorname{dist}(Du_{i+1}(\cdot), K)||_{L^1(\Omega_j^i)}/\operatorname{meas}\Omega_j^i \le 1/i. \tag{4}$$

The only point is that (4) holds for Ω but for Ω_j^i. However (3) could be applied to Ω_j^i as well since

Proposition 2.3. *Assume $\tilde{\Omega}, \Omega$ are open subsets of $\mathbf{R}^n$ with $\operatorname{meas}(\partial\Omega) = 0$ and with $0 \in \Omega$. Assume also $u \in W_0^{1,p}(\Omega; \mathbf{R}^m)$, $1 \le p \le \infty$, is a piece-wise affine function.*

Given $\epsilon > 0$ consider a decomposition of $\tilde{\Omega}$ into subsets $x_i + \epsilon_i\Omega$ with $\epsilon_i < \epsilon$, $i \in \mathbf{N}$, and a set of zero measure, see e.g. [52, p. 109]. Define $\tilde{u} \in W_0^{1,p}(\tilde{\Omega}; \mathbf{R}^m)$ as follows:

$$\tilde{u}(x) = \epsilon_i u((x - x_i)/\epsilon_i) \quad \text{for } x \in (x_i + \epsilon_i\Omega),\ i \in \mathbf{N},$$

$$\tilde{u}(x) = 0 \quad \text{otherwise.}$$

Then $\tilde{u} \in W_0^{1,p}(\tilde{\Omega}; \mathbf{R}^m)$ is a piece-wise affine function and for each continuous and nonnegative function $L : \mathbf{R}^{m\times n} \to \mathbf{R}$ and each $A \in \mathbf{R}^{m\times n}$ we have

$$\int_\Omega L(A + Du(x))dx/\operatorname{meas}\Omega = \int_{\tilde{\Omega}} L(A + D\tilde{u}(x))dx/\operatorname{meas}\tilde{\Omega},$$

$$||\tilde{u}||_{L^p(\tilde{\Omega})}/\operatorname{meas}\tilde{\Omega} \le \epsilon||u||_{L^p(\Omega)}/\operatorname{meas}\Omega.$$

The construction is a standard tool for producing weakly convergence sequences with prescribed distribution of the gradient and is widely used though we do not know to whom it should be attributed. However Bogolubov already used a version of this construction to prove his relaxation theorem, [9]. Recall that the relaxation theorem says that the formal lower semicontinuous envelope $\tilde{J}$ of an integral functional J, which is defined as

$$\tilde{J}(u) := \inf\left\{\liminf_{k\to\infty} J(u_k) : u_k \rightharpoonup u \text{ in } W^{1,1}\right\},$$

is itself an integral functional with the integrand L^c obtained as convexification of the original integrand with respect to the gradient variable (this was proved in the one-dimensional case under the condition of superlinear growth with respect to the gradient variable).

Note that the paper of Bogolubov was a serious event at that time and it was even awarded by a premium of the Bologna Academy of Sciences by recommendation of Tonelli. In our days the relaxation result is sometimes attributed to other authors, see [17, 18], though without indicating a reference.

Therefore (3) implies via (4) the existence of a sequence obtained by perturbation which satisfies (3).

The advantage is that sequences obtained by perturbation always converge strongly and, therefore, if in addition (3) holds then the limit function is a solution of the problem (1) and this proves Theorem 2.1. The fact of strong convergence of sequences obtained by perturbation relies on classical facts of Functional Analysis and of the Calculus of Variations.

Fact 2.4. *Let $L^p(\Omega; \mathbf{R}^m)$ be a space of measurable functions $g : \Omega \to \mathbf{R}^m$ with $J(g) := \int_\Omega |g(x)|^p dx < \infty$, $1 < p < \infty$. Let g_k be a sequence bounded in L^p-norm, where $||g||_{L^p} := J(g)^{1/p}$. Then there exists a subsequence (not relabeled) which converge weakly to g_∞. In this case $\liminf_{k\to\infty} ||g_k||_{L^p} \geq ||g_\infty||_{L^p}$. In case $||g_k||_{L^p} \to ||g_\infty||_{L^p}$ we also have $||g_k - g||_{L^p} \to 0$, $k \to \infty$.*

As usual the weak convergence means convergence of the values of linear continuous functionals, where each such functional has a representation $l(g) = \int_\Omega g(x)f(x)dx$, with $f \in L^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$. See e.g. the book [56] of S. L. Sobolev.

An extended version of this fact in context of Calculus of Variations is

Fact 2.5. *Let $L : \mathbf{R}^{m\times n} \to \mathbf{R}$ be a strictly convex function (called integrand). Let $g_k \rightharpoonup g$ in L^1 ("$\rightharpoonup$" means the weak convergence and we assume $J(g) := \infty$ if $L(g) \notin L^1$). Then*

$$\liminf_{k\to\infty} J(g_k) \geq J(g).$$

In case $J(g_k) \to J(g) < \infty$ we also have

$$g_k \to g, \quad L(g_k) \to L(g) \quad \text{in } L^1.$$

This result was proved first by Bogolubov [9] in the one-dimensional case; at those times Functional Analysis and Calculus of Variations were developing parallely. For the multi-dimensional version of this result (at the time when the theory of Sobolev spaces was established) see e.g. the paper [40] of Reshetnyak and see [46] for a characterization of this property at a given function.

We use Fact 2.4 to prove

Theorem 2.6. *Assume that a sequence of piece-wise affine functions $u_i : \Omega \to \mathbf{R}^m$ is obtained by perturbation. Assume also that $\{u_i, i \in \mathbf{N}\}$ is bounded in $W^{1,p}(\Omega; \mathbf{R}^m)$, $1 < p < \infty$. Then the sequence u_i converges strongly in $W^{1,p}(\Omega; \mathbf{R}^m)$, i.e. $u_i \to u_\infty$ in $W^{1,p}(\Omega; \mathbf{R}^m)$ as $i \to \infty$.*

Proof. There exists a subsequence u_i (not relabeled) which converges weakly in $W^{1,p}$ to $u_\infty \in W^{1,p}$ (since weak convergence of the gradients in L^p holds due to the Fact 2.4). This is again a sequence obtained by perturbation. Note that the L^p-norm of the gradients Du_i increases. In fact given $i \in \mathbf{N}$ there exists a decomposition of Ω into disjoint open sets Ω^i_j, $j \in \mathbf{N}$, and a set of zero measure, where $u_i : \Omega^i_j \to \mathbf{R}^m$ is an affine function, $j \in \mathbf{N}$, and $u_{i+k} = u_k$ on $\partial\Omega^i_j$, $j, k \in \mathbf{N}$. Therefore by the Jensen inequality we have $J(u_{i+k}) \geq J(u_i)$, $k \in \mathbf{N}$, where $J(u) = \int_\Omega |Du(x)|^p dx$, as well as $J(u_\infty) \geq J(u_i)$ since this holds in each particular set Ω^i_j, $j \in \mathbf{N}$. Because of the lower semicontinuity result we also have $J(u_i) \to J(u_\infty)$ and then $||Du_i - Du_\infty||_{L^p} \to 0$ (cf. Fact 2.4) and $||u_i - u_\infty||_{W^{1,p}} \to 0$.

Since arbitrary subsequence of the original sequence contains a subsequence converging in $W^{1,p}$ the sequence itself converges in this norm. □

Remark. The same arguments show that sequence u_i converges in $W^{1,1}$ provided it is weakly precompact in $W^{1,1}$ (use Fact 2.5 with appropriate L which has superlinear growth). In [42, Theorem 2.5] ([43, Theorem 1.2]) we proved this by contradiction showing that for any integral functional J with strictly convex integrand L such that $J(u_i)$, $i \in \mathbf{N}$, are equa-bounded lack of the strong convergence leads to $J(u_i) \to \infty$, $i \to \infty$, which is a contradiction.

