Journal of Convex Analysis
Volume 19 (2012), No. 1, 141–157

Strong Laws of Large Numbers for Double Arrays of Independent Set-Valued Random Variables in Banach Spaces*

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Received: May 5, 2010
Revised manuscript received: November 19, 2010

In this paper, we establish some strong laws of large numbers for double arrays of independent set-valued random variables in separable Banach spaces under various conditions.

Keywords: Double array, triangular array, strong law of large numbers, independent, compactly uniformly integrable, stochastically dominated, uniformly p-smooth, martingale

2000 Mathematics Subject Classification: 28B20, 60G42, 46A17, 54A20

1. Introduction

The theory of set-valued random variables has been extensively studied and applied in the areas of information science, probability and statistics. In particular, the strong law of large numbers (SLLN) for sums of independent set-valued random variables in Banach space have been studied by several authors. Taylor and Inoue [16] established the SLLN for compactly uniformly integrable independent sequences of compact-valued random variables $\{X_n : n \geq 1\}$ by assuming that $\sum_{n=1}^{\infty} n^{-p} E\|X_n\|^p < \infty$ (Chung type condition). Then, they also proved another SLLN result by replacing Chung type condition by the condition that $\{X_n : n \geq 1\}$ is stochastically dominated (see [17]). Developing the work in [17], Fu and Zhang [7] obtained the SLLN for triangular arrays of rowwise independent and compactly uniformly integrable compact-valued random variables. In this paper, we state several new variants of SLLN for double arrays of independent set-valued random variables under various assumptions. The paper is organized as follows. In Section 2 we state and summarize basic results in set-valued integration and probability. Section 3 is concerned with the SLLN for double array of independent compact valued

*This work was supported by the Vietnam's National Foundation for Science and Technology Development (NAFOSTED), No. 101.02.32.09.

ISSN 0944-6532 / \$ 2.50

and compactly uniformly integrable random variables. In Section 4 we present the SLLN for double array of independent convex weakly compact valued random variables.

2. Preliminaries

Throughout this paper $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(X, \|.\|)$ is a real separable Banach space and X^* is its topological dual. Let $\mathrm{c}(X)$ (resp. $\mathrm{cc}(X)$) (resp. $\mathrm{cwk}(X)$) (resp. $\mathrm{k}(X)$) (resp. $\mathrm{ck}(X)$) be the set of nonempty closed (resp. closed convex) (resp. convex weakly compact) (resp. compact) (resp. convex compact) subsets of X. For $A \in \mathrm{c}(X)$, the distance function and the support function associated with A are defined respectively by

$$d(x, A) = \inf\{\|x - y\| \ : y \in A\}, \quad (x \in X)$$

$$\delta^*(x^*, A) = \sup\{\langle x^*, y\rangle \ : y \in A\}, \quad (x^* \in X^*).$$

We also define

$$|A| = \sup\{\|x\| \ : x \in A\}$$

and denote by d_H the Hausdorff distance defined on the $\mathrm{c}(X)$ associated with the topology of the norm in X

$$d_H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}.$$

If $d_H(A, B) < \infty$ then

$$d_H(A, B) = \max\left\{\inf\{\lambda > 0 : B \subset \overline{U}(A; \lambda)\}, \inf\{\lambda > 0 : A \subset \overline{U}(B; \lambda)\}\right\},$$

where $\overline{U}(A; \lambda) = \{x \in X : d(x, A) \leq \lambda\}$ and $\overline{U}(B; \lambda) = \{x \in X : d(x, B) \leq \lambda\}$.

A closed valued mapping $F : \Omega \to \mathrm{c}(X)$ is $\mathcal{F}$-measurable if for every open set U in X the set

$$F^-(U) := \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$$

is a member of $\mathcal{F}$. A function $f : \Omega \to X$ is a $\mathcal{F}$-measurable selection of F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. A Castaing representation of F is a sequence $(f_n)_{n\in\mathbf{N}}$ of $\mathcal{F}$-measurable selections of F such that

$$F(\omega) = \mathrm{cl}\{f_n(\omega), n \in \mathbf{N}\} \ \ \forall w \in \Omega$$

where the closure is taken with respect to the topology of associated with the norm in X. It is known that a nonempty closed-valued mapping $F : \Omega \to \mathrm{c}(X)$ is $\mathcal{F}$-measurable iff it admits a Castaing representation. Since $\mathcal{F}$ is complete, the $\mathcal{F}$-measurability is equivalent to the measurability in the sense of graph, namely the graph of F is a member of $\mathcal{F} \otimes \mathcal{B}(X)$, here $\mathcal{B}(X)$ denotes the Borel tribe on X. Further the Effros σ-field $\mathcal{E}$ of $\mathrm{c}(X)$ is generated by the subsets

$$U^- := \{F \in \mathrm{c}(X) : F \cap U \neq \emptyset\}$$

where U is an open subset in X. Then a mapping $\Gamma : \Omega \to \mathrm{c}(X)$ is $\mathcal{F}$-measurable if and only if, for any $B \in \mathcal{E}$, one has $\Gamma^{-1}(B) \in \mathcal{F}$. The *distribution* P_Γ of the $\mathcal{F}$-measurable mapping $\Gamma : \Omega \to \mathrm{c}(X)$ on the measurable space $(\mathrm{c}(X), \mathcal{E})$ is defined by

$$P_\Gamma(B) = P\{\Gamma^{-1}(B)\}, \quad \forall B \in \mathcal{E}.$$

Two $\mathcal{F}$-measurable mappings Γ and Δ are said to be *equidistributed* (or to have the same distribution) if

$$P_\Gamma = P_\Delta.$$

Two $\mathcal{F}$-measurable mappings Γ and Δ are *independent,* if the following equality holds

