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A Multiplier Rule for Stable Problems in Vector Optimization

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Recently, by using the derivatives of scalarized maps, associated with a vector optimization problem, new multiplier rules have been proven. The first objective of this paper is to show that those rules do not hold in infinite dimensional setting without imposing additional restrictions, even when the ordering cone has a nonempty interior. In this paper, we employ the weak-interior of the ordering cone to propose a new condition. Under this condition, we show that the original problem is equivalent to an scalarized finite-dimensional problem. As a consequence we prove a multiplier rule in infinite dimensional setting for stable data. The proof of these results rely on a new estimate about the dual cones of weakly-solid cones. Several counterexamples showing that the hypotheses are essential are given.

Keywords: Multiplier rules, stable maps, set-valued analysis, contingent epiderivatives, contingent cones, weak-minimizers

1. Introduction

Throughout this paper, X and Z are real normed spaces, $S \subset X$, and $K \subset Z$ is a closed, and convex cone. By Z^* we denote the topological dual of Z, and by K^+ we represent the positive dual cone of K, defined, as usual, by

$$K^+ = \{\lambda^* \in Z^* : \lambda(k) \geq 0, \text{ for every } k \in K\}.$$

Furthermore, $C \subset \mathbb{R}^m$ is a closed, convex, and pointed cone with a nonempty interior, that is, $\operatorname{int} C \neq \emptyset$.

In this work, we focus on the following constrained vector optimization problem

$$(P) \qquad \text{minimize } f(x) \text{ such that } g(x) \in -K, \quad x \in S,$$

where $f : X \to \mathbb{R}^m$ and $g : X \to Z$ are two given single-valued maps. By

$$S_P = \{x \in S : g(x) \in -K\},$$

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we specify the set of all feasible points of problem (P).

In the vector optimization problem (P), we seek a weak minimizer. Recall that a point $\bar{x} \in S_P$ is said to be a weak minimizer of (P), if and only if,

$$(f(\bar{x}) - \operatorname{int} C) \cap f(S_P) = \emptyset.$$

Clearly, when $m = 1$ and $C = \mathbb{R}_+$, we recover the classical constrained optimization problem.

Two general multiplier rules for problem (P) can be stated as follows:

(a) If $\bar{x}$ is a weak minimizer of (P), then there exist $\lambda^* \in C^+$ and $\mu^* \in K^+$ with $(\lambda^*, \mu^*) \neq 0$ such that

$$\lambda^* \circ Df(\bar{x})(u) + \mu^* \circ Dg(\bar{x})(u) \geq 0 \text{ for all } u \in T(S, \bar{x}),\ \mu^* \circ g(\bar{x}) = 0. \quad (1)$$

(b) If $\bar{x}$ is a weak minimizer of (P), then there exist $\lambda^* \in C^+$ and $\mu^* \in K^+$ with $(\lambda^*, \mu^*) \neq 0$ such that

$$D\left(\lambda^* \circ f\right)(\bar{x})(u) + D\left(\mu^* \circ g\right)(\bar{x})(u) \geq 0 \text{ for all } u \in T(S, \bar{x}),\ \mu^* \circ g(\bar{x}) = 0. \quad (2)$$

Here $T(S, \bar{x})$ denotes the contingent cone to S at $\bar{x}$, and the notation D in (1) and (2), represents a suitable derivative. (A precise definition of the derivatives to be employed in this work will be given shortly.) Clearly, in (1), the multiplier rule is given as a scalarization of the derivatives of the involved maps, whereas in (2), the multiplier rule is given in terms of the derivatives of the scalarized maps.

A vast amount of literature is devoted to studies validating the above multipliers rules. Multiplier rule (2) appears more appealing than (1). This is partly because the maps $\lambda^* \circ f$ and $\mu^* \circ g$ are scalar-valued, and, the conditions that ensure the existence of their derivatives $D\left(\lambda^* \circ f\right)(\bar{x})$ and $D\left(\mu^* \circ g\right)(\bar{x})$ are rather mild. It turns out that if $m = 1$, $C = \mathbb{R}_+$, and the maps f and g are Fréchet differentiable and continuously Fréchet differentiable at $\bar{x}$, respectively, and the following regularity condition holds

$$Dg(\bar{x})(\operatorname{cone}(S - \bar{x})) + \operatorname{cone}(g(\bar{x}) + K) = Z,$$

then (a) holds with $\lambda^* \neq 0$ (see [12, Theorem 1.6]) where $\operatorname{cone}(S - \bar{x})$ is the cone generated by $S - \{\overline{x}\}$ whose closure is $T(S, \overline{x})$. Furthermore, in this particular case, (b) is equivalent to (a). We remark that this equivalence between (a) and (b) holds even for an ordering cone K with possibly empty interior. However, this result is not true in general, even for differentiable and convex maps (see [3, Example 3.20]). In fact, for any weaker differentiability notion, in order to prove the existence of multipliers (by means of the Hahn-Banach theorem), it is necessary to assume that the ordering cone K has a nonempty interior.

