Journal of Convex Analysis
Volume 19 (2012), No. 2, 393–402

Lineability Issues Involving Vector-Valued Measurable and Integrable Functions

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Received: November 18, 2009
Revised manuscript received: February 17, 2011

We construct, over many Banach spaces, infinite dimensional vector spaces of scalarly measurable functions that are not strongly measurable, and infinite dimensional vector spaces of ω^*-scalarly measurable functions that are not scalarly measurable. A similar result will be proved for the set of McShane-integrable functions which are not Bochner-integrable.

Keywords: Strongly measurable, scalarly measurable, ω^*-scalarly measurable, McShane integrable functions, Bochner integrable functions

1. Introduction

Let (Ω, Σ, μ) be a probability space and X be a real Banach space. A function $f : \Omega \to X$ is said to be:

(1) strongly measurable if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions from Ω to X such that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for almost every $\omega \in \Omega$;
(2) scalarly measurable if $x^* \circ f$ is measurable for every $x^* \in X^*$.

If $f : \Omega \to X^*$, then f is said to be ω^*-scalarly measurable if $x \circ f$ is measurable for every $x \in X$. Obviously, every strongly measurable function is scalarly measurable, and every scalarly measurable function with values in a dual Banach space is ω^*-scalarly measurable. It is also obvious that if X is reflexive, then ω^*-scalar measurability coincides with scalar measurability. On the other hand, the Pettis' Measurability Theorem (see [1, p. 42]) assures that if (Ω, Σ, μ) is a probability space and X is a real Banach space, then a function $f : \Omega \to X$ is strongly measurable if and only if f is scalarly measurable and essentially separably valued (that is, there exists $E \in \Sigma$ with $\mu(E) = 0$ such that $f(\Omega \setminus E)$ is a separable subset of X). As a consequence, strong measurability and scalar measurability coincide on separable real Banach spaces. In [1] one also finds the following two examples.

ISSN 0944-6532 / $ 2.50

Example 1.1.

(1) There exists a function $\psi : [0,1] \to \ell_2[0,1]$ that is scalarly measurable but not strongly measurable.

(2) There exists a function $\psi : [0,1] \to \ell_\infty$ that is ω^*-scalarly measurable but not scalarly measurable.

The above discussion motivates the following question:

Question 1.2. Let (Ω, Σ, μ) be a probability space and let X be a real Banach space. Then:

(1) If X is not separable, does there exist a scalarly measurable function $\psi : \Omega \to X$ that is not strongly measurable?

(2) If X is not reflexive, does there exist a ω^*-scalarly measurable function $\psi : \Omega \to X^*$ that is not scalarly measurable?

In what follows, we will provide partial positive solutions to Question 1.2. We will also utilize this occasion to show that in each infinite dimensional Banach space X, there are X-valued McShane-integrable functions which are not Bochner-integrable (it is well known that these two classes of functions coincide with the class of Lebesgue-integrable functions as long as X is finite-dimensional).

2. Scalarly measurable functions that are not strongly measurable

In this section, we construct, on every non-separable reflexive real Banach space X, an infinite dimensional vector space every non-zero element of which is a scalarly measurable X-valued function that is not strongly measurable. However, we would like first to introduce the following definition of crucial importance for the main result in this section.

Definition 2.1. A probability space (Ω, Σ, μ) is said to be amenable if $\{\omega\}$ is measurable and has measure zero for every $\omega \in \Omega$.

Observe that if (Ω, Σ, μ) is an amenable probablity space, then Ω is necessarily uncountable. Therefore, if X is a real Banach space whose density character, $\operatorname{dens}(X)$, satisfies that $\operatorname{dens}(X) \geq \operatorname{card}(\Omega)$, then X is not separable.