In case the sequence u_i is only bounded in $W^{1,1}$ convergence of the gradients in measure still holds, see [47].

The idea of the method of "convex integration for Lipschitz functions" by Müller & Šverák was: given a sequence obtained by perturbation we renormalize it via Proposition 2.3 in order to obtain strong convergence of gradients via control of L^∞-norm of the perturbations by modulus of continuity of Du_i, see [37], [44, 45] for two different ways to do this. This could be formalized in the following way.

A function $\xi : \Omega \to \mathbf{R}^l$ is called piece-wise constant if Ω can be decomposed into measurable subsets Ω_i, $i \in \mathbf{N}$, in each of which the function is constant, and a set of zero measure.

Definition 2.7. Let Ω, $\tilde{\Omega}$ be bounded measurable subsets of $\mathbf{R}^n$. Let the sequences $\xi_i : \Omega \to \mathbf{R}^l$, $\tilde{\xi}_i : \tilde{\Omega} \to \mathbf{R}^l$ consist of piece-wise constant functions. Then we say that $\tilde{\xi}_i$ is a renormalization of ξ_i if for each finite collection of indexes $i_1, \ldots, i_k \in \mathbf{N}$ and each collection of vectors $a_{i_1}, \ldots, a_{i_k} \in \mathbf{R}^l$ of the same length we have

$$\begin{aligned} &\frac{1}{\operatorname{meas}\Omega}\{\{\xi_{i_1} = a_{i_1}\} \cap \cdots \cap \{\xi_{i_k} = a_{i_k}\}\} \\ = &\frac{1}{\operatorname{meas}\tilde{\Omega}}\{\{\tilde{\xi}_{i_1} = a_{i_1}\} \cap \cdots \cap \{\tilde{\xi}_{i_k} = a_{i_k}\}\}. \end{aligned}$$

The result of [37], [44, 45] reads as

Theorem 2.8. *Let $u_i \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ be a sequence obtained by perturbations which is bounded in $W^{1,\infty}(\Omega; \mathbf{R}^m)$. Then there exists a sequence $\tilde{u}_i$ gradients of which is renormalization of Du_i and which converges in $W^{1,1}$.*

It is easy to see that

Lemma 2.9. *Let $\xi_i : \Omega \to \mathbf{R}^l$ be a sequence bounded in L^1. Let $\tilde{\xi}_i$ be its renormalization. Then ξ_i converges in L^1 if and only if $\tilde{\xi}_i$ converges in L^1.*

Therefore Theorem 2.6 could be derived as a corollary of Theorem 2.8. This was overlooked by the authors of [36–39] who considered their constructions as a development of a geometrical theory by Gromov of constructing nontrivial immersions, [28], to the case of Lipschitz functions. As we see a more general fact, which is Theorem 2.6, could be obtained on basis of standard facts of Functional Analysis and of Calculus of Variations.

There was another suggestion [42, §5] to apply theory of martingales, see [62, §7.3], in order to derive Theorem 2.6. Facts 2.4, 2.5 are of more elementary nature in order to replace them in this fashion.

3. Sequences obtained by almost maximal perturbations. Universal functional

In this section we also observe that sequences obtained by perturbation can be used to derive various further nontrivial facts on problems (1), (3). For this we use sequences obtained by **almost maximal perturbations**.

Let $L : \mathbf{R}^{m\times n} \to \mathbf{R}$ be a strictly convex C^1-regular function. Let $u : \Omega \to \mathbf{R}^m$ be an admissible for the problem (1), (3) piece-wise affine function. Then Ω can be decomposed into open sets Ω_i with $\text{meas}(\partial\Omega_i) = 0$, $i \in \mathbf{N}$, and a set of zero measure, moreover the restriction of u to each Ω_i is an affine function l_{A_i} (with the gradient equal to A_i).

We define the functional $u \to \text{osc}_L(u)$ as follows

$$\text{osc}_L(u) := \frac{1}{\text{meas}\,\Omega} \sup\{J(u+\phi) - J(u) : u+\phi \text{ is a piece-wise affine admissible for the problem (1), (3) function with } \phi \in W_0^{1,\infty}(\Omega_i;\mathbf{R}^m),\ i \in \mathbf{N}\}.$$

Proposition 2.3 implies that $\text{osc}_L(u)$ does not depend on decomposition of Ω into sets Ω_i.

Definition 3.1. A sequence of admissible for the problem (1), (3) piece-wise affine functions u_i is called obtained by almost maximal perturbations provided u_i is obtained by perturbations and $\text{osc}_L(u_i) \to 0$ as $i \to \infty$.

It could be shown that $\text{osc}_L(u_i) \to 0$ for any strictly convex L provided this holds for a particular one.

Sequences obtained by almost maximal perturbations converges in $W^{1,1}$, see Theorem 2.6. It could be proved that the limit function solves problem (1), (3), where K is any compact set satisfying the property (3).

Proposition 3.2. *Let U be a bounded set and let u_i, $i \in \mathbf{N}$, be a sequence obtained by almost maximal perturbations. Then $u_i \to u_\infty$ in $W^{1,1}$ and $Du_\infty \in K$ a.e. in Ω, where K is any set satisfying* (3).

Proof. Theorem 2.6 allows to state that $u_i \to u_\infty$ in $W^{1,p}$. Let K be such a compact set that (3) holds. Assume that $Du_\infty \notin K$ in a set of positive measure. Then there exists $\epsilon > 0$ such that

$$\text{meas}\{x \in \Omega : \text{dist}(Du_i(x), K) \geq \epsilon\} \geq \epsilon \tag{5}$$

for sufficiently large $i \in \mathbf{N}$.

Since u_i is piece-wise affine the set Ω_i of $x \in \Omega$ in (5) is a union of open sets Ω^i_j, $j \in \mathbf{N}$, in each of which u_i is affine, i.e. $Du_i = A^i_j$ in Ω^i_j.

By definition of $\text{osc}_L(u_i)$

$$\text{osc}_L(u_i) \geq \frac{1}{\text{meas}\,\Omega} \sum_j \int_{\Omega^i_j} \{L(A^i_j + D\phi^i_j(x)) - L(A^i_j)\}dx, \tag{6}$$

where $\phi_j^i \in W_0^{1,\infty}(\Omega_j^i; \mathbf{R}^m)$ are such that $u + \phi$ is a piece-wise affine admissible for the problem (1), (3) function.

Then

$$\operatorname{osc}_L(u_i) \geq \frac{1}{\operatorname{meas}\Omega} \sum_j \int_{\Omega_j^i} \{L(A_j^i + D\phi_j^i) - L(A_j^i) - \langle DL(A_j^i), D\phi_j^i(x)\rangle\} dx$$
$$\geq \frac{1}{\operatorname{meas}\Omega} \sum_j \int_{\Omega_j^i} \theta(|D\phi_j^i(x)|) dx, \tag{7}$$

where

$$\theta(\epsilon) := \inf\{L(B) - L(A) - \langle DL(A), B - A\rangle : B, A \in U, |B - A| \geq \epsilon\}$$

is a nondecreasing positive function.

We can take $\phi_j^i : \Omega_j^i \to \mathbf{R}^m$ in such a way that

$$\operatorname{dist}(D\phi_j^i(x), K) \leq \epsilon/2 \;\text{ a.e. in } \Omega_j^i,\; j \in \mathbf{N}.$$

Then the right-hand side in (7) can be estimated from below by

$$\frac{1}{\operatorname{meas}\Omega}\theta(\epsilon/2) \operatorname{meas}\Omega_i \geq \frac{1}{\operatorname{meas}\Omega}\theta(\epsilon/2)\epsilon.$$

This contradicts to the requirement $\operatorname{osc}_L(u_i) \to 0$ as $i \to \infty$.

