Journal of Convex Analysis
Volume 19 (2012), No. 1, 159–170

Rectifiability of Special Singularities of Non-Lipschitz Functions*

Khai T. Nguyen
Università di Padova, Dipartimento di Matematica Pura ed Applicata, via Trieste 63, 35121 Padova, Italy
khai@math.unipd.it

Davide Vittone
Università di Padova, Dipartimento di Matematica Pura ed Applicata, via Trieste 63, 35121 Padova, Italy
vittone@math.unipd.it

Received: June 22, 2010
Revised manuscript received: December 21, 2010

We prove rectifiability results for special singularities of non-Lipschitz functions, namely for those points where the set of Fréchet horizon supergradients contains a nonzero vector subspace.

Keywords: Normal vectors, exterior sphere condition, rectifiability

1. Introduction

Since the celebrated result by H. Rademacher there has been a lot of interest in the study of the set of singularities of real functions. Given a function $f : \Omega \longrightarrow \mathbb{R}$ defined on an open set $\Omega \subseteq \mathbb{R}^N$, a singularity of f is a point where f is not differentiable. Rademacher's theorem states that, if f is locally Lipschitz, then the set of singularities $\Sigma(f)$ of f has null Lebesgue measure. In general, however, sets with null Lebesgue measure can be very irregular and possess almost no structure. A natural question is then that of investigating the properties of the singular set for special classes of functions.

When f is convex or concave, the properties of $\Sigma(f)$ were first investigated in [17] and then developed in [28], [27], [25], [26], [2] and [3]. The basic approach in such papers is that of estimating the size of $\Sigma(f)$. We mention here a result which is essentially due to L. Zajíček and was later extended to semiconcave functions by G. Alberti, L. Ambrosio and P. Cannarsa [1]. By $\partial^F f(x)$ we denote here the Fréchet supergradient of f at x (see Definition 2.3).

Theorem 1.1 ([1]). *Let f be locally semiconcave. Then, for any $k = 1, 2, \ldots, N$ the singular set $\Sigma^k(f) := \{x \in \Omega \mid \dim \partial^F f(x) = k\}$ is countably $(N-k)$-rectifiable. In particular, $\Sigma(f)$ is countably $(N-1)$-rectifiable and $\Sigma^N(f)$ is at most countable.*

*The authors are supported by MIUR (Italy) and University of Padova.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

The importance of semiconcave/semiconvex functions arises evident in problems of optimal control, see [6], [7], [8], [9]. In particular, the minimum time function T of nonlinear smooth control dynamics with target satisfying internal sphere condition is semiconcave [7]. In the same paper, the authors also proved that T is semiconvex for the linear dynamics with convex target. In both cases, a strong controllability assumption on the systems (namely, T being locally Lipschitz) was demanded. However, as shown by simple examples (e.g., the well known rocket car), the Lipschitz continuity of T does not hold in general.

In order to study the regularity of non Lipschitz functions the notion of set with positive reach was used in [14]. This notion was first introduced by H. Federer [18] and then analyzed independently by several authors under different names, for example *φ-convex* [10], *proximally smooth* [11], and *prox-regular* sets [23]. More precisely, lower (respectively, upper) semicontinuous functions whose epigraph (resp., hypograph) has positive reach still enjoy some regularity property of semiconvex (resp. semiconcave) functions, including the rectifiability of $\Sigma^k(f)$ and almost everywhere second order differentiability. However, the Hausdorff dimension of $\Sigma^{P,\infty}(f) := \{x \in \Omega \mid \partial^{P,\infty} f \neq \varnothing\}$ can be larger than $(N-1)$ (see [14, Example 5.2]), where by $\partial^{P,\infty} f$ we denote the set of proximal horizon supergradients of f (see [14]). A natural idea for proving the positive reach property for the epigraph/hypograph of f is the representation of generalized sub/super gradient of f. By using this approach, the minimum time function T was studied in [15] and [16] under a weak controllability condition requiring T to be only continuous. In both papers, the wedgedness (see Rockafellar [24]) for normal cone to the epigraph/hypograph of T is required. It was proved in [21] that the wedgedness assumption guarantees positive reach for sets satisfying an exterior sphere condition. This result was used in [22] to investigate the relationships among functions whose hypograph satisfies the exterior sphere condition, functions with positive reach hypograph and semiconcave functions. In general, however, the exterior sphere condition is weaker than the positive reach property (see [21, Section 2]). Recently, the set of *bad points* BP_f (see Section 2 for the definition), in which wedgedness fails, was studied in [19] under the exterior sphere condition on the hypograph of the function f. More precisely, it was proved that BP_f is closed in Ω and has zero Lebesgue measure. Consequently, the hypograph of $f_{|\Omega\setminus BP_f}$ has positive reach and thus f is twice differentiable almost everywhere in Ω.