Theorem 2.2. *Let (Ω, Σ, μ) be an amenable probability space and let X be a reflexive real Banach space such that* $\operatorname{dens}(X) \geq \operatorname{card}(\Omega)$. *There exists an infinite dimensional vector space every non-zero element of which is a scalarly measurable function from Ω to X that is not strongly measurable.*

Proof. Since X is reflexive, X can be equivalently renormed to have a Markushevich basis $\left[(x_i)_{i\in I} \subset X, (x_i^*)_{i\in I} \subset \mathsf{S}_{X^*}\right]$ whose dual basis $\left[(x_i^*)_{i\in I} \subset \mathsf{S}_{X^*}, (x_i)_{i\in I} \subset X\right]$ is also a Markushevich basis for X^* (see [3, Theorem 11.20 and Theorem 11.23]). Since $\overline{\operatorname{span}}\{x_i : i \in I\} = X$, we have that $\operatorname{card}(I) \geq \operatorname{dens}(X) \geq \operatorname{card}(\Omega)$. Let us write

$$I = \dot{\bigcup_{n \in \mathbb{N}}} I_n,$$

where $\operatorname{card}(I_n) = \operatorname{card}(I)$ for every $n \in \mathbb{N}$. We can find an injective map $\psi_n : \Omega \to \{x_i : i \in I_n\}$ for every $n \in \mathbb{N}$. We will show now that, given $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ not all zero, the function $\lambda_1\psi_1 + \cdots + \lambda_n\psi_n$ is scalarly measurable but not strongly measurable.

(1). We will show first that $\lambda_1\psi_1 + \cdots + \lambda_n\psi_n$ is not strongly measurable. To do that, we will show that $\lambda_1\psi_1 + \cdots + \lambda_n\psi_n$ is not essentially separably valued (see [1, Chapter II, Theorem 2]). Let $E \subset \Omega$ such that $\mu(E) = 0$ and assume that the set $(\lambda_1\psi_1 + \cdots + \lambda_n\psi_n)(\Omega \setminus E)$ is separable. Since $\|x_i^*\| = 1$ for all $i \in I$, we have that

$$\begin{aligned} &\|(\lambda_1\psi_1(u) + \cdots + \lambda_n\psi_n(u)) - (\lambda_1\psi_1(v) + \cdots + \lambda_n\psi_n(v))\| \\ \geq\ &\sup\{|\lambda_1|, \ldots, |\lambda_n|\} \end{aligned}$$

for all $u \neq v \in \Omega$. As a consequence, $\Omega \setminus E$ is countable. This contradicts the fact that $\mu(\Omega \setminus E) = 1$ because (Ω, Σ, μ) is amenable.

(2). Finally, we will show that $\lambda_1\psi_1 + \cdots + \lambda_n\psi_n$ is scalarly measurable. Let $x^* \in X^*$. Since $\overline{\operatorname{span}}\{x_i^* : i \in I\} = X^*$, there exists a sequence $(y_m^*)_{m \in \mathbb{N}} \subset \operatorname{span}\{x_i^* : i \in I\}$ which is convergent to x^*. Since

$$\begin{aligned} &\|(y_m^* \circ (\lambda_1\psi_1 + \cdots + \lambda_n\psi_n))(\omega) - (x^* \circ (\lambda_1\psi_1 + \cdots + \lambda_n\psi_n))(\omega)\| \\ \leq\ &\|y_m^* - x^*\| \, \|\lambda_1\psi_1(\omega) + \cdots + \lambda_n\psi_n(\omega)\| \end{aligned}$$

for all $m \in \mathbb{N}$ and all $\omega \in \Omega$, we deduce that

$$(y_m^* \circ (\lambda_1\psi_1 + \cdots + \lambda_n\psi_n))_{m \in \mathbb{N}}$$