$$P_{(\Gamma,\Delta)} = P_\Gamma \otimes P_\Delta.$$

A closed-valued $\mathcal{F}$-measurable mapping is also called *closed-valued random variable.* We denote by $\mathcal{L}^1_X(\mathcal{F})$ the space of X-valued $\mathcal{F}$-measurable and Bochner-integrable functions defined on Ω. A $\mathrm{c}(X)$-valued $\mathcal{F}$-measurable $F : \Omega \to \mathrm{c}(X)$ is *integrable* if the set $S^1_F(\mathcal{F})$ of all $\mathcal{F}$-measurable and integrable selections of F is nonempty and it is called *integrably bounded* if $|F| \in \mathcal{L}^1_{\mathbf{R}}(\mathcal{F})$. Given two closed valued integrable random variables F and G, then F and G are independent iff for any $n \in \mathbf{N}$, for any $f = (f_1, f_2, ...f_n)$ in $[\mathcal{S}^1_F(\mathcal{F}_F)]^n$ and for any $g = (g_1, g_2, ...g_n)$ in $[\mathcal{S}^1_G(\mathcal{F}_G)]^n$, f and g are independent, where $\mathcal{F}_F$ is a σ-algebra generated by F. We refer to Hess ([8], Theorem 1-2) for a complete study of the independence of integrable set-valued random variables.

The *expectation* $E[F]$ of a closed valued integrable random variable F is defined by

$$E[F] := \mathrm{cl}\{Ef : f \in \mathcal{S}^1_F\}$$

where the closure is taken in X and Ef is the usual expectation of $f \in \mathcal{S}^1_F$.

If F is a $\mathrm{cwk}(X)$-valued random variable with $|F| \in \mathcal{L}^1_{\mathbf{R}}$, shortly $F \in \mathcal{L}^1_{\mathrm{cwk}(X)}(\mathcal{F})$, then the expectation of F, denote by $E[F]$

$$E[F] = \{Ef : f \in \mathcal{S}^1_F\}$$

is convex weakly compact. See ([4], Theorem V-14).

If F is a $\mathrm{ck}(X)$-valued random variable with $|F| \in \mathcal{L}^1_{\mathbf{R}}$, shortly $F \in \mathcal{L}^1_{\mathrm{ck}(X)}(\mathcal{F})$, then the expectation of F is given by $E[F]$

$$E[F] = \{Ef : f \in \mathcal{S}^1_F\}$$

is convex and norm compact. See ([4], Theorem V-15).

Given a sub-σ-algebra $\mathcal{A}$ of $\mathcal{F}$ and an integrable $\mathcal{F}$-measurable $\mathrm{cc}(X)$-valued mapping $F : \Omega \to \mathrm{cc}(X)$. Hiai and Umegaki [10] showed the existence of a $\mathcal{A}$-measurable $\mathrm{cc}(X)$-valued integrable mapping denoted by $E[F|\mathcal{A}]$ such that

$$\mathcal{S}^1_{E[F|\mathcal{A}]}(\mathcal{A}) = \mathrm{cl}\{E[f|\mathcal{A}] : f \in \mathcal{S}^1_F(\mathcal{F})\}$$

the closure being taken in $\mathcal{L}^1_X(\mathcal{F})$, and $E[F|\mathcal{A}]$ is the *conditional expectation* of F relative to $\mathcal{A}$. We summarize the properties of conditional expectations as follows.

Proposition 2.1. *If F and G are two closed convex valued integrable random variables in X, and $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{F}$, then we have the following properties:*

(a) $E[\mathrm{cl}\{F+G\}|\mathcal{A}] = \mathrm{cl}\{E[F|\mathcal{A}] + E[G|\mathcal{A}]\}$ *a.s.*

(b) *If r is a real $\mathcal{A}$-measurable function such that rF is integrable, then*

$$E[rF|\mathcal{A}] = rE[F|\mathcal{A}] \quad a.s.$$

(c) *If g is a bounded scalarly $\mathcal{A}$-measurable function from Ω to X^*, then*

$$\delta^*(g, E[F|\mathcal{A}]) = E(\delta^*(g, F)|\mathcal{A}) \quad a.s.$$

In particular $\delta^*(x^*, E[F|\mathcal{A}]) = E(\delta^*(x^*, F)|\mathcal{A})$ a.s. for every $x^* \in X^*$.

(d) *Let F be $\mathcal{A}$-measurable and r be a $\mathcal{F}$-measurable positive function such that rF is integrable; then*

$$E[rF|\mathcal{A}] = E(r|\mathcal{A})F \quad a.s.$$

In the case where $F \in \mathcal{L}^1_{\mathrm{cwk}(X)}(\mathcal{F})$, and the dual X^* is strongly separable, we present a specific version of conditional expectation that we summarize below.

Proposition 2.2. *Assume that X^* is strongly separable and $F \in \mathcal{L}^1_{\mathrm{cwk}(X)}(\mathcal{F})$. Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$. Then there is a unique (for the equality a.s.) $\mathcal{A}$-measurable* $\mathrm{cwk}(X)$*-valued mapping $Y := E[F|\mathcal{A}]$ satisfying the properties:*

$$(i) \qquad \mathcal{S}^1_{E[F|\mathcal{A}]}(\mathcal{A}) = \{E(f|\mathcal{A}) : f \in \mathcal{S}^1_F(\mathcal{F})\}; \tag{1}$$

$$(ii) \qquad \forall v \in L^\infty_{X^*}(\mathcal{A}), E\delta^*(v, Y) = E\delta^*(v, F); \tag{2}$$

$$(iii) \qquad d(0, E[F|\mathcal{A}]) \le E(d(0, F)|\mathcal{A}). \tag{3}$$

The existence of $\mathrm{cwk}(X)$-valued conditional expectation for $\mathrm{cwk}(X)$-valued random variable was stated in ([2], Theorem 3). A unified approach for general conditional expectation of $\mathrm{cc}(X)$-valued integrable multifunctions is given in [18] allowing to recover both the $\mathrm{cc}(X)$-valued conditional expectation of $\mathrm{cc}(X)$-valued integrable mapping in the sense of [10] and the $\mathrm{cwk}(X)$-valued conditional expectation of $\mathrm{cwk}(X)$-valued integrably bounded multifunctions given in [2].

Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$ and $\sigma(F)$ be the σ-algebra generated by F. If $\mathcal{A}$ and $\sigma(F)$ are independent, then

$$E[F|\mathcal{A}] = E[F] \in \mathrm{cwk}(X).$$

A sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ of $\mathrm{cc}(X)$-valued mapping is adapted if each X_n is $\mathcal{F}_n$-measurable. When X^* is strongly separable, an adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{\mathrm{cwk}(X)}(\mathcal{F})$ is a $\mathcal{L}^1$-bounded $\mathrm{cwk}(X)$-valued martingale, submartingale, or supermartingale, if $E[X_{n+1}|\mathcal{F}_n] =, \supset$, or $\subset X_n$ a.s. for all $n \in \mathbf{N}$ and $\sup_{n\in\mathbf{N}} E|X_n| < \infty$. Note that this definition has a meaning thanks to Proposition 2.2.

For more information on multivalued conditional expectation and related subjects we refer to [1, 3, 4, 10, 13, 18, 19]. We refer to [4] for the theory of Measurable

Multifunctions and Convex Analysis, and to [6, 12] for basic theory of martingales and adapted sequences.

Let us recall the maximal inequalities for positive submartingales. The first one has been obtained by Doob and the second one is a special version of Marcinkiewicz.

Theorem 2.3. *Let $\{X_k, \mathcal{F}_k : 1 \le k \le n\}$ be a positive submartingale. Then for $p \ge 1$ we have the maximal inequalities:*

$$E\left(\max_{1\le k\le n} X_k\right)^p \le q^p E X_n^p, \quad \text{if } p > 1,\ q = \frac{p}{p-1};$$

$$E\left(\max_{1\le k\le n} X_k\right) \le \frac{e}{e-1}(1 + E(X_n \log^+ X_n)), \quad \text{if } p = 1.$$

A real separable Banach space is of Rademacher type $p\,(1 \le p \le 2)$ if and only if there exists a constant $0 < C < \infty$ such that

$$E\left\|\sum_{j=1}^{n} f_j\right\|^p \le C \sum_{j=1}^{n} E\|f_j\|^p$$

for every finite collection $\{f_1, ..., f_n\}$ of independent integrable random variables with mean 0. The details of definition and proofs, we refer the reader to [11].

A collection $\{F_{mn} : m \ge 1, n \ge 1\}$ of weakly compact valued random variables is *stochastically dominated* by a real valued random variable F if for some constant $C < \infty$

$$P\{|F_{mn}| \ge t\} \le CP\{|F| \ge t\}, \quad t \ge 0,\ m \ge 1,\ n \ge 1.$$

This condition is satisfied when the collection $\{F_{mn} : m \ge 1, n \ge 1\}$ is identically distributed.

A double array $\{F_{mn} : m \ge 1, n \ge 1\}$ of compact valued random variables is said to be *compactly uniformly integrable* (CUI) if for every $\varepsilon > 0$ there exists a compact subset $\mathcal{K}_\varepsilon$ such that

$$\sup_{m\ge1,n\ge1} E|F_{mn} I_{[F_{mn} \notin \mathcal{K}_\varepsilon]}| < \varepsilon.$$

For notational convenience, for $a, b \in \mathbf{R}$, $\max\{a, b\}$ is denoted by $a \vee b$ and the symbol C denotes a generic constant $(0 < C < \infty)$.

Now we proceed to state our main results.

3. SLLN for compact valued independent random variables in Banach spaces

In this section, SLLN will be obtained for double array of CUI set-valued random variables in arbitrary separable Banach space. We need some lemmas which will be used later.

Lemma 3.1. *Let $A \in \mathrm{k}(X)$ and $\{a_{m\,n} : m \geq 1, n \geq 1\}$ be a double array of nonnegative constants. If*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i\,j} \leq m^{\alpha}n^{\beta} \quad \textit{and} \quad \frac{1}{m^{\alpha}n^{\beta}} \max_{\substack{1\leq k\leq m\\ 1\leq l\leq n}} a_{k\,l} \to 0$$

$$\textit{as } m \vee n \to \infty \textit{ for some } \alpha > 0,\ \beta > 0,$$

then

$$d_H\left(\frac{1}{m^{\alpha}n^{\beta}}\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i\,j}A,\ \frac{1}{m^{\alpha}n^{\beta}}\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i\,j}\,\mathrm{co}\,A\right) \to 0 \quad \textit{as } m \vee n \to \infty.$$

Proof. We put

$$\max_{\substack{1\leq k\leq m\\ 1\leq l\leq n}} a_{k\,l} = \delta_{m\,n} \quad \text{and} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i\,j} = \gamma_{m\,n}.$$

Since $\inf\{\lambda > 0 : \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i\,j}A \subset \overline{U}(\gamma_{m\,n}\,\mathrm{co}\,A; \lambda)\} = 0$ for each m, n, it is sufficient to prove that $\inf\{\lambda > 0 : \gamma_{m\,n}\,\mathrm{co}\,A \subset \overline{U}(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i\,j}A; \lambda)\} = o(m^{\alpha}n^{\beta})$ as $m \vee n \to \infty$.

For $\varepsilon > 0$, we will show that

$$\gamma_{m\,n}\,\mathrm{co}\,A \subset \overline{U}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i\,j}A; C\delta_{m\,n} + m^{\alpha}n^{\beta}\varepsilon\right),$$

for mn sufficiently large and a constant $C > 0$.

Indeed, since $\mathrm{co}\,A$ is compact, there exists $b_1, b_2, ..., b_h$ belong to $\mathrm{co}\,A$ such that $\mathrm{co}\,A \subset \bigcup_{p=1}^{h}\{x : \|x - b_p\| < \varepsilon\}$. We consider only the elements b_p of the form $b_p = \eta_1 a_1 + \eta_2 a_2 + ... + \eta_{t_p} a_{t_p}$, $0 < \eta_j < 1, \sum_{j=1}^{t_p} \eta_j = 1$, $a_j \in A$.