The situation is far more complex for non-differentiable maps. For nonsmooth maps, first of all, we need to choose a suitable differentiability notion to establish meaningful multiplier rules. Furthermore, in general, the equivalence between (a) and (b) does not hold. In fact, it is well possible that the scalarization used in (b) is not even well-defined. One of the commonly used approaches in this case is to employ variational analysis tools such as subdifferentials or set-valued derivatives as appropriate differentiability notions for the multiplier rules. In this context, the graphical derivatives, defined by means of tangent cones to graph/epigraph of the involved maps, turn out to be indispensable tools (see [2]). The use of these derivatives also extends to establish multiplier rules for a set-valued extension of (P), where f and g are set-valued maps. In an interesting paper, Corley [5] proved a set-valued analogue of (a) by using the contingent derivatives and the circatangent derivatives (see also [1]). In an important extension of this work, Götz and Jahn [8, 11] proved a version of (a) by employing the notion of contingent epiderivatives which was proposed by Jahn-Rauh [14]. The existence conditions for contingent epiderivatives are rather stringent. However, the equivalence between (a) and (b) is known to hold when the contingent epiderivatives of the maps f and g exist.

Motivated by the ongoing research, recently in [10], the authors showed that the multiplier rule (b) holds when Z is finite-dimensional, the ordering cone K has a nonempty interior, and the involved maps satisfy a mild stability condition without assuming the epidifferentiability of f and g. The results were shown to be true for optimization problems with set-valued maps. Unfortunately, such a result can not be extended to infinite dimensional spaces. Example 4.3 given below, posed in a real separable Hilbert space Z, describes the associated difficulties. Therefore, additional assumptions must be imposed to establish a multiplier rule for stable maps admitting values in infinite-dimensional image spaces. The primary objective of this paper is to remedy this difficulty. To be specific, we will show that (b) holds for stable data provided that the ordering cone has non-empty weak interior. (Recall that the interior of the cone K in the weak topology is referred to as its weak-interior.)

The organization of this paper is as follows: In Section 2, we show that the problem (P) is equivalent to a finite-dimensional problem, given that the cone K has a nonempty weak-interior (cf. Theorem 3.5). By employing this result, we prove, in Section 3, a new multiplier rule for stable data (Theorem 4.4). Let us underline that in this work we restrict our attention entirely to single-valued maps. However, an extension of all of our results to optimization problem with set-valued data can be carried out by combining the techniques used in [10] with the results of this paper.

2. Preliminaries

In this section, we collect some definitions and notations for their later use in this paper. By

$$B(\bar{z}, \epsilon) := \{z \in Z : \|z - \bar{z}\| \leq \epsilon\},$$

we denote the closed ball, centered at $\bar{z} \in Z$ and with radius $\epsilon > 0$. The strong convergence and the weak convergence are specified by $\to$ and $\rightharpoonup$, respectively. Although our primary focus in this work is on single-valued maps, we recall some concepts from set-valued analysis which will be used to define our differentiability notions. The domain, the graph, and the epigraph of a set-valued map $F : X \rightrightarrows Z$

are given by

$$\operatorname{dom}(F) = \{x \in X : \ F(x) \neq \emptyset\},$$
$$\operatorname{graph}(F) = \{(x,z) \in X \times Z : z \in F(x)\},$$
$$\operatorname{epi}(F) = \{(x,z) \in X \times Z : z \in F(x) + K\},$$

respectively. Given $\lambda^* \in Z^*$ the notation $\lambda^* \circ F : X \rightrightarrows \mathbb{R}$ denotes the set-valued map defined by

$$\lambda^* \circ F(x) = \{\lambda^*(z) : z \in F(x)\},$$

whereas $F + K : X \rightrightarrows Z$ defines the set-valued map

$$(F+K)\,(x) = \{z + k : \ z \in F(x), \ k \in K\}.$$

We assume the usual convention that $\lambda(\emptyset) = \emptyset$ and $\emptyset + K = \emptyset$.

Let $A \subset Z$ be arbitrary. The contingent cone to A at $a \in A$, denoted by $T(A,a)$, is defined by:

$$T(A,a) = \{u \in Z : \exists (t_n) \subset \mathbb{R}_+ \setminus \{0\}, \exists (a_n) \subset A \text{ such that } a_n \to a, \ t_n(a_n - a) \to u\}.$$

The contingent derivative of a set-valued map $F : X \rightrightarrows Z$ at $(\bar{x},\bar{z}) \in \operatorname{graph}(F)$ is the set-valued map $D_cF(\bar{x},\bar{y}) : X \rightrightarrows Z$ satisfying the following identity:

$$\operatorname{graph}(D_cF(\bar{x},z)) = T(\operatorname{graph}(F),(\bar{x},\bar{z})).$$

Details of this useful concept are available in [2, 20].

If the map F is single-valued, we write $F : X \to Z$. In this case, instead of $D_cF(\bar{x},F(\bar{x}))$, we will denote the contingent derivative by $D_cF(\bar{x})$. In this paper, we will follow the same convention for other derivatives as well.