converges almost everywhere to $x^* \circ (\lambda_1\psi_1 + \cdots + \lambda_n\psi_n)$. To finish, we will show that $y_m^* \circ (\lambda_1\psi_1 + \cdots + \lambda_n\psi_n)$ is a simple function for all $m \in \mathbb{N}$. For an arbitrarily fixed $m \in \mathbb{N}$, there are $\gamma_1, \ldots, \gamma_k \in \mathbb{R}$ and $j_1, \ldots, j_k \in I$, different from each other, such that $y_m^* = \gamma_1 x_{j_1}^* + \cdots + \gamma_k x_{j_k}^*$. Observe now that the range of the function

$$\left(\gamma_1 x_{j_1}^* + \cdots + \gamma_k x_{j_k}^*\right) \circ (\lambda_1\psi_1 + \cdots + \lambda_n\psi_n)$$

is contained in the finite set

$$\{0\} \cup \left\{ \sum_{j \in A, i \in B} \gamma_j \lambda_i : A \subseteq \{1, \ldots, k\},\, B \subseteq \{1, \ldots, n\} \right\},$$

and hence $y_m^* \circ (\lambda_1\psi_1 + \cdots + \lambda_n\psi_n)$ is a simple function. □

3. ω^*-scalarly measurable functions that are not scalarly measurable

In this section, we construct, on the dual of every Banach space admitting a quotient isomorphic to ℓ_1, an infinite dimensional vector space every non-zero element of which is a ω^*-scalarly measurable function that is not strongly measurable. As in the previous section, we would like first to introduce a definition of vital importance before presenting the main result in this section.

Definition 3.1. A probability space (Ω, Σ, μ) is said to be normal if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions from Ω to $\{0,1\}$ with no measurable cluster points in $\{0,1\}^\Omega$ (with respect to the pointwise convergence topology).

An example of a normal probability space is the closed interval $[0,1]$ with the Lebesgue measure (see [4]). We refer the reader to [4] for a wider perspective on this type of probabilty spaces and their connection with perfect probability spaces.

Theorem 3.2. *Let (Ω, Σ, μ) be a normal probability space and let X be a real Banach space admitting a quotient isomorphic to ℓ_1. There exists an infinite dimensional vector space every non-zero element of which is a ω^*-scalarly measurable function from Ω to X^* that is not scalarly measurable.*

Proof. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions from Ω to $\{0,1\}$ with no measurable cluster points in $\{0,1\}^\Omega$ (with respect to the pointwise convergence topology). Since $\{0,1\}^\Omega$ is a compact space, we deduce that $(f_n)_{n\in\mathbb{N}}$ possesses a subnet $\left(f_{\sigma(i)}\right)_{i\in I}$ convergent to a non-measurable function $f : \Omega \to \{0,1\}$, where $\sigma : I \to \mathbb{N}$ is a directed map. Let $p : X \to \ell_1$ be a surjective, continuous, linear operator. At this point we want to single out that $p^*\left(\mathsf{B}_{\ell_\infty}\right)$ is ω^*-compact in X^*, therefore it is closed in X^* as well as $p^*\left(\ell_\infty\right)$, and hence the Open Mapping Theorem assures that $(p^*)^{-1} : p^*\left(\ell_\infty\right) \to \ell_\infty$ is continuous. Let us write

$$\mathbb{N} = \dot{\bigcup_{k\in\mathbb{N}}} N_k,$$

where $\operatorname{card}(N_k) = \operatorname{card}(\mathbb{N})$ for all $k \in \mathbb{N}$. More precisely, we write the N_k's as strictly increasing sequences of natural numbers, $N_k = \left\{n_j^k : j \in \mathbb{N}\right\}$ for every $k \in \mathbb{N}$. Now we can define the following function for every $k \in \mathbb{N}$,

$$\begin{array}{rccl} \tau_k : & \ell_\infty & \to & \ell_\infty \\ & (a_n)_{n\in\mathbb{N}} & \mapsto & \tau_k\left((a_n)_{n\in\mathbb{N}}\right) = (b_n)_{n\in\mathbb{N}}, \end{array}$$

where

$$b_n = \begin{cases} 0 & \text{if } n \notin N_k, \\ a_j & \text{if } n = n_j^k, \end{cases}$$

for every $n \in \mathbb{N}$. Next, for every $k \in \mathbb{N}$, we define the function

$$\begin{array}{rccl} \psi_k : & \Omega & \to & X^* \\ & t & \mapsto & \psi_k(t) = (p^* \circ \tau_k)\left((f_n(t))_{n\in\mathbb{N}}\right). \end{array}$$