In this paper we prove some rectifiability result for the set of bad points BP_f of f. We partition the set BP_f (see (6)) into sets $BP_{f,k}$, $k = 1, 2, \dots, N$, where, roughly speaking, the suffix k corresponds to the dimension of the largest vector space contained in the set $\partial^\infty f$ of Fréchet horizon supergradients of f (see Definition 2.3). We are able to prove that $BP_{f,k}$ is countably $(N-k)$-rectifiable.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be upper semi-continuous. Then the set $BP_{f,k}$ is countably $(N-k)$-rectifiable.*

Moreover, we are able to refine the main result of [19], namely Theorem 3.1 therein.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be continuous. If the hypograph of f satisfies the θ-exterior sphere condition for some $\theta > 0$, then the set* *of bad points BP_f is locally $(N-1)$-rectifiable. In particular, $\mathcal{H}^{N-1}(BP_f \cap K)$ is finite for any compact set $K \subset \mathbb{R}^N$.*

Finally, in Section 4 we provide an example showing that, in general, the set $BP_{f,k}, k \geq 2$ may not have finite $(N-k)$-Hausdorff measure even under the exterior sphere condition.

2. Notations and preliminary results

Let us briefly recall the main notions that will be used throughout the paper. We refer to the monograph [12] for a more general account about Nonsmooth Analysis.

Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be upper semi-continuous. The hypograph of f is denoted by

$$\text{hypo}(f) = \{(x, \beta) \mid x \in \Omega, \beta \leq f(x)\}. \tag{1}$$

The vector $(-v, \lambda) \in \mathbb{R}^N \times \mathbb{R}$ is a *Fréchet normal vector* to hypo(f) at $(x, f(x))$ iff

$$\limsup_{\text{hypo}(f) \ni (y,\beta) \to (x,f(x))} \left\langle (-v, \lambda), \frac{(y, \beta) - (x, f(x))}{|y - x| + |\beta - f(x)|} \right\rangle \leq 0. \tag{2}$$

We denote by $N^F_{\text{hypo}(f)}(x, f(x))$ the set of Fréchet normal vectors to hypo(f) at $(x, f(x))$.

Remark 2.1. If $(-v, \lambda) \in N^F_{\text{hypo}(f)}(x, f(x))$ then $\lambda \geq 0$.

We say that $(-v, \lambda)$ is a *proximal normal vector* to hypo(f) at $(x, f(x))$ if there exists a constant α such that

$$\langle (-v, \lambda), (y, \beta) - (x, f(x)) \rangle \leq \alpha(\|y - x\|^2 + |\beta - f(x)|^2) \quad \forall y \in \Omega,\ \beta \leq f(x)\,.$$

We denote by $N^P_{\text{hypo}(f)}(x, f(x))$ the set of proximal normal vectors to hypo(f) at $(x, f(x))$. Moreover, we say that $(-v, \lambda) \in N^P_{\text{hypo}(f)}(x, f(x))$ is realized by a ball of radius $\theta > 0$ if

$$\langle (-v, \lambda), (y, \beta) - (x, f(x)) \rangle \leq \frac{\|(-v, \lambda)\|}{2\theta}(\|y - x\|^2 + |\beta - f(x)|^2) \quad \forall y \in \Omega,\ \beta \leq f(x)\,.$$

We say that hypo(f) satisfies the *θ-exterior sphere condition* if for any $x \in \Omega$ there exists $(-v, \lambda) \in N^P_{\text{hypo}(f)}(x, f(x))$ realized by a ball of radius θ.

Remark 2.2. It is easily seen that $N^P_{\text{hypo}(f)}(x, f(x)) \subseteq N^F_{\text{hypo}(f)}(x, f(x))$.

Let us introduce some concepts of generalized differential for f at $x \in \Omega$ associated with hypo(f).

Definition 2.3. Let $x \in \Omega$ and $v \in \mathbb{R}^N$. We say that v is a *Fréchet supergradient* of f at x if $(-v, 1) \in N^F_{\text{hypo}(f)}(x, f(x))$. We denote by $\partial^F f(x)$ the set of Fréchet supergradients of f at x.

We say that v is a *Fréchet horizon supergradient* of f at x if $(-v,0)\in N^F_{\mathrm{hypo}(f)}(x,f(x))$. The set of Fréchet horizon supergradients of f at x is denoted by $\partial^\infty f(x)$.