This proves that $Du_\infty \in K$ a.e. in Ω. □

An immediate consequence of Proposition 3.2 is

Theorem 3.3. *Let U be a bounded set. There exists a set S for which each problem*

$$u \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m), \quad A \in U,\; Du(\cdot) \in S \text{ a.e.},$$

is solvable and moreover $S \subset K$ for each compact set K that satisfies (3).

Proof. Given $A \in U$ we define a sequence $u_i \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$, $i \in \mathbf{N}$, obtained by almost maximal perturbations. Then $u_i \to u_\infty$ in $W^{1,1}$ by Theorem 2.6. Let $S_A := \{Du_\infty(x) : x \in \Omega$ is a Lebesque point of $Du_\infty\}$. Define $S := \cup_{A\in U} S_A$. In case K is a compact set which satisfies (3) we have by Proposition 3.2 that $Du_\infty \in K$ a.e. In particular $S \subset K$. □

The set S itself could be nonclosed. In fact in the scalar case the problem (1), (3) could be solved with extreme points of U, which we define as $\operatorname{extr} U$, see [13], [11]. Here an advantage is that in addition to the fact that this set is minimal for which the inclusions are solvable all the solutions are **stable**, i.e. *in case u is such a solution and u_i is a sequence of admissible for the problem piece-wise affine functions with $u_i \rightharpoonup u$ in $W^{1,1}$ we have $u_i \to u$ in $W^{1,1}$*, see [2], [4], [41], [59]; in context of the Calculus of Variations this property was studies parallely in [16], [46], [60], [61].

In the vector-valued case we also can identify explicitly a minimal set S for which the inclusions (1), (3) are solvable provided $U \subset \mathbf{R}^{m\times n}$ is convex, bounded and open. This set was considered in [35] and is defined as the of all extreme points of U and those faces of ∂U that do not contain rank-one connections, i.e. there are no A, B with $\operatorname{rank}(A-B)=1$; the set was denoted as $\operatorname{gr}\operatorname{extr} U$. The only possibility for a solution $u \in W^{1,\infty}$ of the problem (1), (3) to be stable is $Du \in \operatorname{gr}\operatorname{extr} U$ a.e. in Ω, see Theorem 3.22 of [30]. The same is true for nonhomogeneous differential inclusions $(x,u) \to U(x,u)$ with convex equa-bounded sets $U(x,u)$ which depends continuously on (x,u) in the Hausdorff metric, see Theorem 3.34 of [30]. Therefore to resolve the inclusions with the gradient extreme points one has to construct a solution which is stable.

The issue of minimal subset of U which we can approximate modifying gradients of the functions whose gradients approach U was discussed in [63, 64] for general U. However the question of generating a solution of the inclusion with this subset was not addressed.

In case we want to generate stable solutions explicitly we can not use arbitrary sequences obtained by perturbation, but we have to involve sequences obtained by almost maximal perturbations. We will discuss this issue in §4.

Kirchheim was first to arise and simultaneously to prove existence of stable solutions to the problem (1), (3), see [29], since his method of Baire-1 functions immediately allows to derive density of the points of continuity in the completion of the set of admissible piece-wise affine functions in the weak topology. Let (V, L^∞) be the closure in L^∞-norm of the set of piece-wise affine admissible functions. The map

$$u \in (V, L^\infty) \to Du \in (V, L^1)$$

is Baire-1 one since $u \to Du_\epsilon$, where u_ϵ is the ϵ-mollification of u, is a continuous map and $Du_\epsilon \to Du$ in L^1 as $\epsilon \to 0$ for each $u \in (V, L^\infty)$. Then the set of points of continuity of this map is residual. These points of continuity are automatically stable solutions of the problem (1), (3). In fact if w_i is a sequence for which (3) does not hold and if $w_i \to u$ in $W^{1,1}$ as $i \to \infty$ then we can take associated oscillations $\tilde{w}_i$ around w_i, see Proposition 2.3. And since $||D(\tilde{w}_i - w_i)||_{L^1} \geq \epsilon > 0$ we have $Dw_i \to Du$, but $D\tilde{w}_i \not\to Du$ in L^1 as $i \to \infty$, when we can achieve $D\tilde{w}_i \rightharpoonup Du$, $i \to \infty$ – a contradiction to the fact that u is a point of continuity.

This is, of course, a very powerful method and it could be applied to show that in each weakly compact subset of L^1 points of continuity are dense. To apply weak convergence methods we suggested a straightforward idea to measure maximal oscillations produced by admissible functions at a given one, [42, 43].

Let V be a weakly compacts subset in $L^1(\Omega; \mathbf{R}^m)$. Then we can find a strictly convex nonnegative C^1-regular integrand L such that the family $\{L(\xi) : \xi \in V\}$ is equiintegrable. We define

$$\operatorname{ind}_L(\xi) := \sup_{\xi_i \to \xi, \xi_i \in V} \limsup_{i\to\infty} \frac{1}{\operatorname{meas}\Omega} \int_\Omega [L(\xi_i) - L(\xi)]dx.$$

Any sequence $\xi_i \in V$, $i \in \mathbf{N}$, with $\xi_i \rightharpoonup \xi$ in L^1 and with

$$\int_\Omega L(\xi(x))dx \to \int_\Omega L(\xi(x))dx + \mathrm{ind}_L(\xi)\,\mathrm{meas}\,\Omega, \quad i \to \infty,$$

is called a sequence *generating maximal oscillations* (associated with ξ).

Then we have

Theorem 3.4 (Theorem 1.2 of [42], Theorems 1.6, 1.7 of [43]). *Let S be a weakly compact subset of L^1 and let ρ be a metric equivalent to the weak convergence in S. Then:*

1. *The functional $\xi \in (S, \rho) \to \mathrm{ind}_L(\xi)$ is upper semicontinuous. In particular $\mathrm{ind}_L(\xi_i) \to 0$, $i \to \infty$, for each sequence $\xi_i \in S$ that generates maximal oscillations associated with a $\xi \in S$.*
2. *Given $\epsilon > 0$ there exists $\delta > 0$ such that $\limsup_{i\to\infty} ||\xi_i - \xi||_{L^1} \leq \epsilon$ provided $\xi, \xi_i \in S$, $\mathrm{ind}_L(\xi) \leq \delta$, $\xi_i \rightharpoonup \xi$ in L^1 as $i \to \infty$. In particular $\mathrm{ind}_L(\xi) = 0$ implies $\xi_i \to \xi$ in L^1 provided $\xi_i \rightharpoonup \xi$ in L^1 and $\xi_i \in S$, $i \in \mathbf{N}$.*

We can use properties *1.* to prove density of stable functions (i.e. with $\mathrm{ind}_L(\xi) = 0$, cf. Fact 2.5 in §1) following Marcellini-Dacorogna suggestion to use Baire lemma, [23, 24]. Alternatively we can use *2.* to extract strongly converging sequence ξ_i with $\mathrm{ind}(\xi_i) \to 0$, $i \to \infty$, and with control of $\rho(\xi_{i'}, \xi_i)$, $i' \geq i$, to infer $\mathrm{ind}(\xi_\infty) = 0$ where ξ_∞ is a limit of ξ_i as $i \to \infty$. To do this we take $\delta_i > 0$ such that $\delta_{i+1} \leq \delta_i/2$, $i \in \mathbf{N}$, and we select ξ_i iteratively in such a way that $\mathrm{ind}_L(\xi) \leq \mathrm{ind}_L(\xi_i) + 1/2^i$ provided $\rho(\xi, \xi_i) \leq \delta_i$ with $\rho(\xi_{i+1}, \xi_i) \leq \delta_i/2$, $\mathrm{ind}_L(\xi_i) \leq 1/2^i$. Then the limit function ξ_∞ satisfies all the estimates $\mathrm{ind}_L(\xi_\infty) \leq \mathrm{ind}_L(\xi_i) + 1/2^i \leq 1/2^{i-1}$, $i \in \mathbf{N}$, i.e. ξ_∞ is a stable element. Therefore the functional $\xi \to \mathrm{ind}_L(\xi)$ presents the link between two methods to use Baire category lemma for an appropriate functional and to construct strongly convergent sequences of approximate solutions. This resolve the issue of competition of the methods raised in [25]: *they are equivalent.* Moreover the result of Kirchheim [29] is recovered this way.