We will show now that, given $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ not all zero, the function $\lambda_1\psi_1 + \cdots + \lambda_k\psi_k$ is ω^*-scalarly measurable but not scalarly measurable.

(1) Firstly, let us prove that $\lambda_1\psi_1 + \cdots + \lambda_k\psi_k$ is not scalarly measurable. Let $m \in \{1, \ldots, k\}$ such that $\lambda_m \neq 0$. Consider the vector subspace

$$M = \left\{(c_n)_{n\in\mathbb{N}} \in \ell_\infty : \lim_{i\in I}\left(b_{\sigma(i)}\right) \text{ exists, where } (b_j)_{j\in\mathbb{N}} = \left(c_{n_j^m}\right)_{j\in\mathbb{N}}\right\}.$$

Observe that $\tau_m\left((f_n(t))_{n\in\mathbb{N}}\right)\in M$ for all $t\in\Omega$. Define

$$\begin{array}{rccl}\varphi: & M & \to & \mathbb{R}\\ & (c_n)_{n\in\mathbb{N}} & \mapsto & \varphi\left((c_n)_{n\in\mathbb{N}}\right)=\lim\limits_{i\in I}\left(b_{\sigma(i)}\right),\end{array}$$

again where $(b_j)_{j\in\mathbb{N}}=(c_{n_j^m})_{j\in\mathbb{N}}$. Observe that

$$\varphi\left(\tau_m\left((f_n(t))_{n\in\mathbb{N}}\right)\right)=f(t)$$

for all $t\in\Omega$ and

$$\varphi\left(\tau_p\left((a_n)_{n\in\mathbb{N}}\right)\right)=0$$

for all $p\neq m$ and all $(a_n)_{n\in\mathbb{N}}\in\ell_\infty$. Since $\left|\varphi\left((c_n)_{n\in\mathbb{N}}\right)\right|\leq\left\|(c_n)_{n\in\mathbb{N}}\right\|_\infty$ for all $(c_n)_{n\in\mathbb{N}}\in M$, in virtue of the Hahn-Banach Theorem we can extend φ to a continuous linear functional $\phi:\ell_\infty\to\mathbb{R}$. Again by the Hahn-Banach Theorem, $\phi\circ(p^*)^{-1}$ can be extended linearly and continuously to a functional $x^{**}\in X^{**}$. If we look at the composition $x^{**}\circ(\lambda_1\psi_1+\cdots+\lambda_k\psi_k)$, then for every $t\in\Omega$ we have that

$$\begin{aligned}(x^{**}\circ(\lambda_1\psi_1+\cdots+\lambda_k\psi_k))(t)&=\lambda_m(x^{**}\circ\psi_m)(t)\\&=\lambda_m\lim_{i\in I}\left(f_{\sigma(i)}(t)\right)\\&=\lambda_m f(t),\end{aligned}$$

that is, $x^{**}\circ(\lambda_1\psi_1+\cdots+\lambda_k\psi_k)=\lambda_m f$ is not measurable.