The following notion of contingent epiderivative will play a central role in this work:

Definition 2.1. Let $L = \operatorname{dom}(D_c(f+\mathbb{R})(\bar{x},f(\bar{x})))$. Given $S \subset X$, the contingent epiderivative of a single-valued map $f : S \to \mathbb{R}$ at $\bar{x} \in S$ is the single-valued map $D_\uparrow f(\bar{x},f(\bar{x})) : L \to \mathbb{R}$ whose epigraph coincides with the contingent cone to the epigraph of f at $(\bar{x},f(\bar{x}))$, that is,

$$\operatorname{epi}(D_\uparrow f(\bar{x})) = T(\operatorname{epi}(f),(\bar{x},f(\bar{x}))).$$

We remark that contingent derivatives and contingent epiderivatives depend on the effective domain of the underlying map. Furthermore, the following expression is known to hold for the contingent epiderivative:

$$D_\uparrow f(\bar{x})(u) = \liminf_{\substack{w\to u,\, t\to 0^+\\ \bar{x}+tw\in S}} \frac{f(\bar{x}+tw)-f(\bar{x})}{t}.$$

By considering the weak topology on the image space Z, we have a corresponding notion of τ^w-contingent derivative. Given $(x_n,y_n),(\bar{x},\bar{y}) \subset X \times Z$, by $(x_n,y_n) \underset{s,w}{\to} (\bar{x},\bar{y})$ we understand that $x_n \to \bar{x}$ and $y_n \rightharpoonup \bar{y}$.

Let $A \subset X \times Z$ a nonempty set of the product space and let $(\bar{x}, \bar{y}) \in A$. The weak contingent cone of A at $(\bar{x}, \bar{z})$, denoted by $T^w(A, (\bar{x}, \bar{z}))$, is defined by

$$T^w(A, (\bar{x}, \bar{z})) = \{v = (v_1, v_2) \in X \times Z : \exists (t_n) \subset \mathbb{R}_+ \backslash \{0\}, ((x_n, z_n)) \subset A \text{ with } (x_n, z_n) \underset{s,w}{\rightarrow} (\bar{x}, \bar{z}),\ t_n(x_n - \bar{x}, z_n - \bar{z}) \underset{s,w}{\rightarrow} (v_1, v_2)\}.$$

Using the above notion of the weak contingent cone, the τ^w-contingent derivative of a set-valued map $F : X \rightrightarrows Z$ at $(\bar{x}, \bar{z}) \in \text{graph}(F)$ is the set-valued map $D_c^w F(\bar{x}, \bar{z}) : X \rightrightarrows Z$ such that

$$\text{graph}(D_c^w F(\bar{x}, \bar{z})) = T^w(\text{graph}(F), (\bar{x}, \bar{z})).$$

Details of this notion are available in [22]. A related notion of weak tangent cone can be found in Borwein [4].

We conclude this section by recalling the following useful notion of stability / calmness:

Definition 2.2. Given $S \subset X$, a map $f : S \to Z$ is said to be stable at $\bar{x} \in S$, if there are constants ϵ, $M > 0$ such that

$$\|f(x) - f(\bar{x})\| \leq M \|x - \bar{x}\|, \quad \text{for every } x \in B(\bar{x}, \epsilon) \cap S.$$

3. Discretization of the Problem

Given a finite family of linearly independent functionals $\{\mu_1^*, ..., \mu_n^*\} \subset Z^*$, we define a discretization of the closed, and convex cone $K \subset Z$ by

$$\mathbb{K}_n = T^*(K),$$

where $T^* : Z \to \mathbb{R}^n$ is a bounded, linear operator, defined by

$$T^*(z) = (\mu_1^*(z), ..., \mu_n^*(z)), \quad \text{for every } z \in Z.$$

Using this notion, a scalarization of (P) is defined as follows:

$$(P^*) \qquad \text{minimize } f(x) \text{ such that } (T^* \circ g)(x) \in -\mathbb{K}_n, \quad x \in S.$$

Notice that the above formulation more resembles a discretization of the problem where we discretize the ordering cone. In contrast, a standard scalarization approach would reduce the problem to a scalar problem (see [13, 16, 7] and references therein).

In the following, the weak-interior of the cone K will be denoted by w-int(K). The following result gives a characterization of the discretization cone.