In fact Marcellini & Dacorogna used a weak convergence approach to apply Baire category lemma to appropriate functionals and this was an approach very close to ours in its methodology. However they were keen to use quasiconvexity to construct appropriate functionals as integral functionals, see [23]. Later they switch to an abstract functional, see [24]. However they tried to obtain *all* solutions as a result of application of Baire lemma to the functional, i.e. they wished that all solutions would stay in level sets of a nonnegative upper semicontinuous with respect to the weak convergence functional. At the same time existence of unstable solutions immediately requires to consider larger target sets K than is required in (3), we discussed this in all our papers [42, 43], [44, 45], [35]. Very likely that this approach would finally develop to the same results as ours. In fact all necessary hints already existed in the paper by Bressan & Flores [11], where the authors proved in the scalar case the density result for continuity points in the set of admissible functions, see also [21], simply showing that this result is valid for the class of functions with the gradients staying in extreme points of U (i.e. K in (3) is equal to $\mathrm{extr}\,U$). The next natural step was to deal with abstract continuity points in the situation when K can not be associated with extreme points of the set U in (1), (3), as Kirchheim did in [29].

To conclude we emphasize that the idea originated in the theory of differential inclusions is to control maximal oscillations on neighbour functions via this oscillation on an original one and the difference in the weak metric since maximal oscillations is an upper semicontinuous functional in this space. This idea turned out to be fruitful and we applied it back to Calculus of Variations, see [48]. For general integral functionals (without any requirements of growth) we showed that the set of points where both lower semicontinuity and convergence with the functional properties hold in a set dense in the weak topology. The functional which presents the formal lower semicontinuous envelope also has the property of convergence with the functional at these points. Moreover the values the integral functional assumes at this set completely determine the lower semicontinuous envelope. Under various additional assumptions on integrands these properties can be characterized for a given function in terms of properties of the integrand (lower semicontinuity corresponds to quasiconvexity at $Du(\cdot)$ a.e., the convergence with the functional property is associated with exposed points of the integrand), see [49–51].

4. Generating stable solutions of differential inclusions

We finished previous section by the remark that a natural expectation to generate stable solutions by explicit construction is to use sequences obtained by almost maximal perturbations. However we did not succeed to clarify whether all the associated solutions are stable. To do this we renormalize the sequences.

Theorem 4.1. *Let U be a bounded, convex and open set and let f be a piece-wise affine admissible function. Let also $u_i \in f + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ be a sequence obtained by almost maximal perturbations. Then there exists a sequence $\tilde{u}_i$ gradients of which is the renormalization of Du_i and which converges to a stable solution u_∞ of the problem* (1)*,* (3)*.*

Theorem 3.22 of [30] mentioned in the previous section implies that $Du_\infty(\cdot) \in \operatorname{gr}\operatorname{extr} U$ a.e. in Ω.

We prove first two auxiliary propositions.

Proposition 4.2. *Let U be a bounded subset of $\mathbf{R}^{m\times n}$ and let $L : \mathbf{R}^{m\times n} \to \mathbf{R}$ be a strictly convex C^1-regular function. Let $u_i : \Omega \to \mathbf{R}^m$ be a sequence of admissible (i.e. with $Du_i \in U$ a.e.) piece-wise affine functions and let $\tilde{u}_i : \Omega \to \mathbf{R}^m$ be another sequence of admissible piece-wise affine functions gradients $D\tilde{u}_i$ of which present renormalization of Du_i. Then*

$$\operatorname{osc}_L(u_i) = \operatorname{osc}_L(\tilde{u}_i), \quad i \in \mathbf{N}. \tag{8}$$

Proof. For each $i \in \mathbf{N}$ the set Ω can be decomposed into open sets Ω^i_j, $j \in \mathbf{N}$, and a set of zero measure in such a way that $u_i : \Omega^i_j \to \mathbf{R}^m$ is an affine function $l_{A^i_j}$, $j \in \mathbf{N}$. Then

$$\operatorname{osc}_L(u_i) = \sum_j \operatorname{osc}_L(l_{A^i_j}) \operatorname{meas} \Omega^i_j, \quad i \in \mathbf{N}.$$

$\mathrm{osc}_L(\tilde{u}_i)$ is equal to the same value since the sequence $D\tilde{u}_i$ is the renormalization of Du_i.

Therefore (8) holds. □

Lemma 4.3. *Let U be an open, bounded and convex set and let $L : \mathbf{R}^{m\times n} \to \mathbf{R}$ be strictly convex C^1-regular function. Assume that $u : \Omega \to \mathbf{R}^m$ is an admissible piece-wise affine function. Then*

$$\mathrm{osc}_L(u) = \mathrm{ind}_L(Du). \tag{9}$$

Proof. Ω can be decomposed into open sets Ω_i, $i \in \mathbf{N}$, and a set of zero measure in such a way that $u : \Omega_i \to \mathbf{R}^m$ is an affine function l_{A_i}, $i \in \mathbf{N}$.

Let $\phi \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ be such a piece-wise affine function that $u + \phi$ is admissible and $\phi \in W_0^{1,\infty}(\Omega_i; \mathbf{R}^m)$, $i \in \mathbf{N}$. Then we can apply Proposition 2.3 in each Ω_i to generate a sequence $u + \phi_j$ of piece-wise affine admissible functions with $\phi_j \rightharpoonup^* 0$ in $W^{1,\infty}(\Omega; \mathbf{R}^m)$ and with

$$\int_\Omega L(Du(x) + D\phi_j(x))dx = \int_\Omega L(Du(x) + D\phi(x))dx, \quad j \in \mathbf{N}. \tag{10}$$

Then

$$\mathrm{ind}_L(Du) \geq \frac{1}{\mathrm{meas}\,\Omega} \lim_{j\to\infty} \{J(u + \phi_j) - J(u)\} = \frac{1}{\mathrm{meas}\,\Omega}\{J(u + \phi) - J(u)\}. \tag{11}$$

Since $u+\phi$ in (11) is an arbitrary admissible piece-wise function with $\phi \in W_0^{1,\infty}(\Omega_i; \mathbf{R}^m)$, $i \in \mathbf{N}$, we obtain

$$\mathrm{ind}_L(Du) \geq \mathrm{osc}_L(u). \tag{12}$$

To prove the inverse inequality we consider a sequence of admissible (not mandatory piece-wise affine) functions u_j with $u_j \rightharpoonup^* u$ in $W^{1,\infty}(\Omega; \mathbf{R}^m)$ as $j \to \infty$.