The largest vector subspace contained in $N^F_{hypo(f)}(x,f(x))$ will be denoted by

$$NL(x) = \{\ \xi \in N^F_{hypo(f)}(x,f(x))\ |\ -\xi \in N^F_{hypo(f)}(x,f(x))\}. \tag{3}$$

From Remark 2.1, one can see that $NL(x) \subseteq \{(v,0)\ |\ -v \in \partial^\infty f(x)\}$. Let us define

$$V_x := \{v \in \mathbb{R}^N\ |\ (v,0) \in NL(x)\}; \tag{4}$$

clearly, V_x is the largest vector space contained in $\partial^\infty f(x)$ and $\dim V_x = \dim NL(x)$. We say that $v \in V_x$ is realized by a ball of radius θ if $(v,0) \in N^P_{\mathrm{hypo}(f)}(x,f(x))$ is realized by a ball of radius θ.

The set of *bad points* BP_f of f is defined by

$$BP_f = \{x \in \Omega\ |\ NL(x) \neq \{0\}\}. \tag{5}$$

According to the dimension of $NL(x)$, for $k = 1,2,\dots,N$ we introduce

$$BP_{f,k} = \{x \in BP_f\ |\ \dim NL(x) = k\} = \{x \in BP_f\ |\ \dim V_x = k\}. \tag{6}$$

It is clear that $BP_f = \bigcup_{k=1}^N BP_{f,k}$.

Let $k \geq 0$ and $A \subset \mathbb{R}^N$ be fixed. The *k-dimensional Hausdorff measure* of A is defined as

$$\mathcal{H}^k(A) := \lim_{\delta\to 0^+} \mathcal{H}^k_\delta(A) = \sup_{\delta>0} \mathcal{H}^k_\delta(A)$$

where for any $\delta > 0$ we set

$$\mathcal{H}^k_\delta(A) := \inf\left\{\sum_{i\in I}(\mathrm{diam}\ A_i)^k\ |\ A \subset \bigcup_{i\in I} A_i,\ \mathrm{diam}\ A_i < \delta\right\}.$$

The *Hausdorff dimension* of A is

$$\mathcal{H}\text{-}\dim(A) := \inf\{k \geq 0\ |\ \mathcal{H}^k(A) = 0\} = \sup\{k \geq 0\ |\ \mathcal{H}^k(A) = \infty\}.$$

It is well known (see e.g. [18, 20]) that $\mathcal{H}^k$ is a Borel measure on $\mathbb{R}^N$; $\mathcal{H}^0$ is the counting measure.

Let $k \in \mathbb{N}$; we say that $A \subset \mathbb{R}^N$ is *countably k-rectifiable* if

$$A\ \subset\ \mathcal{N} \cup \bigcup_{i=1}^\infty S_i$$

where S_i are suitable Lipschitz k-dimensional surfaces and $\mathcal{N}$ is a $\mathcal{H}^k$-negligible set. We say that A is *k-rectifiable* if it is countably k-rectifiable and $\mathcal{H}^k(A) < \infty$, while A is *locally k-rectifiable* if $A \cap K$ is is k-rectifiable for any compact set $K \subset \mathbb{R}^N$.

Any countably k-rectifiable set A satisfies $\mathcal{H}$-dim$(A) = k$. It is well known that, if $f : A \subset \mathbb{R}^k \to \mathbb{R}^N$ is Lipschitz continuous, then $f(A)$ is countably k-rectifiable; if A is bounded, then $f(A)$ is k-rectifiable.

In what follows, given $A \subset \mathbb{R}^N$ we define its ϵ-neighborhood $(A)_\epsilon$ by

$$(A)_\epsilon := \{x \in \mathbb{R}^N \mid \text{there exists } y \in A \text{ such that } \|x - y\| < \epsilon\}.$$

Let $\mathcal{K}$ denote the set of closed subsets of $S^{N-1} \subset \mathbb{R}^N$; for $A, B \in \mathcal{K}$ we introduce the *Hausdorff distance* $d_H(A, B)$ by

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subset (B)_\epsilon \text{ and } B \subset (A)_\epsilon\}.$$

It turns out (see e.g. [5]) that $(\mathcal{K}, d_H)$ is a complete compact metric space.

We will denote by $G(N, k)$ the Grassmann manifold of all k-dimensional vector subspaces of $\mathbb{R}^N$; we endow $G(N, k)$ with the distance

$$d_G(V_1, V_2) := d_H(V_1 \cap S^{N-1}, V_2 \cap S^{N-1}).$$

The metric space $(G(N, k), d_G)$ is compact and, in particular, the following property holds

$$\forall\, R > 0\ \exists\, V_1, V_2, \ldots, V_m \in G(N, k) \ \text{ such that } G(N, k) \subset \bigcup_{i=1}^{m} B_G(V_i, R), \tag{7}$$

where $B_G(V_i, R)$ denote the open ball (with respect to d_G) with center V_i and radius R.