Theorem 3.1. *If* w-int$(K) \neq \emptyset$, *then there exists a family* $\{\mu_1^*, ..., \mu_n^*\} \subset Z^*$ *such that the associated discretization cone* $\mathbb{K}_n$ *is a closed, and convex cone, and has a nonempty interior. Furthermore, the following characterization holds:*

$$K^+ = \left\{ \sum_{i=1}^{n} \mu_i \mu_i^* : (\mu_1, ..., \mu_n) \in \mathbb{K}_n^+ \right\}. \tag{3}$$

Proof. We begin by establishing (3). We fix an element $k \in$ w-int(K). Then there exist $\varepsilon > 0$, a family $\{\mu_1^*, ..., \mu_n^*\} \subset Z^*$, and a weak neighborhood of 0 of the form

$$\mathcal{U} = \{z \in Z : |\mu_i^*(z)| < \varepsilon,\ i = 1, \dots, n\},$$

such that $k + \mathcal{U} \subset K$. We can assume that $\{\mu_1^*, ..., \mu_n^*\}$ is linearly independent. By denoting $\mathcal{N} = \bigcap_{i=1}^n \ker \mu_i^*$, we have

$$k + \mathcal{N} \subset k + \mathcal{U} \subset K.$$

Therefore, for every $\mu^* \in K^+$, we have

$$\mu^*(k + \mathcal{N}) \subset \mu^*(K) \subset \mathbb{R}_+.$$

Being a linear subspace, $\mathcal{N}$ is symmetric, and hence $\mu^*(\mathcal{N}) = 0$. This, in view of [19, Lemma 1.9.11], implies that μ^* is a linear combination of $\{\mu_1^*, ..., \mu_n^*\}$. In other words, there exists $(\mu_1, ..., \mu_n) \in \mathbb{R}^n$ such that

$$\mu^* = \sum_{i=1}^n \mu_i \mu_i^*. \tag{4}$$

Denoting by $\langle \cdot, \cdot \rangle_Z$, the duality pairing between Z^* and Z, we define a bounded operator $R : \mathbb{R}^n \to Z^*$ by

$$R(\mu) = \sum_{i=1}^n \mu_i \mu_i^*,$$

where $\mu = (\mu_1, ..., \mu_n) \in \mathbb{R}^n$. Its adjoint operator $R^* : Z^{**} \to (\mathbb{R}^n)^*$ is given by

$$R^*(z^{**}) = z^{**} \circ R \quad \text{for every } z^{**} \in Z^{**}.$$

Now if we denote by $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ the usual inner product in $\mathbb{R}^n$ and by $J : Z \to Z^{**}$ the natural embedding map, defined by $J(z) = \langle \cdot, z \rangle_Z$, we have

$$\begin{aligned} \langle \mu, T^* z \rangle_{\mathbb{R}^n} &= \sum_{i=1}^n \mu_i \mu_i^*(z) \\ &= \langle R(\mu), z \rangle_Z \\ &= J(z) \circ R(\mu) \\ &= (R^* \circ J(z))(\mu), \end{aligned}$$

for every $\mu \in \mathbb{R}^n,\ z \in Z$. Therefore, $R^* \circ J = T^*$, and we have proven the following identity

$$\langle R(\mu), z \rangle_Z = \langle \mu, T^* z \rangle_{\mathbb{R}^n} \quad \text{for all } \mu \in \mathbb{R}^n,\ z \in Z. \tag{5}$$

By (4), $K^+ \subset \text{range}(R)$, and hence we can define the subset $\mathbb{D} := R^{-1}(K^+)$. Therefore:

$$K^+ = \left\{ \sum_{i=1}^n \mu_i \mu_i^* :\ (\mu_1, ..., \mu_n) \in \mathbb{D} \right\}. \tag{6}$$

It is easily seen that $\mathbb{D}$ is a closed and convex cone in $\mathbb{R}^n$. Let us now prove that $\mathbb{D} = \mathbb{K}_n^+$. In fact, given $\mu = (\mu_1, ..., \mu_n) \in \mathbb{K}_n^+$, we have $\langle \mu, T^* z\rangle_{\mathbb{R}^n} \geq 0$, and by (5), we obtain $\langle R(\mu), z\rangle_{\mathbb{R}^n} \geq 0$ for every $z \in K$. This implies that $R(\mu) \in K^+$ and therefore $\mu \in \mathbb{D} = R^{-1}(K^+)$. Conversely, given $\mu \in \mathbb{D}$, we have $R(\mu) \in K^+$, implying $\langle R(\mu), z\rangle_{\mathbb{R}^n} \geq 0$, and by (5) $\langle \mu, T^* z\rangle_{\mathbb{R}^n} \geq 0$ for every $z \in K$. This, moreover, implies that $\mu \in \mathbb{K}_n^+$. Therefore $\mathbb{K}_n^+ = \mathbb{D}$. Combining this with (6) yields (3).