We have to prove that

$$\frac{1}{\mathrm{meas}\,\Omega} \limsup_{j\to\infty} \{J(u_j) - J(u)\} \leq \mathrm{osc}_L(u). \tag{13}$$

We prove first (13) for the case when u is an affine admissible function, i.e. $u = l_A$, $A \in U$. For $\lambda \in]0, 1[$ consider another sequence

$$u_j^\lambda := l_A + \lambda(u_j - l_A), \quad j \in \mathbf{N}. \tag{14}$$

The range of the gradients of u_j^λ belongs to $A + \lambda(U - A)$ and is compactly supported in U. Then we can modify u_j^λ as $\tilde{u}_j^\lambda$ to meet the requirements

$$\tilde{u}_j^\lambda\big|_{\partial\Omega} = l_A, \quad j \in \mathbf{N}, \qquad ||D\tilde{u}_j^\lambda - Du_j^\lambda||_{L^1} \to 0, \quad j \to \infty, \tag{15}$$

and keeping $\tilde{u}_j^\lambda$ to be admissible functions. In fact consider $\phi_k \in C_0^\infty(\Omega)$ with $0 \leq \phi_k \leq 1$, $\phi_k = 1$ in Ω_k, where $\mathrm{meas}(\Omega \setminus \Omega_k) \leq 1/k$, and define

$$\tilde{u}_j^\lambda = l_A + \phi_{k(j)}\lambda(u_j - l_A). \tag{16}$$

Then

$$D\tilde{u}_j^\lambda = A + \phi_{k(j)}\lambda(Du_j - A) + D\phi_{k(j)}\lambda(u_j - l_A) \tag{17}$$

and for an appropriately selected sequence $k(j) \to \infty$ as $j \to \infty$ we have

$$||D\tilde{u}_j^\lambda - Du_j^\lambda||_{L^1} \to 0, \quad j \to \infty, \qquad D\tilde{u}_j^\lambda \subset A + \frac{\lambda}{2}(U - A), \quad j \in \mathbf{N}. \tag{18}$$

Then (15) also holds due to (17), (18). We can further consider appropriate mollifications of $\tilde{u}_j^\lambda$ and then piece-wise affine functions $\bar{u}_j^\lambda$ associated with appropriate triangulations of Ω such that

$$||D\bar{u}_j^\lambda - D\tilde{u}_j^\lambda||_{L^1} \to 0, \quad D\bar{u}_j^\lambda \in U \text{ a.e. in } \Omega. \tag{19}$$

Then for appropriately selected sequence $\lambda(j) \to 1 - 0$ as $j \to \infty$ we have

$$||D\bar{u}_j^{\lambda(j)} - Du_j||_{L^1} \to 0, \quad D\bar{u}_j^{\lambda(j)} \in U \text{ a.e. in } \Omega, \ \bar{u}_j^{\lambda(j)}\big|_{\partial\Omega} = l_A. \tag{20}$$

However by definition of the functional $u \to \operatorname{osc}(u)$ we have

$$\operatorname{osc}(l_A) \geq \frac{1}{\operatorname{meas}\Omega}\left\{J\left(\bar{u}_j^{\lambda(j)}\right) - J(l_A)\right\}, \quad j \in \mathbf{N}. \tag{21}$$

Therefore (20), (21) result in (13) for $u = l_A$.

We now switch to the case of arbitrary piece-wise affine admissible function $u \in W^{1,\infty}(\Omega; \mathbf{R}^m)$. Ω can be decomposed into open sets Ω_i, $i \in \mathbf{N}$, and a set of zero measure with $u : \Omega_i \to \mathbf{R}^m$ affine, i.e. $u\big|_{\Omega_i} = l_{A_i}$, $i \in \mathbf{N}$. For each particular Ω_i we already proved

$$\frac{1}{\operatorname{meas}\Omega_i} \limsup_{j\to\infty}\{J(u_j; \Omega_i) - J(u; \Omega_i)\} \leq \operatorname{osc}_L(l_{A_i}). \tag{22}$$

Then (13) follows since

$$\operatorname{osc}_L(u_i) = \sum_i \operatorname{osc}_L(l_{A_i}) \operatorname{meas}\Omega_i.$$

Therefore we established the inequality

$$\operatorname{ind}_L(Du) \leq \operatorname{osc}(u). \tag{23}$$

Together with (12) this proves the lemma. □

Now we are prepared to prove Theorem 3.1.

Proof of Theorem 3.1. Let $u_i : \Omega \to \mathbf{R}^m$ be a sequence obtained by almost maximal perturbations. We can renormalize it as $\tilde{u}_i$ iteratively via Proposition 2.3 in order to meet the requirements

$$\rho(D\tilde{u}_{i+1}, D\tilde{u}_i) \leq \delta_i/2, \tag{24}$$

where ρ is a metric equivalent to the weak convergence in the completion of admissible functions (e.g. $\rho = ||\cdot||_{L^\infty}$). Here $\delta_{i+1} \leq \delta_i/2$ and

$$\mathrm{ind}_L(D\bar{u}) \leq \mathrm{ind}_L D\tilde{u}_i + 1/2^i \tag{25}$$

provided $\rho(D\bar{u}, D\tilde{u}_i) \leq \delta_i$, $i \in \mathbf{N}$.

Then we have by Propositions 3.2, 3.3

$$\mathrm{ind}_L(D\tilde{u}_i) = \mathrm{osc}_L(\tilde{u}_i) = \mathrm{osc}_L(u_i) \to \infty, \quad i \to \infty, \tag{26}$$

Since $\tilde{u}_i \to u_\infty$ in $W^{1,1}$ (cf. Theorem 2.6) and since

$$\rho(Du_i, Du_\infty) \leq \delta_i, \quad i \in \mathbf{N}, \tag{27}$$

we infer via (25), (26) that

$$\mathrm{ind}_L(Du_\infty) \leq \mathrm{ind}_L(D\tilde{u}_i) + 1/2^i \leq \mathrm{osc}(u_i) + 1/2^i, \quad i \in \mathbf{N},$$

i.e. $\mathrm{ind}_L(Du_\infty) = 0$. Then u_∞ is a stable solution of the differential inclusion (1), (3). □

Remark. We would prove that the original sequence u_i converge to a stable solution u_0 of the problem (1), (3) provided we prove that $Du_i(x)$, $D\tilde{u}_i(x)$ consist of the same sequences for a.a. $x \in \Omega$ (in this case $Du_0 \in \mathrm{gr\,extr}\, U$ a.e. since $Du_\infty \in \mathrm{gr\,extr}\, U$ a.e.). Definition 2.5 of renormalization seems suggest this. However we were unable to prove this rigorously. Therefore this is an open problem.

5. Nonhomogeneous Differential Inclusions and Higher Regularity of Solutions

In this section we stop at nonhomogeneous version of problems (1), (3). We observe that for each Lipschitz piece-wise affine admissible boundary data f and for each $\epsilon > 0$ there exists a solution u_ϵ of the differential inclusion such that $||f - u_\epsilon||_{L^\infty} \leq \epsilon$ and $u_\epsilon \in C^{0,\alpha}$ for any $\alpha \in]0,1[$. This means that solutions of differential inclusions could be almost as regular as Lipschitz functions in spite integrability of the gradient is determined by the inclusion and can not be improved. The latter observation was made for a particular α in the paper [3] for special homogeneous inclusions (1), (3) with unbounded U, K. There the authors noticed that when taking sequences obtained by perturbation (see Definition 2.2) and applying Proposition 2.3, i.e. when renormalizing the gradients, we can fit an arbitrary bound on $C^{0,\alpha}$-norm of the perturbation since it is Lipschitz and decompositions in renormalization could be taken as fine as necessary. A more careful application of this observation leads to bounds in all $C^{0,\alpha}$-norms simultaneously, in particular the results of [3] could be improved both in the case of lower and upper bounds. This is an essential feature of solutions of general differential inclusions and it is obtained on basis of the explicit construction suggested by Definition 2.2. It is unclear, and seems impossible, how to derive this result via other approaches since they are insufficiently explicit.

Let us state the result.