For each m, n such that mn sufficiently large, we put $c_1 = a_{1\,1}, ..., c_n = a_{1\,n}, c_{n+1} = a_{2\,1}, ..., c_{2n} = a_{2\,n}, ..., c_{(m-1)n+1} = a_{m\,1}, ..., c_{mn} = a_{m\,n}$. Then $\{c_j : 1 \leq j \leq mn\}$ is a sequence of nonnegative constant and $\max_{1\leq j\leq mn} c_j = \delta_{m\,n}$, $\sum_{j=1}^{mn} c_j = \gamma_{m\,n}$. There exists integers $0 = s_0 < s_1 < ... < s_{t_p} = mn$ such that

$$\left|\sum_{r=s_{j-1}+1}^{s_j} c_r - \eta_j\gamma_{m\,n}\right| \leq 2\delta_{m\,n}, \quad \text{for all } j = 1, 2, ..., t_p.$$

By putting

$$d_p = \sum_{j=1}^{t_p}\ \sum_{r=s_{j-1}+1}^{s_j} c_r a_j \in \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i\,j}A,$$

we have

$$\|d_p - \gamma_{mn} b_p\| \le \sum_{j=1}^{t_p} \left| \sum_{r=s_{j-1}+1}^{s_j} c_r - \eta_j \gamma_{mn} \right| \|a_j\| \le 2t_p \delta_{mn} |A| \le C\delta_{mn}.$$

Let $b \in \operatorname{co} A$, there exists b_{p_0} such that $\|\gamma_{mn} b - \gamma_{mn} b_{p_0}\| < \gamma_{mn}\varepsilon \le m^\alpha n^\beta \varepsilon$. Hence, $\|\gamma_{mn} b - d_{p_0}\| \le C\delta_{mn} + m^\alpha n^\beta \varepsilon$. It follows that

$$\inf\left\{\lambda > 0 : \gamma_{mn} \operatorname{co} A \subset \overline{U}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} A; \lambda\right)\right\} \le C\delta_{mn} + m^\alpha n^\beta \varepsilon = o(m^\alpha n^\beta) + \varepsilon.$$

The lemma is proved. □

The two following lemmas are direct corollaries from Theorem 2.1 and Theorem 2.4 in [14]. Their proofs are obtained easily by noting that a p-uniformly smooth Banach space is also q-uniformly smooth Banach space ($1 \le q \le p \le 2$) and the real line $\mathbf{R}$ is a 2-uniformly smooth Banach space.

Lemma 3.2. *For every double array of independent random variables $\{V_{ij} : i \ge 1, j \ge 1\}$ and every choice of constant $\alpha > 0$, $\beta > 0$, the condition*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E|V_{mn}|^p}{m^{\alpha p} n^{\beta p}} < \infty \quad \textit{for some } 1 \le p \le 2$$

implies

$$\frac{1}{m^\alpha n^\beta} \sum_{i=1}^{m}\sum_{j=1}^{n} (V_{ij} - EV_{ij}) \to 0 \quad \textit{a.s. as } m \vee n \to \infty.$$

Lemma 3.3. *Let $\{V_{ij} : i \ge 1, j \ge 1\}$ be a double array of independent random variables. Suppose that $\{V_{ij} : i \ge 1, j \ge 1\}$ is stochastically dominated by a random variable V. If $E\left(|V|(\log^+ |V|)^2\right) < \infty$, then*

$$\frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} (V_{ij} - EV_{ij}) \to 0 \quad \textit{a.s. as } m \vee n \to \infty.$$

Now we will establish the SLLN for double array of independent and CUI set-valued random variables in $\mathrm{k}(X)$, where X is arbitrary separable Banach space.

Theorem 3.4. *Let $\{F_{ij} : i \ge 1, j \ge 1\}$ be a double array of independent and CUI set-valued random variables in $\mathrm{k}(X)$. Then the strong law of large numbers*

$$d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E[\operatorname{co} F_{ij}]\right) \to 0 \quad \textit{a.s. as } m \vee n \to \infty$$

holds, if

$$(i) \qquad \textit{The double series } \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E|F_{ij}|^p}{(ij)^p} < \infty \quad \textit{for some } 1 \le p \le 2, \qquad \textit{or}$$

(*ii*) *The collection* $\{F_{ij} : i \geq 1, j \geq 1\}$ *is stochastically dominated by a random variable* F *and* $E\left(|F|\,(\log^+|F|)^2\right) < \infty$.

Proof. For every $\varepsilon > 0$ there exists a compact subset $\mathcal{K}$ of $\mathrm{k}(X)$ such that $E|F_{mn}I_{[F_{mn}\notin\mathcal{K}]}| < \varepsilon$ for all $m, n \in \mathbf{N}$. By the compactness of $\mathcal{K}$ there exists $\{K_1, K_2, ..., K_p\} \subset \mathcal{K}$ such that $\mathcal{K} \subset \bigcup_{t=1}^{p}\{A : d_H(K_t, A) < \varepsilon\} := \bigcup_{t=1}^{p} B(K_t, \varepsilon)$.

Now let us denote

$$G_{mn} = I_{[F_{mn}\in\mathcal{K}]}G'_{mn},$$

where

$$G'_{mn} = I_{[F_{mn}\in B(K_1,\varepsilon)]}K_1 + \sum_{t=2}^{p} I_{[[F_{mn}\in B(K_t,\varepsilon)]\cap[\cup_{j=1}^{t-1}[F_{mn}\in B(K_j,\varepsilon)]]^c]}K_t.$$