It remains to show that $\mathbb{K}_n$ has a nonempty interior. Due to the definition $\mathbb{K}_n = T^*(K)$ and the conditions w-int$(K) \neq \emptyset$, we have int$(K) \neq \emptyset$. This, in view of the linearity of T^*, confirms that int$(\mathbb{K}_n) \neq \emptyset$. This completes the proof. □

The following example illustrates the above result:

Example 3.2. Let Z be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of Z, and let $(e_n^*)_{n\in\mathbb{N}} \equiv (\langle e_n, \cdot\rangle)_{n\in\mathbb{N}} \subset Z^*$ be the associated family of biorthogonal functionals. We consider the following ordering cone:

$$K = \{z \in Z : e_1^*(z) \geq 0,\ e_1^*(z)^2 \geq e_2^*(z)^2 + e_3^*(z)^2\}.$$

This cone is a closed convex cone with nonempty weak interior. Furthermore, the family

$$\{\mu_1^*, ..., \mu_n^*\} = \{e_1^*, ..., e_3^*\}$$

verifies Theorem 3.1. In this case, $T^* \equiv (e_1^*, e_2^*, e_3^*) : Z \to \mathbb{R}^3$ and the discretization cone is given by the “ice-cream" cone

$$\mathbb{K}_3 = T^*(K) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0,\ z_1^2 \geq z_2^2 + z_3^2\}. \qquad \square$$

Remark 3.3. Notice that the family $\{\mu_1^*, ..., \mu_n^*\}$, and hence the discretization cone, in Theorem 3.1 is not unique. In fact, it can even have different cardinality. For instance, $\{\mu_1^*, ..., \mu_n^*\} = \{e_1^*, ..., e_4^*\}$, with associated (nonpointed) cone

$$\mathbb{K}_4 = \{(z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : z_1 \geq 0,\ \ z_1^2 \geq z_2^2 + z_3^2\},$$

also verify Theorem 3.1.

Remark 3.4. In general, the containment $\ker T^* \subset K \cap -K$ holds. For an infinite-dimensional Z, $\ker T^*$ is not reduced to zero. On the contrary, we would have $Z^* = \langle \mu_1^*, ..., \mu_n^* \rangle$. Therefore, every closed and convex cone in an infinite-dimensional space with nonempty weak interior is necessarily nonpointed.

The following result makes use of the above characterization.

Theorem 3.5. *If* w-int$(K) \neq \emptyset$*, then there exists a linearly independent family of functionals* $\{\mu_1^*, ..., \mu_n^*\}$ *such that the problems* (P) *and* (P^*) *are equivalent. In other words, the feasible sets, as well as the set of solutions of* (P) *and* (P^*) *coincide.*

Proof. As w-int$(K) \neq \emptyset$, we can consider the family $\{\mu_i^*, ..., \mu_n^*\} \subset Z^*$ and $\mathbb{K}_n$ the discretization cone provided by Theorem 3.1. We begin by showing that $S_P = S_{P^*}$, that is,

$$\{x \in S : g(x) \in -K\} = \{x \in S : T^* \circ g(x) \in -\mathbb{K}_n\}.$$

The containment $S_P \subseteq S_{P^*}$ is a straightforward consequence of the definition of $\mathbb{K}_n$. On the other hand, given $\tilde{x} \in S_{P^*}$, there exists $k \in K$ such that $T^*(g(\tilde{x}) + k) = 0$, implying $g(\tilde{x}) + k \in \ker T^*$. This, in view of Remark 3.4, ensures that $g(\tilde{x}) + k \in K \cap -K$. Therefore $g(\tilde{x}) \in [K \cap -K] - K \subset -K$, which implies that $\tilde{x} \in S_P$. The proof is complete. □

4. A Multiplier Rule for Stable Data

Throughout this section, we will follow the notations defined in the Section 3 above. For a family $\{\mu_1^*, ..., \mu_n^*\} \subset Z^*$, let $\mathbb{K}_n$ be the corresponding discretization cone of K.

We begin by formulating the following three statements:

(i) An element $\bar{x}$ is a weak minimizer of (P).

(ii) There exist $\lambda^* \in C^+$ and $\mu^* \in K^+$ with $(\lambda^*, \mu^*) \neq 0$, such that

$$D_{\uparrow}(\lambda^* \circ f)(\bar{x})(u) + D_{\uparrow}(\mu^* \circ g)(\bar{x})(u) \geq 0, \quad \text{for every } u \in T(S, \bar{x}), \tag{7}$$

$$\mu^*(g(\bar{x})) = 0. \tag{8}$$

(iii) There exists $0 \neq (\lambda^*, (\mu_1, ..., \mu_n)) \in C^+ \times \mathbb{K}_n^+$ such that

$$D_{\uparrow}(\lambda^* \circ f)(\bar{x})(u) + \sum_{i=1}^{n} \mu_i D_{\uparrow} (\mu_i^* \circ g) (\bar{x})(u) \geq 0, \quad \text{for every } u \in T(S, \bar{x}),$$

$$\sum_{i=1}^{n} \mu_i (\mu_i^* \circ g) (\bar{x}) = 0.$$

We also need to recall the following convexity notions for its later use.

Definition 4.1. A map $f : S \subset X \to Z$ is said to be contingently K-convex (respectively τ^w-contingently K-convex) at $(\bar{x}, f(\bar{x})) \in \text{graph}(f)$, if $T(\text{epi}(f), (\bar{x}, f(\bar{x})))$ (respectively $T^w(\text{epi}(f), (\bar{x}, f(\bar{x})))$ is a convex subset of $X \times Z$.