Let $U : \Omega \to 2^{\mathbf{R}^{m\times n}}$ be a multi-valued function with bounded values for a.e. $x \in \Omega$ and let

$$|U(\cdot)| \leq g(\cdot), \quad \text{where } g \in L^1(\Omega) \tag{28}$$

and where $|U| := \sup\{|v| : v \in U\}$. Let $K : \Omega \to 2^{\mathbf{R}^{m\times n}}$ be a closed-valued lower semicontinuous multi-function, i.e. for each $x \in \Omega$ the set $K(x)$ is closed and given $x_0 \in \Omega$, $v \in K(x_0)$ we have $\operatorname{dist}(v, K(x)) \to 0$ as $x \to x_0$.

Theorem 5.1. *Let the multi-valued functions U, K satisfy the above made assumptions. Assume also that for a.e. $x_0 \in \Omega$ and for each $\epsilon > 0$ and each $A \in U(x_0)$ there exists a piece-wise affine function $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with*

$$(A + D\phi(\cdot)) \in U(x_0) \text{ a.e.}, \ ||\operatorname{dist}(A + D\phi(\cdot); K(x_0))||_{L^1(\Omega)} \leq \epsilon \operatorname{meas}\Omega \tag{29}$$

and with

$$(A + D\phi(\cdot)) \in U(x) \text{ a.e. in } \Omega \text{ for a.e. } x \in B(x_0, \delta), \tag{30}$$

where $\delta > 0$.

Then given a Lipschitz piece-wise affine function f which is admissible (i.e. with $Df(x) \in U(x)$ a.e.) and given $\eta > 0$ there exists a function $u_\eta \in f + W_0^{1,1}(\Omega; \mathbf{R}^m)$ such that

$$u_\eta\big|_{\partial\Omega} = f\big|_{\partial\Omega}, \quad ||u_\eta - f||_{L^\infty} \leq \eta, \quad Du_\eta(\cdot) \in K(\cdot) \ \text{ a.e.}, \ u_\eta \in C^{0,\alpha} \ \forall\alpha \in]0,1[. \tag{31}$$

Proof. Of course we will exploit the fact that a solution to the problem (31) could be constructed as a limit of a sequence u_i obtained by perturbation (see Definition 2.2) with

$$\int_\Omega \operatorname{dist}(Du_i(x), K(x))dx \to 0, \quad i \to \infty.$$

Assumptions (29), (30) and the assumption of lower semicontinuity of the multi-valued mapping $x \to K(x)$ allow to do this. In addition we can select such a sequence to be bounded in $C^{0,\alpha}$ for all $\alpha \in]0,1[$ and with $||u_{i+1} - u_i||_{L^\infty} \leq \eta/2^i$, $i \in \mathbf{N}$. Therefore the limit function $u_\infty \in C^{0,\alpha}$, $\forall\alpha \in]0,1[$, $Du_\infty(x) \in K(x)$ a.e. and the inequality $||u_\infty - f||_{L^\infty} \leq \eta$ holds.

Enough to consider the case $f = l_A$.

Assume that given $k \in \mathbf{N}$ we have constructed first k iterations and then $u_k \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ is piece-wise affine Lipschitz and satisfies

$$Du_k(\cdot) \in U(\cdot) \ \text{ a.e. in } \Omega, \tag{32}$$

$$\int_\Omega \operatorname{dist}(Du_k(x), K(x))dx \leq \operatorname{meas}\Omega/k. \tag{33}$$

We want to construct $u_{k+1} \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ which is piece-wise affine, meets both (32) and (33) with $1/(k+1)$ instead $1/k$ and the estimates

$$||u_{k+1} - u_k||_{C^{0,1-1/i}} \leq 1/2^k, \quad i \in \{1, \ldots, k\}, \qquad ||u_{k+1} - u_k||_{L^\infty} \leq \eta/2^k. \tag{34}$$

Given an open subset $\tilde{\Omega} \subset \Omega$ with $Du_k = \tilde{A}$ in $\tilde{\Omega}$ and given a point $x_0 \in \tilde{\Omega}$ which fits (29), (30) with $\epsilon > 0$ and $\delta = \delta(\epsilon) > 0$ and with $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$, $B(x_0, \delta) \subset \tilde{\Omega}$, we can renormalize ϕ via Proposition 2.3 as $\tilde{\phi} \in W_0^{1,\infty}(B(x_0, \delta); \mathbf{R}^m)$ in such a way that

$$\int_{B(x_0,\epsilon)} \operatorname{dist}((\tilde{A} + D\tilde{\phi}(x)), K(x_0))dx \leq \epsilon \operatorname{meas} B(x_0, \delta)$$

and $(\tilde{A} + D\tilde{\phi}(\cdot)) \in U(x_0)$ a.e., $(\tilde{A} + D\tilde{\phi}(\cdot)) \in U(x)$ a.e. for a.e. $x \in B(x_0, \delta)$. Moreover we can take $\delta > 0$ sufficiently small to satisfy automatically

$$\int_{\tilde{B}} \operatorname{dist}(\tilde{A} + D\tilde{\phi}(x), K(x))dx \leq \frac{1}{2(k+1)} \operatorname{meas} \tilde{B},$$

$$\operatorname{meas}(B(x_0, \delta) \setminus \tilde{B}) \leq \epsilon \operatorname{meas} B(x_0, \delta), \quad \tilde{B} \subset B(x_0, \delta). \tag{35}$$

Indeed if

$$\int_{\Omega} \operatorname{dist}(\tilde{A} + D\phi(x), K(x_0))dx$$

is sufficiently small then the set

$$\bar{\Omega} := \{x \in \Omega : \operatorname{dist}(\tilde{A} + D\phi(x), K(x_0)) \leq \epsilon\}$$

is sufficiently large to guarantee that

$$\operatorname{meas}(\Omega \setminus \bar{\Omega}) \leq \frac{\epsilon}{2} \operatorname{meas} \Omega. \tag{36}$$

The set $\bar{\Omega}$ can be decomposed as $\bar{\Omega} = \cup_{i=1}^{\infty} \Omega_i \cup S$, where Ω_i are open, $\operatorname{meas} S = 0$, and $D\phi(\cdot) = B_i$ in Ω_i, $i \in \mathbf{N}$. We can take a finite collection of Ω_i, $i \in \{1, \ldots, M\}$, denoted as Ω_ϵ, such that the inequalities

$$\operatorname{dist}(\tilde{A} + B_i, K(x_0)) \leq \epsilon, \quad i \in \{1, \ldots, M\}, \tag{37}$$

hold and for the remaining set we have

$$\operatorname{meas}(\bar{\Omega} \setminus \Omega_\epsilon) \leq \frac{\epsilon}{2} \operatorname{meas} \Omega. \tag{38}$$

By lower semicontinuity of the map $x \to K(x)$ we can find $\delta = \delta(\epsilon) > 0$ such that

$$\max_{i \in \{1,\ldots,M\}} \operatorname{dist}(\tilde{A} + B_i, K(x)) \leq 2\epsilon, \quad x \in B(x_0, \delta). \tag{39}$$

Then for $\tilde{\phi} \in W_0^{1,\infty}(B(x_0, \delta); \mathbf{R}^m)$ associated with ϕ via Proposition 2.3 and for $2\epsilon \leq \frac{1}{2(k+1)}$ we have (see (36), (38) and (39)) $\tilde{A} + D\tilde{\phi}(\cdot) \in U(\cdot)$ a.e. in $B(x_0, \delta)$ and (35) holds. Moreover this holds for any $\tilde{\phi} \in W_0^{1,\infty}((B(x_0, \delta); \mathbf{R}^m)$ obtained via Proposition 2.3 from ϕ and we will use this flexibility to fit further requirements on $\tilde{\phi}$.