3. Rectifiability results for the set of bad points

Let $V \in G(N, k)$ be fixed; each $z \in \mathbb{R}^N$ can be written in a unique way as $z = z_V + z_{V^\perp}$ where $z_V \in V$ and $z_{V^\perp} \in V^\perp$. For $\alpha \in (0, 1)$ we denote by $C_\alpha(V)$ the open cone along V of aperture $1/\alpha$ defined by

$$C_\alpha(V) := \{z \in \mathbb{R}^N \mid \|z_V\| > \alpha \|z\|\}.$$

If $x \in \mathbb{R}^N$ we set

$$C_\alpha(x, V) := x + C_\alpha(V) = \{z \in \mathbb{R}^N \mid \|(z - x)_V\| > \alpha \|z - x\|\};$$

It is easily seen that

$$z \in C_\alpha(x, V) \iff \exists\, v \in V \cap S^{N-1} \ \text{ such that } \langle v, z - x\rangle > \alpha \|z - x\|. \tag{8}$$

We also point out the following implication:

$$d_G(V_1, V_2) < R \implies C_{\alpha+R}(x, V_1) \subset C_\alpha(x, V_2) \tag{9}$$

which holds provided $\alpha + R < 1$. To prove (9) it is enough to notice that for any $z \in C_{\alpha+R}(x, V_1)$

$$\begin{aligned}&\text{there exists } v_1 \in V_1 \cap S^{N-1} \text{ such that } \langle v_1, z - x\rangle > (\alpha + R)\|z - x\|\\ &\text{there exists } v_2 \in V_2 \cap S^{N-1} \text{ such that } \|v_1 - v_2\| \le R\end{aligned}$$

whence

$$\langle v_2, z-x\rangle = \langle v_1, z-x\rangle - \langle v_1 - v_2, z-x\rangle > \alpha \|z-x\|,$$

i.e., $z \in C_\alpha(x, V_2)$.

For any fixed $\rho > 0$, let us introduce the sets

$$BP^\rho_{f,k} = \Big\{ x \in BP_{f,k} \mid \left\langle v_x, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{\|v_x\|}{8} \quad \forall v_x \in V_x, y \in B(x,\rho), \beta < f(y) \Big\}. \tag{10}$$

Remark 3.1. If $\rho_1 > \rho_2 > 0$ then $BP^{\rho_1}_{f,k} \subseteq BP^{\rho_2}_{f,k}$.

As the following Lemma shows, the sets $BP^\rho_{f,k}$ give a partition of $BP_{f,k}$.

Lemma 3.2. *We have*

$$BP_{f,k} = \cup_{\rho>0} BP^\rho_{f,k}. \tag{11}$$

In particular, from Remark 3.1 *it holds*

$$BP_{f,k} = \cup_{i\in\mathbb{N}\setminus\{0\}} BP^{1/i}_{f,k}. \tag{12}$$

Proof. Fix $x \in BP_{f,k}$ and let $v_1, v_2, \ldots, v_k$ be an orthonormal basis for V_x. By the definition of V_x we have $-v_i \in V_x$ for all $i \in \{1, 2, \ldots, k\}$. Recalling (4), (3) and (2), there exists a constant $\rho_x > 0$ such that $B(x, \rho_x) \subset \Omega$ and for all $i \in \{1, 2, \ldots, k\}$ one has

$$\left\langle v_i, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{1}{8\sqrt{k}} \quad \text{and} \quad \left\langle -v_i, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{1}{8\sqrt{k}}$$

for all $y \in B(x, \rho_x)$ and $\beta \leq f(y)$. Thus

$$\left| \left\langle v_i, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \right| \leq \frac{1}{8\sqrt{k}} \tag{13}$$

for all $y \in B(x, \rho_x)$ and $\beta \leq f(y)$.

Fix $v_x \in V_x$; we have $v_x = \sum_{i=1}^k \alpha_i v_i$ for suitable $\alpha_i \in \mathbb{R}$. From (13), we get

$$\left\langle v_x, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{\sum_{i=1}^k |\alpha_i|}{8\sqrt{k}}$$

for all $y \in B(x, \rho_x)$ and $\beta \leq f(y)$. On the other hand,

$$\|v_x\| = \left(\sum_{i=1}^k \alpha_i^2 \right)^{1/2} \geq \frac{\sum_{i=1}^k |\alpha_i|}{\sqrt{k}}.$$

Therefore

$$\left\langle v_x, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{\|v_x\|}{8}$$

for all $y \in B(x, \rho_x)$ and $\beta \leq f(y)$. Thus $x \in BP^{\rho_x}_{f,k}$ and the proof is accomplished. $\square$

In view of a rectifiability result for the sets $BP_{f,k}$, we begin with a technical result.