It is easy to check that $G_{mn} = \sum_{t=1}^{p} K_t I_{[G_{mn}=K_t]}$ for all $m, n \in \mathbf{N}$ and we have

$$\begin{aligned}
& d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij},\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E[\mathrm{co}\, F_{ij}]\right) \\
\leq\, & d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij},\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}I_{[F_{ij}\in\mathcal{K}]}\right) && \text{(I)} \\
& + d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}I_{[F_{ij}\in\mathcal{K}]},\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} G_{ij}\right) && \text{(II)} \\
& + d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{p} K_t I_{[G_{ij}=K_t]},\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{p} K_t P(G_{ij}=K_t)\right) && \text{(III)} \\
& + d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{p} K_t P(G_{ij}=K_t),\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{p} \mathrm{co}\, K_t P(G_{ij}=K_t)\right) && \text{(IV)} \\
& + d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E[\mathrm{co}\, G_{ij}],\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E[\mathrm{co}\, F_{ij}I_{[F_{ij}\in\mathcal{K}]}]\right) && \text{(V)} \\
& + d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E[\mathrm{co}\, F_{ij}I_{[F_{ij}\in\mathcal{K}]}],\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E[\mathrm{co}\, F_{ij}]\right). && \text{(VI)}
\end{aligned}$$

Let us estimate the above parts as follow:

For (I), we have

$$\begin{aligned}
& d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij},\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}I_{[F_{ij}\in\mathcal{K}]}\right) \\
\leq\, & \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(|F_{ij}I_{[F_{ij}\notin\mathcal{K}]}| - E|F_{ij}I_{[F_{ij}\notin\mathcal{K}]}|\right) + \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E|F_{ij}I_{[F_{ij}\notin\mathcal{K}]}| \\
\leq\, & \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(|F_{ij}I_{[F_{ij}\notin\mathcal{K}]}| - E|F_{ij}I_{[F_{ij}\notin\mathcal{K}]}|\right) + \varepsilon.
\end{aligned}$$

Since $\{|F_{ij}I_{[F_{ij}\notin\mathcal{K}]}| : i \geq 1, j \geq 1\}$ is a double array of independent random variables, moreover, if the condition (i) is satisfied then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F_{ij}I_{[F_{ij}\notin\mathcal{K}]}|^p}{(ij)^p} \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F_{ij}|^p}{(ij)^p} < \infty,$$

or if the condition (ii) is satisfied then

$$P\left(|F_{ij}I_{[F_{ij}\notin\mathcal{K}]}| \geq t\right) \leq P\left(|F_{ij}| \geq t\right) \leq P(|F| \geq t).$$

Thus from Lemma 3.2 and Lemma 3.3, we obtain

$$\limsup_{m\vee n\to\infty} d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}I_{[F_{ij}\in\mathcal{K}]}\right) \leq \varepsilon \quad \text{a.s.}$$

For (II), by the definition of G_{mn}, we have

$$\begin{aligned} &d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}I_{[F_{ij}\in\mathcal{K}]}, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}G_{ij}\right) \\ &\leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}d_H(F_{ij}I_{[F_{ij}\in\mathcal{K}]}, G_{ij}) < \varepsilon. \end{aligned}$$

For (III), we have

$$\begin{aligned} &d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{p}K_tI_{[G_{ij}=K_t]}, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{p}K_tP(G_{ij}=K_t)\right) \\ &\leq \sum_{t=1}^{p}|K_t|\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(I_{[G_{ij}=K_t]} - P(G_{ij}=K_t)\right)\right|. \end{aligned}$$

Note that $\{I_{[G_{ij}=K_t]} - P(G_{ij}=K_t) : i \geq 1, j \geq 1\}$ is the collection of independent and bounded random variables with means zero. Thus from Lemma 3.3, it follows that (III) converges to 0 as $m \vee n \to \infty$.

For (IV), applying Lemma 3.1 with $\alpha = \beta = 1$, $a_{ij} = P(G_{ij} = K_t)$, we have

$$\text{(IV)} \leq \sum_{t=1}^{p}d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}K_tP(G_{ij}=K_t), \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\operatorname{co}K_tP(G_{ij}=K_t)\right) \to 0$$

as $m \vee n \to \infty$.

For (V), by the definition of G_{mn}, we obtain

$$d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}E[\operatorname{co}G_{ij}], \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}E[\operatorname{co}F_{ij}I_{[F_{ij}\in\mathcal{K}]}]\right)$$
$$\leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}d_H(E[\operatorname{co}G_{ij}], E[\operatorname{co}F_{ij}I_{[F_{ij}\in\mathcal{K}]}])$$
$$\leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}Ed_H(G_{ij}, F_{ij}I_{[F_{ij}\in\mathcal{K}]}) < \varepsilon.$$

For (VI), we have

$$(\mathrm{VI}) \leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}d_H\left(E[\operatorname{co}F_{ij}I_{[F_{ij}\in\mathcal{K}]}], E[\operatorname{co}F_{ij}]\right)$$
$$\leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}Ed_H(F_{ij}I_{[F_{ij}\in\mathcal{K}]}, F_{ij}) < \varepsilon.$$

Combining the above parts, we obtain

$$\limsup_{m\vee n\to\infty} d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}E[\operatorname{co}F_{ij}]\right) \leq 4\varepsilon \quad \text{a.s.}$$

The proof is completed. □

Remark. By setting $F_{mn} = \{0\}$ a.s. for all $m \geq 2$, $n \geq 1$, from Theorem 3.4, we recover a related result in [16].

4. SLLN for convex weakly compact valued independent random variables in separable Banach space

In this section we assume that the strong dual of X is separable in order to have the weak compactness property of conditional expectation (see Proposition 2.2), apart from this fact our proofs work with any separable Banach space.