We have to fit bounds (31) in $B(x_0, \delta)$ for what enough to fit them only in the norm $C^{0,1-1/k}$. Of course $\tilde{\phi}$ can be taken with $||\tilde{\phi}||_{L^\infty} \leq \eta/2^k$. In case we take further renormalization of $\tilde{\phi}$ in $B(x_0, \delta)$ (not relabeled) the inclusions (29), (30) still hold and given a set $\Omega_j^i \subset B(x_0, \delta)$ where $\tilde{\phi}\big|_{\partial\Omega_j^i} = 0$ we have

$$|\tilde{\phi}(x)| \leq |U(x_0)| \operatorname{dist}(x, \partial\Omega_j^i) \leq \operatorname{dist}(x, \partial\Omega_j^i)^{1-1/k}(1/2^k), \quad x \in \Omega_j^i,$$

provided $\operatorname{diam}\Omega_j^i$ is sufficiently small. This gives us an estimate on maximal diameter of decomposition of $B(x_0, \delta)$ when applying the renormalization.

Finally we can find a finite collection of balls $B_j := B(x_j, \delta_j)$, $j \in \{1, \ldots, M_1\}$, such that

$$\int_{\Omega\setminus\cup B_j} 2g(x)dx \leq \epsilon \operatorname{meas}\Omega \tag{40}$$

(see (28)) and $\tilde{\phi}_j + W_0^{1,\infty}(B(x_j, \delta_j); \mathbf{R}^m)$ which satisfies (35) with A_j instead of $\tilde{A}$, where $u_k\big|_{B(x_j,\delta_j)} = l_{A_j}$, $j \in \{1, \ldots, M_1\}$.

For each ball $B(x_j, \delta_j)$, $j \in \{1, \ldots, M_1\}$, we have due to (35)

$$\begin{aligned} &\int_{\tilde{B}_j} \operatorname{dist}(A_j + D\tilde{\phi}_j(x), K(x))dx \leq \frac{1}{2(k+1)} \operatorname{meas}\tilde{B}_j, \\ &\qquad \text{where } \operatorname{meas}(B_j \setminus \tilde{B}_j) \leq \epsilon \operatorname{meas} B_j,\ \tilde{B}_j \subset B_j. \end{aligned} \tag{41}$$

We define $u_{k+1} = u_k + \tilde{\phi}_j$ in B_j, $j \in \{1, \ldots, M_1\}$, $u_{k+1} = u_k$ in $\Omega \setminus \cup B_j$. Then (40), (41) result in

$$\begin{aligned} &\int_\Omega \operatorname{dist}(Du_{k+1}(x), K(x))dx \\ &\leq \sum_{j=1}^{M_1} \int_{\tilde{B}_j} \operatorname{dist}(A_j + D\tilde{\phi}_j(x), K(x))dx \\ &\qquad + \sum_{j=1}^{M_1} \int_{(B_j\setminus\tilde{B}_j)} \operatorname{dist}(A_j + D\tilde{\phi}_j, K(x))dx + \int_{\Omega\setminus\cup B_j} 2g(x)dx \\ &\leq \frac{1}{2(k+1)} \operatorname{meas}(\cup\tilde{B}_j) + \int_{\cup(B_j\setminus\tilde{B}_j)} 2g(x)dx + \epsilon \operatorname{meas}\Omega, \end{aligned} \tag{42}$$

where

$$\operatorname{meas}\{\cup_{j=1}^{M_1}(B_j \setminus \tilde{B}_j)\} \leq \epsilon \operatorname{meas}(\cup_{j=1}^{M_1} B_j) \leq \epsilon \operatorname{meas}\Omega.$$

Therefore if ϵ is so small that $\epsilon \leq \frac{1}{4(k+1)}$ and

$$\int_{\tilde{\Omega}} 2g(x)dx \leq \frac{1}{4(k+1)} \operatorname{meas}\Omega$$

provided $\operatorname{meas}\tilde{\Omega} \leq \epsilon \operatorname{meas}\Omega$ then (42) result in

$$\int_\Omega \operatorname{dist}(Du_{k+1}(x), K(x))dx \leq \frac{1}{(k+1)} \operatorname{meas}\Omega. \tag{43}$$

We have constructed u_{k+1} in such a way that

$$||u_{k+1} - u_k||_{C^{0,1-1/i}} \leq 1/2^k, \quad i \in \{1, \dots, k\}, \qquad ||u_{k+1} - u_k||_{L^\infty} \leq \eta/2^k. \tag{44}$$

Then the sequence u_i is bounded in $C^{0,1-1/k}$-norm since on the first $k-1$ steps this norm was bounded because all u_i are Lipschitz.

Moreover u_k converge in $W^{1,1}(\Omega; \mathbf{R}^m)$ to u_∞ due to Theorem 2.6. The inequality (43) implies

$$\operatorname{dist}(Du_k(\cdot), K(\cdot)) \to 0 \ \text{ in } L^1, \ k \to \infty,$$

and, therefore,

$$Du_\infty(\cdot) \in K(\cdot) \ \text{ a.e. in } \Omega.$$

In addition (44) guarantees that u_∞ is bounded in all $C^{0,\alpha}$-norms, $\alpha \in]0, 1[$, and $||u_\infty - f||_{L^\infty} \leq \eta$. Then all the requirements of (31) are met. □

References

[1] J. P. Aubin, A. Cellina: Differential Inclusions. Set-Valued Maps and Viability Theory, Grundlehren der Mathematischen Wissenschaften 264, Springer, Berlin (1984).

[2] A. Amrani, C. Castaing, M. Valadier: Méthodes de troncature appliquées à des problémes de convergence faible ou fort dans L^1, Arch. Rational Mech. Anal. 117 (1992) 167–191.

[3] K. Astala, D. Faraco, L. Szekelyhidi: Convex integration and the L^p theory of elliptic equations, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 7(1) (2008) 1–50.

[4] E. J. Balder: On weak convergence implying strong convergence under extremal conditions, J. Math. Anal. Appl. 163 (1992) 147–156.

[5] S. Bertone, A. Cellina: On the existence of variations, possibly with pointwise gradient constraints, ESAIM, Control Optim. Calc. Var. 13 (2007) 331–342.

[6] J. M. Ball, R. D. James: Fine mixtures as minimizers of energy, Arch. Rational Mech. Anal. 100 (1987) 13–52.

[7] J. M. Ball, R. D. James: Proposed experimental tests of a theory of fine microstructure and the two-well problem, Philos. Trans. R. Soc. Lond., Ser. A 338 (1992) 389–450.

[8] K. Bhattacharya, N. Firoozye, R. D. James, R. V. Kohn: Restrictions on microstructure, Proc. R. Soc. Edinb., Sect. A 124 (1994) 843–878.

[9] N. N. Bogolubov: Sur quelques methods novelles dans le calculus des variations, Ann. Mat. Pura Appl. 7(4) (1930) 149–271.

[10] A. Bressan: The most likely path of a differential inclusion, J. Differ. Equations 88 (1990) 155–174.

[11] A. Bressan, F. Flores: On total differential inclusions, Rend. Semin. Mat. Univ. Padova 92 (1994) 9–16.

[12] A. Cellina: On the differential inclusion $\dot{x} \in [-1, 1]$, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. 69 (1980) 1–6.

[13] A. Cellina: On minima of functionals of gradient: sufficient conditions, Nonlinear Analysis 20 (1993) 343–347.

[14] A. Cellina: A view on differential inclusions, Rend. Semin. Mat., Torino 63 (2005) 197–209.

[15] A. Cellina, S. Perrotta: On a problem of potential wells, J. Convex Analysis 2 (1995) 103–115.

[16] A. Cellina, S. Zagatti: A version of Olech's lemma in a problem of the calculus of variations, SIAM J. Control Optimization 32 (1994) 1114–1127.

[17] B. Dacorogna: Nonconvex problems of the calculus of variations and differential inclusions, in: Stationary Partial Differential Equations. Vol. II, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam (2005) 57–126.

[18] B. Dacorogna: Calculus of variations, implicit partial differential equations and microstructure, GAMM-Mitt. 29(2) (2006) 150–171.