(2) Finally, we will prove that $\lambda_1\psi_1+\cdots+\lambda_k\psi_k$ is ω^*-scalarly measurable. Let $x\in X$. There exists $(a_n)_{n\in\mathbb{N}}\in\ell_1$ such that $p(x)=(a_n)_{n\in\mathbb{N}}$. For every $t\in\Omega$, we have that

$$(x\circ(\lambda_1\psi_1+\cdots+\lambda_k\psi_k))(t)=\lambda_1\sum_{j=1}^{\infty}a_{n_j^1}f_j(t)+\cdots+\lambda_k\sum_{j=1}^{\infty}a_{n_j^k}f_j(t).$$

Observe that the sequence

$$\left(\lambda_1\sum_{j=1}^{l}a_{n_j^1}f_j(t)+\cdots+\lambda_k\sum_{j=1}^{l}a_{n_j^k}f_j(t)\right)_{l\in\mathbb{N}}$$

of measurable functions converges (uniformly) to the function $x\circ(\lambda_1\psi_1+\cdots+\lambda_k\psi_k)$, and hence $x\circ(\lambda_1\psi_1+\cdots+\lambda_k\psi_k)$ is measurable. □

4. McShane integrable functions that are not Bochner integrable

Following [5], we recall that a function $f:[0,1]\to X$ is said to be *Mcshane integrable* if there exists $x\in X$ with the following property:
(M): for every $\epsilon>0$, there exists $\delta:I\to\mathbb{R}^+$ such that for all partitions $0=x_0<x_1<\cdots<x_n=1$ of $I=[0,1]$ and all points $t_i\in I$ with $[x_{i-1},x_i]\subset(t_i-\delta(t_i),t_i+\delta(t_i))$, $1\leq i\leq n$, it follows that

$$\left\|\sum_{i=1}^{n}f(t_i)(x_i-x_{i-1})-x\right\|<\epsilon.$$

Here we say that the *McShane Integral* of f is x and, in symbols, we shall write

$$(M)\int_0^1 f(t)dt = x.$$

We shall also say that f is *Bochner-integrable* if it is (strongly) measurable and

$$\int_0^1 \|f(t)\|\, dt < \infty.$$

See [1], Chapter II, Theorem 2.2. Letting $M(X)$ and $B(X)$ denote the spaces of McShane and Bochner integrable functions $f : I \to X$, respectively, it is not difficult to check that $B(X) \subset M(X)$. As already noted in the introduction, the above inclusion can be strengthened to an equality for X with $\dim X < \infty$. That the inclusion is proper for the infinite-dimensional Banach space X can also be seen by checking that $B(X)$ is always complete whereas $M(X)$ is incomplete for such an X. In what follows we shall in fact show that corresponding to each infinite-dimensional Banach space X, there exists an infinite-dimensional vector space every non-zero element of which is a McShane integrable function from I into X that is not Bochner integrable. Let us say that a partition satisfying the condition in the definition of a McShane integrable function shall be called a *δ-fine M-partition* corresponding to the function δ which will be referred to as a *gauge*.

Theorem 4.1. *Let X be an infinite-dimensional Banach space. Then there exists an infinite-dimensional vector space of functions from I into X every non-zero element of which is McShane integrable but not Bochner-integrable.*

Proof. We shall prove the above assertion in several steps:

(1) Let $\sum_{n=1}^{\infty} x_n$ be an unconditionally convergent series in X with $x_n \neq 0$, $n \geq 1$. Choose a sequence $\{K_n\}_{n=1}^{\infty}$ of open subintervals of $I = [0,1]$ such that $K_m \cap K_n = \phi, m \neq n$ and put $K = \cup_{n=1}^{\infty} K_n$, $C = I \setminus K$. Setting $y_n = \frac{x_n}{\lambda(K_n)}$, $n \geq 1$ where λ is the Lebesgue measure on I, we see that the series $\sum_{n=1}^{\infty} \lambda(K_n) y_n$ is unconditionally convergent. We now define

$$f(t) = \sum_{n=1}^{\infty} y_n \chi_n(t), \quad t \in I$$

where χ_n is the indicator function of K_n.