Let us consider the following regularity condition of Kurcyusz-Robinson-Zowe type:

$$D_c (g + K) (\bar{x})(T(S, \bar{x})) + \text{cone}(g(\bar{x}) + K) = Z. \tag{9}$$

Our first result of this section is as follows:

Theorem 4.2. *Let Z be finite-dimensional and* $\text{int}(K) \neq \emptyset$, *let* $S = \text{dom}(f) = \text{dom}(g)$. *If the map (f, g) is contingently $C \times K$-convex and stable at $\bar{x}$, then* (i) $\Rightarrow$ (ii). *Moreover, if* (9) *holds then* $\lambda^* \neq 0$.

Proof. The proof of the result that (i) implies (ii) is based on standard arguments and it can be deduced from known results. We only need to follow the same steps as in [10, Theorem 3.1] and take into account that in [10, Theorem 3.1], the $C \times K$-convexity of (f, g) at $\bar{x}$ can be replaced by the more general condition of contingent $C \times K$-convexity.

Let us now prove that $\lambda^* \neq 0$ under the regularity assumption (9). Notice that if $\lambda^* = 0$, then from (7) and (8), we have that there exists $\mu^* \neq 0$ such that

$$D_\uparrow(\mu^* \circ g)(\bar{x})(u) \geq 0 \text{ for every } u \in T(S, \bar{x}), \qquad (10)$$
$$\mu^*(g(\bar{x})) = 0.$$

From (9), there exist $\tilde{u} \in T(S, \bar{x})$, $w \in D_c(g + K)(\bar{x})(\tilde{u})$, $\alpha \in \mathbb{R}_+$, $k \in K$ such that $z = w + \alpha\,(g(\bar{x}) + k)$, so taking into account that

$$D_\uparrow(\mu^* \circ g)(\bar{x})(\tilde{u}) = \min \mu^*(D_c g(\bar{x})(\tilde{u}))$$

(see [9, Proposition 2.2]) and (10), we have

$$\begin{aligned} \mu^*(w + \alpha\,(g(\bar{x}) + k)) &= \mu^*(w + \alpha k) \\ &\geq D_\uparrow(\mu^* \circ g)(\bar{x})(\tilde{u}) \\ &\geq 0. \end{aligned}$$

This, however, implies that $\mu^*(z) \geq 0$ for every $z \in Z$. Therefore, $\mu^* = 0$, and we obtain a contradiction to the hypothesis. The proof is complete. □

The next example shows that Theorem 4.2 can not be extended to the case when Z is an infinite-dimensional space.

Example 4.3. Let $X = \mathbb{R}$, $m = 1$, $S = C = \mathbb{R}_+$, and as in Example 3.2, let Z be a real separable Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Z is ordered by the following ordering cone:

$$K = \left\{ z \in Z : e_1^*(z) \geq 0,\ e_1^*(z)^2 \geq \sum_{i=2}^{\infty} e_i^*(z)^2 \right\}.$$

Since K is a closed, convex, and pointed cone, we have w-int$(K) = \emptyset$ (see Remark 3.4). Furthermore, K has a nonempty interior. In fact, it can be checked that $e_1 \in \text{int}(K)$.

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x & \text{for } x \in \{n^{-1} : n \in \mathbb{N}\}, \\ x^2 & \text{for } x \in \mathbb{R}_+ \backslash \{n^{-1} : n \in \mathbb{N}\}, \end{cases}$$

and let $g : \mathbb{R}_+ \to \ell^2$ be defined by

$$g(x) = \begin{cases} x\,(-e_1 + 2e_n) & \text{for } x \in \{n^{-1} : n \in \mathbb{N}\}, \\ 0 & \text{for } x \in \mathbb{R}_+ \backslash \{n^{-1} : n \in \mathbb{N}\}. \end{cases}$$

The feasible set of problem (P) is $S_P = \mathbb{R}_+ \backslash \{n^{-1} : n \in \mathbb{N}\}$, so it is easily seen that $\bar{x} = 0$ is a solution of problem (P). Let us ensure that the rest of hypotheses of the theorem are also verified.

By a direct inspection, the map (f,g) is stable at $\bar{x}=0$.

Computing $T(\text{graph}((f,g)+C\times K),(0,(0,0)))$, we will see that (f,g) is contingently $C\times K$-convex at $\bar{x}=0$. Indeed, let

$$(u,(v,w))\in T(\text{graph}((f,g)+C\times K),(0,(0,0)))$$

be arbitrary. Then by the definition of the contingent cone, there exist $(t_n)\subset \mathbb{R}_+\backslash\{0\}$, $(x_n)\subset\mathbb{R}_+$, $(c_n,k_n)\subset C\times K$ such that $(x_n,(f(x_n)+c_n,g(x_n)+k_n))\to(0,(0,0))$ and

$$(u_n,v_n,w_n):=t_n\,(x_n,(f(x_n)+c_n,g(x_n)+k_n))\to(u,(v,w)). \tag{11}$$

In first place, for every $u\in\mathbb{R}_+$, by taking $t_n=u(n+1)/\sqrt{n}$, $x_n=\sqrt{n}/(n+1)$, $(c_n,k_n)=(0,0)$ we have

$$(u_n,v_n,w_n)=(u,(\sqrt{n}/(n+1),0))\to(u,(0,0)).$$

Therefore

$$(u,(0,0))\in T(\text{graph}((f,g)+C\times K),(0,(0,0)))$$

and consequently

$$(u,(0,0))+\{0\}\times C\times K\subset T(\text{graph}((f,g)+C\times K),(0,(0,0))).$$