Lemma 3.3. *Let $a \in \mathbb{R}^N$, $\rho > 0$ and $x, y \in BP^{\rho}_{f,k} \cap B(a, \frac{\rho}{2})$ be such that $d_G(V_x, V_y) < \frac{1}{8}$; then*

$$y \in \mathbb{R}^N \backslash C_{\frac{1}{4}}(x, V_x)\,.$$

Proof. Since $x, y \in B(a, \frac{\rho}{2})$, we have $x \in B(y, \rho)$ and $y \in B(x, \rho)$. Therefore, from (10) if $v_x \in V_x \cap \mathcal{S}^{N-1}$ we have

$$\langle v_x, y - x\rangle \leq \frac{1}{8}(\|y - x\| + |\beta - f(x)|) \text{ for all } \beta \leq f(y). \tag{14}$$

Similarly, for any $v_y \in V_y \cap \mathcal{S}^{N-1}$ we obtain

$$\langle v_y, y - x\rangle \leq \frac{1}{8}(\|y - x\| + |\beta - f(y)|) \text{ for all } \beta \leq f(x). \tag{15}$$

We have to distinguish two cases: if $f(y) \geq f(x)$, we choose $\beta = f(x)$ in (14) to get

$$\langle v_x, y - x\rangle \leq \frac{1}{8}\|y - x\| \quad \forall\, v_x \in V_x \cap S^{N-1}\,.$$

Recalling (8), this implies that $y \notin C_{\frac{1}{4}}(x, V_x)$, as desired.

If $f(y) \leq f(x)$, we choose $\beta = f(y)$ in (15) to get

$$\langle v_y, y - x\rangle \leq \frac{1}{8}\|y - x\| \quad \forall\, v_y \in V_y \cap S^{N-1}.$$

Since $d_G(V_x, V_y) < \frac{1}{8}$, for any $v_x \in V_x \cap S^{N-1}$ there exists $v_y = v_y(v_x) \in V_y \cap S^{N-1}$ such that $\|v_x - v_y\| < \frac{1}{8}$. Therefore, for any $v_x \in V_x \cap S^{N-1}$ it holds

$$\langle v_x, y - x\rangle \leq \langle v_y, y - x\rangle + |\langle v_x - v_y, y - x\rangle| \leq \frac{1}{4}\|y - x\| \tag{16}$$

i.e. $y \notin C_{\frac{1}{4}}(x, V_x)$, as desired. □

We now fix $R := 1/16$ and let $V_1, V_2, \ldots, V_m \in G(N, k)$ be given by (7). We thus divide $BP^{\rho}_{f,k}$ into m sets

$$BP^{\rho}_{f,k} = \bigcup_{j=1}^{m} BP^{\rho,j}_{f,k} \tag{17}$$

where

$$BP^{\rho,j}_{f,k} = \{x \in BP^{\rho}_{f,k} \mid d_G(V_x, V_j) < 1/16\}.$$

For $j = 1, 2, \ldots, m$ we denote by π_j the orthogonal projection $\mathbb{R}^n \to V_j^{\perp}$; clearly, $\pi_j(z) = z_{V_j^{\perp}} = z - z_{V_j}$.

Lemma 3.4. *Assume that $a \in \mathbb{R}^N$ and $\rho > 0$ are such that $BP^{\rho,j}_{f,k} \cap B(a,\rho/2)$ is nonempty. Then the projection $\pi_j : BP^{\rho,j}_{f,k} \cap B(a,\rho/2) \to \pi_j(BP^{\rho,j}_{f,k} \cap B(a,\rho/2))$ is invertible and its inverse map is Lipschitz continuous with Lipschitz constant at most 2.*

Proof. Let $x, y \in BP^{\rho,j}_{f,k} \cap B(a,\rho/2)$ be fixed. We have $d_G(V_x, V_y) < 1/8$ and Lemma 3.3 ensures that $y \notin C_{1/4}(x, V_x)$. Since $d_G(V_x, V_j) < 1/16$, by (9) we deduce that $C_{1/2}(x, V_j) \subseteq C_{5/16}(x, V_j) \subseteq C_{1/4}(x, V_x)$ and, in particular, that $y \notin C_{1/2}(x, V_j)$. This implies that $\|(y-x)_{V_j}\| \le \frac{1}{2}\|y-x\|$, whence

$$\|\pi_j(y) - \pi_j(x)\| = \|\pi_j(y-x)\| = \left\|(y-x) - (y-x)_{V_j}\right\| \ge \tfrac{1}{2}\|y-x\| .$$

This is enough to conclude. □

The rectifiability of the sets $BP^{\rho}_{f,k}$ is now a consequence of Lemma 3.4.