Claim: f is McShane integrable. We show that, in fact, $(M)\int_0^1 f(t)dt = \sum_{n=1}^{\infty} x_n = x$, say .. (*)

To this end, fix $\epsilon > 0$ and choose $m \geq 1$ such that

$$\left\|\sum_{j=1}^{m} y_j \lambda(K_j) - x\right\| = \left\|\sum_{j=1}^{m} x_j - x\right\| < \epsilon/3 \tag{1}$$

$$\left\|\sum_{j\in Q} y_j \lambda(K_j)\right\| = \left\|\sum_{j\in Q} x_j\right\| < \epsilon/3, \tag{2}$$

for all finite subsets $Q \subset \{m+1, m+2, ...\}$.
Further, choose η such that

$$0 < \eta < \frac{\epsilon}{3\left(\sum_{j=1}^{m} \|y_j\|\right)} \tag{3}$$

and an open set $G \supset C$ such that $\lambda(G) < \lambda(C) + \eta$.
Thus, we can define a gauge $\delta : I \to \mathbb{R}^+$ such that

$$S(t, \delta(t)) \subset K_j, \quad \text{if } t \in K_j,\ j \geq 1$$

$$S(t, \delta(t)) \subset C_j, \quad \text{if } t \in C. \tag{4}$$

To show that (*) holds amounts to proving that given a δ-fine M-partition $\{(J_i, t_i); 1 \leq i \leq k\}$ of I, where δ is the gauge defined by (4), then

$$\left\| \sum_{i=1}^{k} f(t_i)\lambda(J_i) - x \right\| < \epsilon \tag{**}$$

By (1), we have

$$\left\| \sum_{i=1}^{k} f(t_i)\lambda(J_i) - x \right\| < \epsilon/3 + \left\| \sum_{i=1}^{k} f(t_i)\lambda(J_i) - \sum_{i=1}^{m} y_i \lambda(k_i) \right\|$$

Put

$$K_1 = \cup_{i=1}^{m} K_i\,, \qquad K_2 = \cup_{i=m+1}^{\infty} K_i\,.$$

Then, we get

$$\begin{aligned}
\sum_{i=1}^{k} f(t_i)\lambda(J_i) &= \sum_{\substack{i=1 \\ t_i \in K_1}}^{k} f(t_i)\lambda(J_i) + \sum_{\substack{i=1 \\ t_i \in K_2}}^{k} f(t_i)\lambda(J_i) \\
&= \sum_{j=1}^{m} \sum_{\substack{i=1 \\ t_i \in K_j}}^{k} f(t_i)\lambda(J_i) + \sum_{j=m+1}^{\infty} \sum_{\substack{i=1 \\ t_i \in K_j}}^{k} f(t_i)\lambda(J_i) \\
&= \sum_{j=1}^{m} y_j \sum_{i=1}^{k} \lambda(J_i) + \sum_{j=m+1}^{\infty} \sum_{\substack{i=1 \\ t_i \in K_j}}^{k} \lambda(J_i).
\end{aligned}$$

This gives:

$$\begin{aligned}
&\left\| \sum_{i=1}^{k} f(t_i)\lambda(J_i) - \sum_{i=1}^{m} y_i \lambda(k_i) \right\| \\
\leq\ &\left\| \sum_{j=1}^{m} y_j \left(\sum_{i=1}^{k} \lambda(J_i) - \lambda(k_j) \right) \right\| + \left\| \sum_{j=m+1}^{\infty} y_j \left(\sum_{\substack{i=1 \\ t_i \in K_j}}^{k} \lambda(J_i) \right) \right\|
\end{aligned} \tag{5}$$

Since $\{(J_i, t_i)\}_{i=1}^k$ is δ-fine, (4) yields that $J_i \subset K_j$, if $t_i \in K_j$.
This leads to

$$\sum_{\substack{i=1\\ t_i \in K_j}}^{k} \lambda(J_i) \leq \lambda(K_j),$$

and, therefore, we can choose $\mu_j \in [0,1]$ such that

$$\sum_{\substack{i=1\\ t_i \in K_j}}^{k} \lambda(J_i) = \mu_j \lambda(K_j), \quad \text{for all } j \geq 1$$