Definition 4.1. Let $1 < p < \infty$. A double array $\{F_{ij} : i \geq 1, j \geq 1\}$ of convex weakly compact valued random variables in $\mathcal{L}^1_{\mathrm{cwk}(X)}(\mathcal{F})$ is of type p if there exists a constant C such that

$$E\left|\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}\right|^p \leq C\sum_{i=1}^{m}\sum_{j=1}^{n}E\,|F_{ij}|^p \quad \text{for all } m \geq 1, n \geq 1.$$

Example 4.2. Let X be a separable Banach space and $K \in \mathrm{cwk}(X)$. Assume that $\{f_{ij} : i \geq 1, j \geq 1\}$ is a collection of independent real valued random elements with $Ef_{ij} = 0$ and $E|f_{ij}|^p < \infty$ for all $i \geq 1, j \geq 1$. Then the collection $\{F_{ij} = f_{ij}K : i \geq 1, j \geq 1\}$ is independent cwk(X)-valued random variables, moreover $0 \in E[F_{ij}]$

for all $i \geq 1, j \geq 1$. Since the real line $\mathbf{R}$ is Rademacher type 2 Banach space, it follows that for all $1 < p \leq 2$ we have

$$E\left|\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}\right|^p = |K|^p\, E\left|\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}\right|^p$$

$$\leq C|K|^p \sum_{i=1}^{m}\sum_{j=1}^{n} E|f_{ij}|^p = C\sum_{i=1}^{m}\sum_{j=1}^{n} E|F_{ij}|^p,$$

for all $m \geq 1, n \geq 1$. Hence the double array $\{F_{ij} : i \geq 1, j \geq 1\}$ is of type p.

Example 4.3. Let X be a separable Banach space and let $\{F_{ij} : i \geq 1, j \geq 1\}$ be a collection of cwk(X)-valued random variables such that $0 \leq |F_{ij}| \leq (ij)^{-\alpha}$ a.s. for each $i \geq 1, j \geq 1$ and for some $\alpha > 1$. We have

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}\right| \leq \sum_{i=1}^{m}\sum_{j=1}^{n} |F_{ij}| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{(ij)^{\alpha}} := C.$$

It shows that for each $1 < p < \infty$ we obtain

$$E\left|\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}\right|^p \leq C^p \leq C'\sum_{i=1}^{m}\sum_{j=1}^{n} E|F_{ij}|^p$$

for each $m \geq 1, n \geq 1$. Then the collection $\{F_{ij} : i \geq 1, j \geq 1\}$ is of type p.

Lemma 4.4. *Let (X, d) be a metric space and $\{x_{mn} : m \geq 1, n \geq 1\} \subset X$. If there exists an element $x \in X$ such that*

$$\lim_{k\vee l\to\infty} d(x_{2^k 2^l}, x) = 0, \quad \text{and} \quad \lim_{k\vee l\to\infty} \max_{\substack{2^k<m\leq 2^{k+1}\\ 2^l<n\leq 2^{l+1}}} d(x_{mn}, x) = 0,$$

then $d(x_{mn}, x) \longrightarrow 0$ as $m \vee n \to \infty$.

Proof. For $m > 1, n > 1$, there exists $k, l \in \mathbf{N}$ such that $2^k < m \leq 2^{k+1}, 2^l < n \leq 2^{l+1}$. We have

$$\begin{aligned} d(x_{mn}, x) &\leq d(x_{mn}, x_{2^k 2^l}) + d(x_{2^k 2^l}, x)\\ &\leq \max_{\substack{2^k<m\leq 2^{k+1}\\ 2^l<n\leq 2^{l+1}}} d(x_{mn}, x_{2^k 2^l}) + d(x_{2^k 2^l}, x)\\ &\leq \max_{\substack{2^k<m\leq 2^{k+1}\\ 2^l<n\leq 2^{l+1}}} d(x_{mn}, x) + 2\, d(x_{2^k 2^l}, x). \end{aligned}$$

When $k \vee l \to \infty$ then $m \vee n \to \infty$ and we obtain the conclusion of lemma. □

The following lemma is a maximal inequality for convex weakly compact valued random variables with metric Hausdorff d_H, and it plays the key role in establishing the next law of large numbers.

Lemma 4.5. *Assume that X is a separable Banach space with strongly separable dual, $\{F_{ij} : i \ge 1, j \ge 1\}$ are independent convex weakly compact valued random variables of type p with $1 < p < \infty$ and $0 \in E[F_{ij}]$, $E|F_{ij}|^p < \infty$ for all $i \ge 1$, $j \ge 1$. Then there exists a positive constant C such that*

$$E\left[\max_{1\le k\le m, 1\le l\le n}\left|\sum_{i=1}^{k}\sum_{j=1}^{l}F_{ij}\right|\right]^p \le C\sum_{i=1}^{m}\sum_{j=1}^{n}E\,|F_{ij}|^p \quad \text{for all } m\ge 1, n\ge 1. \tag{4}$$

Proof. For each $1 \le k \le m$, $1 \le l \le n$, let us set

$$S_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l}F_{ij}, \qquad Y_{ml} = \max_{1\le k\le m}|S_{kl}|,$$
$$\mathcal{F}_{kl} = \sigma\{F_{ij} : 1 \le i \le k, 1 \le j \le l\}.$$

Now for each $k = 1, 2, \ldots, m$ and $2 \le l \le n$, using the conditional expectation properties given by Proposition 2.1, we obtain

$$\begin{aligned} E[S_{kl}|\mathcal{F}_{k\,(l-1)}] &= E[S_{k\,(l-1)} + F_{1\,l} + \cdots + F_{kl}|\mathcal{F}_{k\,(l-1)}] \\ &= E[S_{k\,(l-1)}|\mathcal{F}_{k\,(l-1)}] + E[F_{1\,l}|\mathcal{F}_{k\,(l-1)}] + \cdots + E[F_{kl}|\mathcal{F}_{k\,(l-1)}] \\ &= S_{k\,(l-1)} + E[F_{1\,l}] + \cdots + E[F_{kl}] \supset S_{k\,(l-1)} \text{ a.s.} \end{aligned}$$

It shows that $\{S_{kl}, \mathcal{F}_{kl} : 1 \le l \le n\}$ is a cwk(X)-valued submartingale (see e.g. [1]).
On the other hand, by the well-known equality for cwk(X)-valued conditional expectation (see (2)) we have

$$\delta^*(x^*, E[S_{kl}|\mathcal{F}_{k\,(l-1)}]) = E(\delta^*(x^*, S_{kl})|\mathcal{F}_{k\,(l-1)}) \ \text{ a.s. } \forall x^* \in B^*_{X^*}.$$