This proves

$$\begin{aligned}&\{(u,(v,w))\in\mathbb{R}_+\times Y\times Z:(v,w)\in(0,0)+C\times K\}\\ \subset\ &T(\text{graph}((f,g)+C\times K,(0,(0,0))).\end{aligned}$$

Let us now prove the converse inclusion. Notice that $u\in\mathbb{R}_+$. Furthermore, we can assume without loss of generality the following two cases:

- If $(x_n)\neq(1/n)$, then by a direct computation $(v,w)\in C\times K$
- If $(x_n)=(1/n)$, then

$$(u_n,(v_n,w_n))=t_n\left(\frac{1}{n},\left(\frac{1}{n^2}+c_n,\frac{1}{n}(-e_1+2e_n)+k_n\right)\right)\to(u,(v,w)). \tag{12}$$

In this case, we can also consider two cases. Firstly, if (t_n) is convergent, then $u=0$, and from (12) it is easily seen that $(v,w)\in C\times K$. Secondly, if $(t_n)\to\infty$, we do not obtain tangent vectors as we prove in the following. On the contrary from (12) we have that the sequence $w_n=\frac{t_n}{n}(-e_1+2e_n)+t_nk_n$ converges in norm to an element $w\in Z$. Considering the expansion $k_n=\sum_{j=1}^{\infty}k_n^je_j$ by definition

$$k_n^1\geq 0,\qquad \left(k_n^1\right)^2\geq\sum_{j=2}^{\infty}\left(k_n^j\right)^2. \tag{13}$$

Then we can rewrite (12) in the following way

$$\frac{t_n}{n}(-e_1+2e_n)+t_nk_n=t_n\left(\frac{-1}{n}+k_n^1\right)e_1+t_n\left(\frac{2}{n}+k_n^n\right)e_n+t_n\sum_{j\neq1,n}^{\infty}k_n^je_j\to w. \tag{14}$$

From (14) necessarily $t_n\left(\frac{-1}{n}+k_n^1\right)$ is a norm convergent sequence, and since $\frac{t_n}{n}\to u$ we have $(t_nk_n^1)$ is a convergent sequence. The sequence (e_n) has no norm convergent subsequences, so $t_n\left(\frac{2}{n}+k_n^n\right)\to 0$ and consequently $\frac{2}{n}+k_n^n\to 0$.

Therefore, for a fixed $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $k_n^n\in\left(-\frac{2}{n}-\varepsilon,-\frac{2}{n}+\varepsilon\right)$ and

$$(k_n^n)^2\in\left(\left(-\frac{2}{n}+\varepsilon\right)^2,\left(-\frac{2}{n}-\varepsilon\right)^2\right)\quad\text{for every } n\geq N.$$

From this and (13), we have

$$\left(k_n^1\right)^2\geq (k_n^n)^2\geq\left(-\frac{2}{n}+\varepsilon\right)^2$$

and since $k_n^1\geq 0$, we can assure that

$$k_n^1\geq -\frac{2}{n}+\varepsilon\quad\text{for every } n\geq N.$$

Therefore, $t_nk_n^1\geq 2\frac{t_n}{n}+t_n\varepsilon$, and now taking into account that $\varepsilon>0$ is arbitrary, $\frac{t_n}{n}\to u$ and we are assuming $t_n\to\infty$, we have $(t_nk_n^1)\to\infty$ which contradicts that $(t_nk_n^1)$ is convergent. Therefore, in this case we do not obtain tangent vectors.

Consequently, we have proven

$$\begin{aligned}&T(\text{graph}((f,g)+C\times K,(0,(0,0)))\\&=\{(u,(v,w))\in\mathbb{R}_+\times Y\times Z:(v,w)\in(0,0)+C\times K\},\end{aligned}$$

which is a convex set, so (f,g) is contingently $C\times K$-convex at $\bar{x}=0$.