Combining with (2), this gives

$$\begin{aligned} \left\| \sum_{j=m+1}^{\infty} \left(\sum_{\substack{i=1\\ t_i \in K_j}}^{k} \lambda(J_i) \right) y_j \right\| &= \lim_{N\to\infty} \left\| \sum_{j=m+1}^{N} \left(\sum_{\substack{i=1\\ t_i \in K_j}}^{k} \lambda(J_i) \right) y_j \right\| \\ &= \lim_{N\to\infty} \left\| \sum_{j=m+1}^{N} \mu_j \lambda(K_j) y_j \right\| < \epsilon/3 \end{aligned} \tag{6}$$

To estimate the first term on the RHS of (5), we see that

$$\left\| \sum_{j=1}^{m} y_j \left(\sum_{i=1}^{k} \lambda(J_i) - \lambda(k_j) \right) \right\| \leq \sum_{j=1}^{m} \|y_j\| \left(\left| \lambda(K_j) - \sum_{\substack{i=1\\ t_i \in K_j}}^{k} \lambda(J_i) \right| \right) \tag{7}$$

and since $\{J_i\}_{i=1}^K$ is δ-fine, so that for $1 \leq j \leq k$,

$$K_j \setminus \left(\cup_{t_i \in K_j} J_i \right) \subset K \setminus \left(\cup_{t_i \in K} J_i \right) \tag{8}$$

we get

$$\begin{aligned} \lambda(K_j) - \sum_{\substack{i=1\\ t_i \in K_j}}^{k} \lambda(J_i) &= \lambda(K_j) - \lambda\left(\bigcup_{\substack{i=1\\ t_i \in K_j}}^{k} J_i \right) = \lambda\left(K_j \setminus \left(\bigcup_{\substack{i=1\\ t_i \in K_j}}^{k} J_i \right) \right) \\ &\leq \lambda\left(K \setminus \left(\bigcup_{\substack{i=1\\ t_i \in K}}^{k} J_i \right) \right) = \lambda(K) - \lambda\left(\bigcup_{\substack{i=1\\ t_i \in K}}^{k} J_i \right) \\ &= \lambda\left(\bigcup_{\substack{i=1\\ t_i \in C}}^{k} J_i \right) - \lambda(C) \leq \lambda(G) - \lambda(C) < \eta, \quad \text{(By (3))} \end{aligned}$$

Now (3) and (7) combined with the above estimate gives:

$$\left\| \sum_{j=1}^{m} y_j \left(\sum_{i=1}^{k} \lambda(J_i) - \lambda(k_j) \right) \right\| < \eta \sum_{j=1}^{m} \|y_j\| < \epsilon/3$$

This together with (6) yields that the (RHS) of (5) is less than $2\epsilon/3$ and, therefore,

$$\left\|\sum_{i=1}^{k} f(t_i)\lambda(J_i) - x\right\| < \epsilon$$

or, equivalently,

$$(M)\int_0^1 f(t)dt = x$$

(2) Choose a sequence $\{x_n\} \subset X$, $x_n \neq 0$ for all $n \geq 1$, such that $\sum_n x_n$ is unconditionally convergent but not absolutely convergent. This is possible by Dvoretzky - Rogers theorem (see [2, 1.2]). Let $\{A_\alpha; \alpha \in \Lambda\}$ be a family of subsets of $\mathbb{N}$ such that

(a) $\operatorname{card} A_\alpha = \mathbb{N}_0$, for all $\alpha \in \Lambda$,
(b) $A_\alpha \cap A_\beta$ is a finite set for all $\alpha \neq \beta \in \Lambda$,
(c) $\operatorname{card} \Lambda = c$, cardinality of the continuum.