Theorem 3.5. *The set $BP^{\rho}_{f,k} \cap K$ is $(N-k)$-rectifiable for any $\rho > 0$ and any compact set $K \subset \mathbb{R}^N$; in particular*

$$\mathcal{H}^{N-k}(BP^{\rho}_{f,k} \cap K) < +\infty. \tag{18}$$

Proof. It will be sufficient to show that for any $j = 1, 2, \ldots, m$ the set $BP^{\rho,j}_{f,k} \cap K$ is $(N-k)$-rectifiable. Since K is compact, there exist $a_1, a_2, \ldots, a_h \in \mathbb{R}^N$ such that

$$BP^{\rho,j}_{f,k} \cap K \subset \bigcup_{i=1}^{h} \left(BP^{\rho,j}_{f,k} \cap B(a_i, \rho/2)\right).$$

By Lemma 3.4, for any $i = 1, 2, \ldots, h$ the set $BP^{\rho,j}_{f,k} \cap B(a_i,\rho/2)$ is the image of

$$\pi_j^{-1} : \pi_j\left(BP^{\rho,j}_{f,k} \cap B(a_i,\rho/2)\right) \to \mathbb{R}^N,$$

i.e. of a Lipschitz map defined on a bounded subset of $V_j^{\perp} \equiv \mathbb{R}^{N-k}$ with Lipschitz constant at most 2. In particular, $BP^{\rho,j}_{f,k} \cap B(a_i,\rho/2)$ is $(N-k)$-rectifiable and this allows to conclude. □

We can finally pass to the proof of our main results.

Proof of Theorem 1.2. It is an easy consequence of Lemma 3.2 and Theorem 3.5. □

Before passing to the proof of Theorem 1.3, we would like to discuss the relation between BP_f and the set of bad points BP^P_f considered in [19], namely,

$$BP^P_f := \{x \in \Omega \,|NL^P(x) \neq \{0\}\},$$

where $NL^P(x) = \{\xi \in N^P_{hypo(f)}(x, f(x)) \mid -\xi \in N^P_{hypo(f)}(x, f(x))\}$. From Remark 2.2 it is clear that $BP^P_f \subseteq BP_f$, but in general the two sets do not coincide. For instance, the real function $f(x) = -|x|^{2/3}$ is such that $(\pm 1, 0) \in N^F_{hypo(f)}(0,0)$ but $(\pm 1, 0) \notin N^P_{hypo(f)}(0,0)$. Thus, in general BP^P_f and BP_f do not coincide.

However, the equality $BP_f = BP^P_f$ holds under the assumptions of Theorem 1.3. Indeed, from Corollary 3.1 in [19] it follows that the hypograph of $f_{|\Omega_P}$ has positive reach, where Ω_P is the open set defined by $\Omega_P := \Omega \backslash BP^P_f$. Therefore (see [13, Proposition 6.2 and 4.2] and [18, Theorem 4.8 (12)]) one has

$$N^P_{\mathrm{hypo}(f_{|\Omega_P})}(x, f_{|\Omega_P}(x)) = N^F_{\mathrm{hypo}(f_{|\Omega_P})}(x, f_{|\Omega_P}(x)) \text{ for all } x \in \Omega_P.$$

and thus

$$N^P_{\mathrm{hypo}(f)}(x, f(x)) = N^F_{\mathrm{hypo}(f)}(x, f(x)) \text{ for all } x \in \Omega_P.$$

Consequently, $NL(x) = NL^P(x)$ for all $x \in \Omega_P$. By the definition of BP^P_f, we have $NL^P(x) = \{0\}$ for all $x \in \Omega_P$. This implies that $NL(x) = \{0\}$ for all $x \in \Omega_P$, i.e. $BP_f \cap \Omega_P = \varnothing$. Thus, $BP_f \subseteq BP^P_f$, as claimed.

Proof of Theorem 1.3. Recalling Theorem 1.2, we have $\mathcal{H}^{N-1}(BP_{f,k}) = 0$ for all $k \in \{2, 3, \dots, N\}$. Since

$$BP_f = BP_{f,1} \cup \bigcup_{k=2}^{N} BP_{f,k},$$

the proof will be accomplished after proving that the set $BP_{f,1}$ is locally $(N-1)$-rectifiable. From the definition (6), for every $x \in BP_{f,1}$ the set

$$V_x = \{tv_x \mid v_x \in \mathbb{R}^N, \ \|v_x\| = 1 \text{ and } t \in \mathbb{R}\}$$

is a line along v_x. Therefore by [19, Lemma 4.3], $(\pm v_x, 0) \in N^P_{hypo(f)}(x, f(x))$ are realized by a ball of radius θ, i.e.

$$\langle \pm v_x, y - x \rangle \leq \frac{1}{2\theta}(\|y - x\|^2 + |\beta - f(x)|^2) \ \ \forall y \in \Omega, \beta \leq f(x).$$