Hence we deduce from this equality and Jensen inequality for conditional expectation

$$\begin{aligned} |S_{k\,(l-1)}| \le |E[S_{kl}|\mathcal{F}_{k\,(l-1)}]| &= \sup_{x^*\in B^*_{X^*}}|\delta^*(x^*, E[S_{kl}|\mathcal{F}_{k\,(l-1)}])| \\ &= \sup_{x^*\in B^*_{X^*}}|E(\delta^*(x^*, S_{kl})|\mathcal{F}_{k\,(l-1)})| \le \sup_{x^*\in B^*_{X^*}}E(|\delta^*(x^*, S_{kl})||\mathcal{F}_{k\,(l-1)}) \\ &\le E\left(\sup_{x^*\in B^*_{X^*}}|\delta^*(x^*, S_{kl})||\mathcal{F}_{k\,(l-1)}\right) = E(|S_{kl}||\mathcal{F}_{k\,(l-1)}). \end{aligned}$$

Thus $\{|S_{kl}|, \mathcal{F}_{kl} : 1 \le l \le n\}$ is a nonnegative submartingale for each $k = 1, 2, \ldots, m$. Hence $\{Y_{ml} = \max_{1\le k\le m}|S_{kl}|, \mathcal{F}_{ml} : 1 \le l \le n\}$ is a submartingale, too. So by Doob's inequality (see [5], p. 255 or Theorem 2.3),

$$E\left[\max_{1\le k\le m, 1\le l\le n}|S_{kl}|\right]^p = E\left(\max_{1\le l\le n}Y_{ml}\right)^p \le q^p\,EY^p_{mn}, \quad q = \frac{p}{p-1}. \tag{5}$$

Similarly $\{|S_{k\,n}|, \mathcal{F}_{k\,n} : 1 \le k \le m\}$ is also a submartingale and so by Doob's inequality again

$$EY^p_{m\,n} = E\left[\max_{1\le k\le m} |S_{k\,n}|\right]^p \le q^p\, E|S_{m\,n}|^p$$
$$\le q^p C \sum_{i=1}^{m}\sum_{j=1}^{n} E|F_{i\,j}|^p, \quad q = \frac{p}{p-1}. \qquad (6)$$

The conclusion (4) follows immediately from (5) and (6). □

The SLLN for convex weakly compact valued random variables will be established in the following theorem. In the case of single-valued random variables, this result was proved by Rosalsky and Thanh [15].

Theorem 4.6. *Let X be a real separable Banach space with the strongly separable dual. Let $1 < p < \infty$ and $\{F_{m\,n} : m \ge 1, n \ge 1\}$ be a double array of independent* cwk(X)*-valued random variables of type p with $0 \in E[F_{i\,j}]$ and $E|F_{i\,j}|^p < \infty$ for all $i \ge 1$, $j \ge 1$. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E|F_{i\,j}|^p}{(i^\alpha j^\beta)^p} < \infty, \qquad (7)$$

for some $\alpha > 0$, $\beta > 0$ then

$$d_H\left(\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{i\,j}, \{0\}\right) \to 0 \quad \textit{a.s. as } m \vee n \to \infty. \qquad (8)$$

Proof. First, let $\varepsilon > 0$, by Markov's inequality we have

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P\left\{\left|\frac{1}{2^{k\alpha}2^{l\beta}}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} F_{i\,j}\right| > \varepsilon\right\} \le \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(2^{k\alpha}2^{l\beta})^p\varepsilon^p} E\left|\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} F_{i\,j}\right|^p$$
$$\le C\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E|F_{i\,j}|^p}{(2^{k\alpha}2^{l\beta})^p} \le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=[\log i]}^{\infty}\sum_{l=[\log j]}^{\infty} \frac{E|F_{i\,j}|^p}{(2^{k\alpha}2^{l\beta})^p}$$
$$\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E|F_{i\,j}|^p}{(2^{[\log i]\alpha}2^{[\log j]\beta})^p} \le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E|F_{i\,j}|^p}{(i^\alpha j^\beta)^p} < \infty. \qquad \text{(by (7))}$$

It follows by Borel-Cantelli lemma that

$$\lim_{k\vee l\to\infty} d_H\left(\frac{1}{2^{k\alpha}2^{l\beta}}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} F_{i\,j}, \{0\}\right) = 0 \;\text{ a.s.} \qquad (9)$$

Next, let $\varepsilon > 0$ be arbitrary again, then

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P\left\{\max_{\substack{2^k<m\le 2^{k+1}\\ 2^l<n\le 2^{l+1}}}\left|\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}\right|>\varepsilon\right\}\\
&\le \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P\left\{\max_{\substack{2^k<m\le 2^{k+1}\\ 2^l<n\le 2^{l+1}}}\left|\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}\right|>2^{k\alpha}2^{l\beta}\varepsilon\right\}\\
&\le \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{(2^{k\alpha}2^{l\beta})^p\varepsilon^p}E\left(\max_{\substack{2^k<m\le 2^{k+1}\\ 2^l<n\le 2^{l+1}}}\left|\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}\right|\right)^p \quad \text{(by Markov's inequality)}\\
&\le C\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{(2^{k\alpha}2^{l\beta})^p}E\left(\max_{\substack{1\le m\le 2^{k+1}\\ 1\le n\le 2^{l+1}}}\left|\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}\right|\right)^p\\
&\le C\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}E|F_{ij}|^p}{(2^{k\alpha}2^{l\beta})^p} \quad \text{(by Lemma 4.5)}\\
&\le C\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}E|F_{ij}|^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p}\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=[\log i]}^{\infty}\sum_{l=[\log j]}^{\infty}\frac{E|F_{ij}|^p}{(2^{k\alpha}2^{l\beta})^p}\\
&\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F_{ij}|^p}{(2^{[\log i]\alpha}2^{[\log j]\beta})^p}\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F_{ij}|^p}{(i^\alpha j^\beta)^p}<\infty. \quad \text{(by (7))}
\end{aligned}$$

Again by the Borel-Cantelli lemma, we have that

$$\lim_{k\vee l\to\infty}\max_{\substack{2^k<m\le 2^{k+1}\\ 2^l<n\le 2^{l+1}}} d_H\left(\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij},\{0\}\right)=0 \ \text{ a.s.} \tag{10}$$

Combining (9), (10) and Lemma 4.4, we obtain (8). □

In the next theorem, we obtain the Marcinkiewicz-Zygmund's type law of large numbers for double arrays of convex weakly compact valued random variables.