For each $\alpha \in \Lambda$, define

$$x^{(\alpha)} = \left(x_n^{(\alpha)}\right)_{n=1}^{\infty} \subset X$$

such that

$$x_i^{(\alpha)} = \begin{cases} x_n, & \text{if } i \in A_\alpha \text{ and } i \text{ is the } n^{th}\text{-term of } A_\alpha \\ 0, & \text{otherwise} \end{cases}$$

Further, we define

$$f_\alpha(t) = \sum_{i=1}^{\infty} \frac{x_i^{(\alpha)}}{\lambda(K_i)}\chi_i(t), \quad t \in [0,1].$$

Since $\sum_{i=1}^{\infty} x_i^{(\alpha)}$ is unconditionally convergent in X for each $\alpha \in \Lambda$, it follows by step (1) that $f_\alpha \in M(X)$.

Claim 1: $\{f_\alpha; \alpha \in \Lambda\}$ is linearly independent. Let $\sum_{k=1}^{n} \lambda_k f_{\alpha_k} = 0$ and fix $1 \leq j \leq n$. Now choose $i \in A_{\alpha j}$ such that $i \notin A_{\alpha_l}, l \neq j$. Take $t \in K_i$ and use the definition of f_α to get

$$\frac{1}{\lambda(K_i)}\sum_{k=1}^{n} \lambda_k x_i^{(\alpha_k)} = 0, \quad \text{i.e. } \lambda_j = 0.$$

This shows that $\{f_\alpha : \alpha \in \Lambda\}$ is linearly independent. Thus $E = \operatorname{span}\{f_\alpha : \alpha \in \Lambda\}$ is a c-dimensional vector space such that $E \subset M(X)$.

Claim 2: $E \cap B(X) = \{0\}$. To this end, let

$$f = \sum_{i \in \Lambda_0} \lambda_i f_{\alpha_i}, \quad \lambda_i \in \mathbb{R}, \ \Lambda_0 = \{\alpha_1, \alpha_2, \cdots \alpha_l\} \subset \Lambda$$

Assume, without loss of generality, that $\lambda_1 \neq 0$. Choose an infinite set $A \subset A_{\alpha_1}$ such that $A_{\alpha_1} \setminus A$ is finite and that

$$A \cap \left(\cup_{i=2}^{l} A_{\alpha_i}\right) = \phi.$$

Now, the definition of f_α's gives:

$$\begin{aligned}\int_0^1 \|f(t)\|dt &= \sum_{n=1}^{\infty} \int_{K_n} \left\| \sum_{i\in\Lambda_0} \lambda_i f_{\alpha_i} \right\| dt \\ &= \sum_{n=1}^{\infty} \int_{K_n} \left\| \sum_{i\in\Lambda_0} \lambda_i x_n^{(\alpha_i)} \right\| \frac{dt}{\lambda(K_n)} \\ &= \sum_{n=1}^{\infty} \left\| \sum_{i\in\Lambda_0} \lambda_i x_n^{(\alpha_i)} \right\| \\ &\geq \sum_{n\in A} |\lambda_1| \left\| x_n^{(\alpha_1)} \right\| .\end{aligned}$$

Finally, since $\sum_{n\in A_{\alpha_1}} \|x_n^{(\alpha_1)}\| = \infty$ and $A_{\alpha_1} \setminus A$ is finite, it follows that $\sum_{n\in A} \|x_n^{(\alpha_i)}\| = \infty$, since all the terms $\|x_n\|$ appear in that sum and that $\sum_{n\in\mathbb{N}} \|x_n\| = \infty$. Because $\lambda_1 \neq 0$, we conclude that

$$\int_0^1 \|f(t)\|dt = \infty,$$

which yields that $f \notin B(X)$.

Remark. The conclusion of Step (1) of the proof of the above theorem was also proved by S. J. Dilworth and M. Girardi in their work "Nowhere weak differentiability of the Pettis integral", Quaest. Math. 18 (1995) 365–380. □

Acknowledgements. The authors would like to thank the learned referee for his / her comments and suggestions which have led to a considerable improvement of the paper.

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