From the above inequality, reasoning as in the proof of Lemma 3.3 one can obtain that the following holds. If $a \in \mathbb{R}^N$, $\rho \in (0, \theta/8]$, $x, y \in BP_{f,1} \cap B(a, \frac{\rho}{2})$ are such that $d_G(V_x, V_y) < \frac{1}{8}$, then

$$y \in \mathbb{R}^N \backslash C_{\frac{1}{4}}(x, V_x)\,.$$

From this fact, the local $(N-1)$-rectifiability of $BP_{f,1}$ follows (up to considering $BP_{f,1}$ instead of $BP^\rho_{f,k}$) as in the proof of Theorem 3.5. □

4. A counterexample

By virtue of Theorem 1.3, the set of bad points BP_f is locally $(N-1)$-rectifiable provided the θ-exterior sphere condition holds. On the contrary, an analogous result of $(N-k)$-rectifiability of $BP_{f,k}$ does not hold for $k \geq 2$; in other words, for such k Theorem 1.2 cannot be refined to show that $\mathcal{H}^{N-k}(BP_{f,k} \cap K) < \infty$ for any compact set $K \subset \mathbb{R}^N$. We are going to provide an example of a continuous function $f : (-1,1) \times (-1,1) \to \mathbb{R}$ satisfying the θ-exterior sphere condition with $\theta = 1$ and such that $\mathcal{H}^0(BP_{f,2} \cap K) = +\infty$ for any neighbourhood K of the origin. It will be clear from the construction that what is missing is a uniform control on the radii of exterior balls (recall that, by Theorem 3.5, $BP^{\rho}_{f,k}$ is locally $(N-k)$-rectifiable for any $\rho > 0$).

Let $\Omega := (-1,1) \times (-1,1)$; for $n \in \mathbb{N}$ let us define $x_n^+, x_n^- \in \overline{\Omega}$ by

$$x_n^+ := (2^{-n}, 0), \qquad x_n^- := (-2^{-n}, 0)\,.$$

We also set

$$c_n^+ := \frac{x_n^+ + x_{n+1}^+}{2} = (3{\cdot}2^{-n-2}, 0) \in \Omega, \qquad c_n^- := \frac{x_n^- + x_{n+1}^-}{2} = (-3{\cdot}2^{-n-2}, 0) \in \Omega$$

and

$$r_n := \frac{\|x_n^+ - x_{n+1}^+\|}{2} = \frac{\|x_n^- - x_{n+1}^-\|}{2} = 2^{-n-2}\,.$$

Notice that the closed balls $\overline{B(c_n^{\pm}, r_n)}$ are pairwise disjoint except for the case of consecutive balls, which instead are tangent, i.e., for any $n \geq 1$ one has

$$\overline{B(c_n^+, r_n)} \cap \overline{B(c_{n-1}^+, r_{n-1})} = \{x_n^+\}, \qquad \overline{B(c_n^-, r_n)} \cap \overline{B(c_{n-1}^-, r_{n-1})} = \{x_n^-\}\,.$$

Define $f_1 : \Omega \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} -\sqrt{r_n^2 - \|x - c_n^+\|^2} & \text{if } x \in B(c_n^+, r_n) \\ -\sqrt{r_n^2 - \|x - c_n^-\|^2} & \text{if } x \in B(c_n^-, r_n) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that f_1 is continuous and that $\{x_n^+, x_n^- : n \geq 1\} \subset BP_{f_1}$; more precisely

$$\begin{gathered} (1,0) \in \partial^\infty f_1(x_n^+) \text{ is realized by a ball of radius } r_{n-1}, \\ (-1,0) \in \partial^\infty f_1(x_n^+) \text{ is realized by a ball of radius } r_n, \\ (1,0) \in \partial^\infty f_1(x_n^-) \text{ is realized by a ball of radius } r_n, \\ (-1,0) \in \partial^\infty f_1(x_n^-) \text{ is realized by a ball of radius } r_{n-1}. \end{gathered} \tag{19}$$

For any $x = (\xi, \eta) \in \Omega$ we also define

$$f_2(x) = -\sqrt{2|\eta| - \eta^2} = -\sqrt{1 - (1 - |\eta|)^2}\,.$$

One can easily check that f_2 is continuous on Ω and that $BP_{f_2} = \{(\xi, 0) : \xi \in (-1,1)\}$; more precisely, for any $\xi \in (-1,1)$

$$(0,1), (-1,0) \in \partial^\infty f_2(\xi, 0) \text{ are realized by balls of radius } 1. \tag{20}$$

Notice also that $f_1(x_n^{\pm}) = f_2(x_n^{\pm}) = 0$ for any $n \geq 1$. Therefore, the function $f := \inf\{f_1, f_2\}$ is continuous on Ω and $f(x_n^{\pm}) = f_1(x_n^{\pm}) = f_2(x_n^{\pm}) = 0$. Taking (19) and (20) into account we obtain that

$$(1,0), (-1,0), (0,1), (0,-1) \in \partial^\infty f(x_n^{\pm}) \quad \text{for any } n \geq 1$$

whence

$$\{x_n^+, x_n^- : n \geq 1\} \subset BP_{f,2}$$

which in turn implies $\mathcal{H}^0(BP_{f,2}) = \infty$, as desired.