[19] F. S. De Blasi, G. Pianigiani: A Baire category approach to the existence of solutions of multivalued differential equations in Banach spaces, Funkc. Ekvacioj, Ser. Int. 25 (1982) 153–162.

[20] F. S. De Blasi, G. Pianigiani: Non-convex-valued differential inclusions in Banach spaces, J. Math. Anal. Appl. 157 (1991) 469–494.

[21] F. De Blasi, G. Pianigiani: On the Dirichlet problem for first order partial differential equations. A Baire category approach, NoDEA, Nonlinear Differ. Equ. Appl. 6(1) (1999) 13–34.

[22] C. De Lellis, L. Szekelyhidi: The Euler equations as a differential inclusion, Ann. Math. (2) 170(3) (2009) 1417–1436.

[23] B. Dacorogna, P. Marcellini: General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases, Acta Math. 178 (1997) 1–37.

[24] B. Dacorogna, P. Marcellini: Implicit Partial Differential Equations, Birkhäuser, Boston (1999).

[25] B. Dacorogna, P. Marcellini: On the solvability of implicit nonlinear systems in the vectorial case, in: Nonlinear Partial Differential Equations (Evanston, 1998), G.-Q. Chen et al. (ed.), Contemp. Math. 238, Amer. Math. Soc., Providence (1999) 89–113.

[26] B. Dacorogna, G. Pisante: A general existence theorem for differential inclusions in the vector valued case, Port. Math. (N.S.) 62(4) (2005) 421–436.

[27] L. C. Evans: Quasiconvexity and partial regularity in the calculus of variations, Arch. Ration. Mech. Anal. 95 (1986) 227–252.

[28] M. Gromov: Partial Differential Relations, Springer, Berlin (1986).

[29] B. Kirchheim: Deformations with finitely many gradients and stability of quasiconvex hulls, C. R. Acad. Sci., Paris, Sér. I, Math. 332(3) (2001) 289–294.

[30] B. Kirchheim: Lecture Notes 16, Max Planck Institute for Mathematics in Sciences, Leipzig (2003).

[31] B. Kirchheim, S. Müller, V. Šverák: Studying nonlinear pde by geometry in matrix space, in: Geometric Analysis and Nonlinear Partial Differential Equations, S. Hildebrandt et al. (ed.), Springer, Berlin (2003) 347–395.

[32] J. Kristensen, A. Taheri: Partial regularity of strong local minimizers in the multidimensional calculus of variations, Arch. Ration. Mech. Anal. 170 (2003) 63–89.

[33] S. Müller: Microstructures, phase transitions and geometry, in: European Congress of Mathematics (Budapest, 1996), Vol. 2, A. Balog et al. (ed.), Birkhäuser, Basel (1998).

[34] S. Müller, M. Rieger, V. Šverák: Parabolic systems with nowhere smooth solutions, Arch. Ration. Mech. Anal. 177(1) (2005) 1–20.

[35] S. Müller, M. Sychev: Optimal existence theorems for nonhomogeneous differential inclusions, J. Funct. Anal. 181(2) (2001) 447–475.

[36] S. Müller, V. Šverák: Unexpected solutions of first and second order partial differential equations, Doc. Math., J. DMV, Extra Vol. ICM Berlin 1998 II (1998) 691–702.

[37] S. Müller, V. Šverák: Attainment results for the two-well problem by convex integration, in: Geometric Analysis and the Calculus of Variations, J. Jost (ed.), International Press, Cambridge (1996) 239–251.

[38] S. Müller, V. Šverák: Convex integration with constraints and applications to phase transitions and partial differential equations, J. Eur. Math. Soc. (JEMS) 1(4) (1999) 393–422.

[39] S. Müller, V. Šverák: Convex integration for Lipschitz mappings and counterexamples to regularity, Ann. Math. 157 (2003) 715–742.

[40] Yu. G. Reshetnyak: General theorems on semicontinuity and convergence with a functional, Sib. Math. J. 8 (1967) 801–816 (in English); Sib. Math. Zh. 8 (1967) 1052–1071 (in Russian).

[41] T. Rzežuchowski: Impact of dentability on weak convergence in L^1, Boll. Unione Mat. Ital., VII. Ser., A 6 (1992) 71–80.

[42] M. Sychev: A few remarks on differential inclusions, Proc. R. Soc. Edinb., Sect. A, Math. 136 (2006) 649–668.

[43] M. Sychev: Few remarks on differential inclusions, MPI Preprint 46, MPI, Leipzig (2001).

[44] M. Sychev: Comparing two methods of resolving homogeneous differential inclusions, Calc. Var. Partial Differ. Equ. 13 (2001) 213–229.

[45] M. Sychev: Comparing various methods of resolving nonconvex variational problems, SISSA Preprint 66, SISSA, Triest (1998).

[46] M. Sychev: Necessary and sufficient conditions in theorems of lower semicontinuity and convergence with a functional, Russ. Acad. Sci., Sb., Math. 186 (1995) 847–878.

[47] M. Sychev: Convex integration. The method of approximate solutions, Dokl. Akad. Nauk, Ross. Akad. Nauk 392 (2003) 24–27 (in Russian); Dokl. Math. 68 (2003) 170–172 (in English).

[48] M. Sychev: Sets of lower semicontinuity and stability of integral functionals, J. Math. Pures Appl. 84 (2005) 985–1014.

[49] M. Sychev: Young measure approach to characterization of behaviour of integral functionals on weakly convergent sequences by means of their integrands, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 15 (1998) 755–782.

[50] M. Sychev: A new approach to Young measure theory, relaxation and convergence in energy, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 16 (1999) 773–812.

[51] M. Sychev: Semicontinuity and relaxation theorems for integrands satisfying the fast growth condition, Sib. Math. J. 46 (2005) 540–554.

[52] S. Saks: Theory of the Integral, 2nd Ed., Hafner, New York (1937).

[53] V. Scheffer: An inviscid flow with compact support in space-time, J. Geom. Anal. 3 (1993) 343–401.

[54] Shnirelman: On the nonuniqueness of weak solutions of the Euler equation, Commun. Pure Appl. Math. 50 (1997) 1261–1286.

[55] Shnirelman: Weak solutions with decreasing energy of incompressible Euler equations, Commun. Math. Phys. 210 (2000) 541–603.

[56] S. L. Sobolev: Some Applications of Functional Analysis in Mathematical Physics, Izdat. Leningrad. Gos. Univ., Leningrad (1950).

[57] V. Šverák: Lower semicontinuity of variational integrals and compensated compactness, in: Proceedings of the International Congress of Mathematicians (Zürich, 1994), Vol. 2, S. D. Chatterji (ed.), Birkhäuser, Basel (1995) 1135–1138.

[58] L. Székelyhidi: On the regularity of critical points of polyconvex functionals, Arch. Ration. Mech. Anal. 172 (2004) 133–152.

[59] M. Valadier: Young Measures, in: Methods of Convex Analysis (Varenna, 1989), A. Cellina (ed.), Lecture Notes in Mathematics 1446, Springer, Berlin (1990) 152–189.

[60] G. N. Vasilenko: Convergence with a functional, Sib. Math. J. 27 (1986) 26–35.

[61] A. Visintin: Strong convergence results related to strict convexity, Commun. Partial Differ. Equations 9 (1984) 439–466.

[62] H. G. Tucker: A Graduate Course in Probability, Probability and Mathematical Statistics, Vol. 2, Academic Press, New York (1967).

[63] K. W. Zhang: On the structure of quasiconvex hulls, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 15 (1998) 663–686.

[64] K. W. Zhang: On various semiconvex hulls in the calculus of variations, Calc. Var. Partial Differ. Equ. 6 (1998) 143–160.