Theorem 4.7. *Let X be a real separable Banach space with strongly separable dual. Let $1 < p < \infty$ and $\{F_{mn} : m \ge 1, n \ge 1\}$ be a double array of independent* $\mathrm{cwk}(X)$*-valued random variables of type p with $0 \in E[F_{ij}]$ and $E|F_{ij}|^p < \infty$ for all $i \ge 1$, $j \ge 1$. Suppose that $\{F_{mn} : m \ge 1, n \ge 1\}$ is stochastically dominated by a random variable F.*

(i) If $E(|F|^r \log^+ |F|) < \infty$ for some $r \in (1, p)$, then

$$d_H\left(\frac{1}{(mn)^{\frac{1}{r}}}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij},\{0\}\right)\to 0 \ \ \text{a.s. as } m\vee n\to\infty. \tag{11}$$

(ii) *If* $E(|F|(\log^+|F|)^2) < \infty$, *then*

$$d_H\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}, \{0\}\right) \to 0 \quad a.s.\ as\ m \vee n \to \infty. \tag{12}$$

Proof. (i) Let $\mathbf{F}$ be the distribution function of $|F|$ and $\mathrm{d}(k)$ be the number of divisors of k. For $m \geq 1$, $n \geq 1$, set

$$F'_{mn} = F_{mn}I_{[|F_{mn}| \leq (mn)^{\frac{1}{r}}]}, \qquad F''_{mn} = F_{mn}I_{[|F_{mn}| > (mn)^{\frac{1}{r}}]}.$$

By using the fact that $\sum_{k=i+1}^{\infty} k^{\frac{-p}{r}}\,\mathrm{d}(k) = O\left((i+1)^{1-\frac{p}{r}}\log i\right)$, we obtain the inequalities

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F'_{ij}|^p}{(ij)^{\frac{p}{r}}} \leq C\sum_{k=1}^{\infty}\frac{\mathrm{d}(k)}{k^{\frac{p}{r}}}\int_0^{k^{\frac{1}{r}}} x^p d\,\mathbf{F}(x)\\
&= C\sum_{i=1}^{\infty}\left(\sum_{k=i}^{\infty}\frac{\mathrm{d}(k)}{k^{\frac{p}{r}}}\right)\int_{(i-1)^{\frac{1}{r}}}^{i^{\frac{1}{r}}} x^p d\,\mathbf{F}(x) \leq C\sum_{i=1}^{\infty}\frac{\log i}{i^{\frac{p}{r}-1}}\int_{(i-1)^{\frac{1}{r}}}^{i^{\frac{1}{r}}} x^p d\,\mathbf{F}(x)\\
&\leq CE|F|^r\log^+|F| < \infty.
\end{aligned} \tag{13}$$

On the other hand, if we use the fact that $\sum_{k=1}^{n} k^{\frac{-1}{r}}\,\mathrm{d}(k) = O(n^{1-\frac{1}{r}}\log n)$, then

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F''_{ij}|}{(ij)^{\frac{1}{r}}} \leq C\sum_{k=1}^{\infty}\frac{\mathrm{d}(k)}{k^{\frac{1}{r}}}\int_{k^{\frac{1}{r}}}^{\infty} x d\,\mathbf{F}(x)\\
&= C\sum_{i=1}^{\infty}\left(\sum_{k=1}^{i}\frac{\mathrm{d}(k)}{k^{\frac{1}{r}}}\right)\int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} x d\,\mathbf{F}(x) \leq C\sum_{i=1}^{\infty} i^{1-\frac{1}{r}}\log i\int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} x d\,\mathbf{F}(x)\\
&\leq CE|F|^r\log^+|F| < \infty.
\end{aligned}$$

This implies that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F''_{ij}|^p}{(ij)^{\frac{p}{r}}} < \infty. \tag{14}$$

Combining (13) and (14) we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F_{ij}|^p}{(ij)^{\frac{p}{r}}} \leq C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F'_{ij}|^p}{(ij)^{\frac{p}{r}}} + C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F''_{ij}|^p}{(ij)^{\frac{p}{r}}} < \infty.$$

Applying Theorem 4.6 with $\alpha = \beta = \frac{1}{r}$, we get (11).

(ii) For $m \geq 1$, $n \geq 1$, set

$$F'_{mn} = F_{mn}I_{[|F_{mn}| \leq mn]}, \qquad F''_{mn} = F_{mn}I_{[|F_{mn}| > mn]}.$$

By the same method, it is easy to check that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F'_{ij}|^p}{(ij)^p} \le CE|F|\log^+|F| \le CE|F|(\log^+|F|)^2 < \infty. \tag{15}$$

On the other hand, note that $\sum_{k=1}^{n} k^{-1}\,\mathrm{d}(k) = O(\log^2 n)$, we obtain

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F''_{ij}|}{ij} \le C\sum_{k=1}^{\infty}\frac{\mathrm{d}(k)}{k}\int_k^{\infty} x d\,\mathbf{F}(x) \le CE|F|\,(\log^+|F|)^2 < \infty.$$

It also implies that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F''_{ij}|^p}{(ij)^p} < \infty. \tag{16}$$

(15) and (16) yield

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|F_{ij}|^p}{(ij)^p} < \infty.$$

Applying Theorem 4.6 with $\alpha = \beta = 1$, we get (12). □

Acknowledgements. The authors wish to thank Professor Charles Castaing (Département de Mathématiques, Université Montpellier II, France) for helpful discussions.

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