Acknowledgements. We are grateful to G. Colombo for his interest in the paper and for many stimulating discussions. We also thank the two referees for their careful reading of the paper and their valuable suggestions.

References

[1] G. Alberti, L. Ambrosio, P. Cannarsa: On the singularities of convex functions, Manuscr. Math. 76 (1992) 421–435.

[2] G. Alberti: On the structure of singular sets of convex functions, Calc. Var. Partial Differ. Equ. 2 (1994) 17–27.

[3] P. Albano, P. Cannarsa: Singularities of semiconcave functions in Banach spaces, in: Stochastic Analysis, Control, Optimization and Applications, W. M. McEneaney et al. (ed.), Birkhäuser, Boston (1999) 171–190.

[4] P. Albano, P. Cannarsa: Structural properties of singularities of semiconcave functions, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 28 (1999) 719–740.

[5] L. Ambrosio, P. Tilli: Topics on Analysis in Metric Spaces, Oxford Lecture Series in Mathematics and its Applications 25, Oxford University Press, Oxford (2004).

[6] M. Bardi, I. Capuzzo-Dolcetta: Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhäuser, Boston (1997).

[7] P. Cannarsa, C. Sinestrari: Convexity properties of the minimum time function, Calc. Var. Partial Differ. Equ. 3 (1995) 273–298.

[8] P. Cannarsa, C. Sinestrari: Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control, Birkhäuser, Boston (2004).

[9] P. Cannarsa, P. R. Wolenski: Semiconcavity of the value function for a class of differential inclusions, Discrete Contin. Dyn. Syst. 29 (2011) 453–466.

[10] A. Canino: On p-convex sets and geodesics, J. Differ. Equations 75 (1988) 118–157.

[11] F. H. Clarke, R. J. Stern, P. R. Wolenski: Proximal smoothness and the lower-C^2 property, J. Convex Analysis 2 (1995) 117–144.

[12] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski: Nonsmooth Analysis and Control Theory, Springer, New York (1998).

[13] G. Colombo, V. V. Goncharov: Variational inequalities and regularity properties of closed sets in Hilbert spaces, J. Convex Analysis 8 (2001) 197–221.

[14] G. Colombo, A. Marigonda: Differentiability properties for a class of non-convex functions, Calc. Var. Partial Differ. Equ. 25 (2006) 1–31.

[15] G. Colombo, A. Marigonda, P. R. Wolenski: Some new regularity properties for the minimal time function, SIAM J. Control Optim. 44 (2006) 2285–2299.

[16] G. Colombo, Khai T. Nguyen: On the structure of the minimum time function, SIAM J. Control Optim. 48 (2010) 4776–4814.

[17] P. Erdös: Some remarks on the measurability of certain sets, Bull. Amer. Math. Soc. 51 (1945) 728–731.

[18] H. Federer: Geometric Measure Theory, Springer, Berlin (1969).

[19] Khai T. Nguyen: Hypographs satisfying an external sphere condition and the regularity of the minimum time function, J. Math. Anal. Appl. 372 (2010) 611–628.

[20] P. Mattila: Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability, Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge (1995).

[21] C. Nour, R. J. Stern, J. Takche: Proximal smoothness and the exterior sphere condition, J. Convex Analysis 16 (2009) 501–514.

[22] C. Nour, R. J. Stern, J. Takche: The θ-exterior sphere condition, φ-convexity, and local semiconcavity, Nonlinear Anal. 73 (2010) 573–589.

[23] R. A. Poliquin, R. T. Rockafellar, L. Thibault: Local differentiability of distance functions, Trans. Amer. Math. Soc. 352 (2000) 5231–5249.

[24] R. T. Rockafellar: Clarke's tangent cones and the boundaries of closed sets in $\mathbb{R}^n$, Nonlinear Anal. 3 (1979) 145–154.

[25] L. Veselý: On the multiplicity points of monotone operators on separable Banach spaces, Commentat. Math. Univ. Carol. 27 (1986) 551–570.

[26] L. Veselý: On the multiplicity points of monotone operators on separable Banach spaces, Commentat. Math. Univ. Carol. 28 (1987) 295–299.

[27] L. Zajíček: On the points of multiplicity of monotone operators, Commentat. Math. Univ. Carol. 19 (1978) 179–189.

[28] L. Zajíček: On the differentiation of convex functions in finite and infinite dimentional spaces, Czech. Math. J. 29 (1979) 340–348.