For each $\lambda^*\in C^+\equiv\mathbb{R}_+$, we have

$$D_\uparrow\left(\lambda^*\circ f\right)(0)(u)=-\lambda^*u\quad\text{for every } u\in T(\mathbb{R}_+,0)=\mathbb{R}_+.$$

Taking into account that in this case the biorthogonals $\{e_n^*\}_{n\in\mathbb{N}}$ form a Schauder basis of Z^*, any element $\mu^*\in K^+\backslash\{0\}$ can be expressed as $\mu^*=\sum_{i=1}^\infty\mu_ie_i^*$. Therefore, the function $\mu^*\circ g$ is determined by $(\mu^*\circ g)(n^{-1})=(n^{-1})\left(-\mu_1+2\mu_n\right)$, and $(\mu^*\circ g)(x)=0$ elsewhere. Taking into account that $\mu_n\to 0$, by a direct computation, for every $u\in\mathbb{R}_+$, we obtain

$$\begin{aligned}D_\uparrow\left(\mu^*\circ g\right)(0)(u)&=\min D_c(\mu^*\circ g)(0)(u)\\&=\min\{-\mu_1u,0\}.\end{aligned}$$

Since $e_1\in\text{int}(K)$, we necessarily have $\mu^*(e_1)=\mu_1>0$. Therefore

$$D_\uparrow\left(\mu^*\circ g\right)(0)(u)=-\mu_1u<0\quad\text{for every } u\in\mathbb{R}_+. \tag{15}$$

In particular, at $u=1$, for all $(\lambda^*,\mu^*=\sum_{i=1}^\infty\mu_ie_i^*)\in C^+\times K^+\backslash\{(0,0)\}$, we have

$$D_\uparrow\left(\lambda^*\circ f\right)(0)(1)+D_\uparrow\left(\mu^*\circ g\right)(0)(1)=-(\lambda^*+\mu_1).$$

Therefore, we have the following two cases:

- If $\lambda^* \neq 0$, then clearly $-(\lambda^* + \mu_1) < 0$.
- If $\lambda^* = 0$, then $\mu^* \neq 0$ and by (15) $-(\lambda^* + \mu_1) = -\mu_1 < 0$.

Consequently, we have shown that

$$D_{\uparrow}(\lambda^* \circ f)(0)(1) + D_{\uparrow}(\mu \circ g)(0)(1) = -(\lambda^* + \mu_1) < 0. \qquad \square$$

The above example makes it clear that to obtain a multiplier rule for problem (P) in the infinite-dimensional setting some additional hypotheses must be imposed. Nonetheless, by means of Theorem 3.5, we can establish a multiplier rule like (ii) for stable data without imposing any additional differentiability assumption. In this case, we will need to consider the following regularity condition which is implied by (9):

$$D_c(T^* \circ g + \mathbb{K}_n)(\bar{x})(T(S, \bar{x})) + \text{cone}(T^* \circ g(\bar{x}) + \mathbb{K}_n) = \mathbb{R}^n. \tag{16}$$

Theorem 4.4. *Assume that* w-int$(K) \neq \emptyset$ *and* $S = \text{dom}(f) = \text{dom}(g)$. *Let* $\{\mu_1^*, ..., \mu_n^*\} \subset Z^*$ *be any family verifying conditions of Theorem* 3.1. *If the map* $(f, T^* \circ g)$ *is contingently* $C \times \mathbb{K}_n$*-convex and stable at* $\bar{x}$*, then* (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii)*. Moreover, if* (16) *holds then* $\lambda^* \neq 0$.

Proof. (i) $\Rightarrow$ (iii). Let $\bar{x}$ be a weak minimizer of (P). The result then is direct consequence of Theorem 3.5. By this theorem, (P) is equivalent to (P^*). That is, if $\bar{x}$ is a solution of (P), then $\bar{x}$ solves

$$(P^*) \qquad \text{minimize } f(x) \text{ such that } (T^* \circ g)(x) \in -\mathbb{K}_n, \quad x \in S.$$

Taking into account that $\text{int}(\mathbb{K}_n) \neq \emptyset$, the implication follows from applying Theorem 4.2 (with $Z \equiv \mathbb{R}^n$, $K \equiv \mathbb{K}_n$, $f \equiv f$, $g \equiv T^* \circ g$) to (P^*). From this result there exists $0 \neq (\lambda^*, (\mu_1, ..., \mu_n)) \in C^+ \times \mathbb{K}_n^+$ such that $\sum_{i=1}^n \mu_i (\mu_i^* \circ g)(\bar{x}) = 0$ and

$$\begin{aligned} & D_{\uparrow}(\lambda^* \circ f)(\bar{x})(u) + D_{\uparrow}\left(\sum_{i=1}^n \mu_i \mu_i^* \circ g\right)(\bar{x})(u) \\ &= D_{\uparrow}(\lambda^* \circ f) + \sum_{i=1}^n \mu_i D_{\uparrow}(\mu_i^* \circ g)(\bar{x})(u) \\ &\geq 0 \quad \text{for every } u \in T(S, \bar{x}), \end{aligned}$$

where in the last equality we have applied the sum rule for stable maps established in [10, Theorem 2.3.].

(iii) $\Leftrightarrow$ (ii). This implication follows directly from the characterization of the dual cone K^+ given in Theorem 3.1. $\square$

If the space Z is reflexive, then we can establish the above theorem by imposing convexity directly on the product map (f, g). For this purpose, we need the following technical result:

Proposition 4.5. *Let Z be a reflexive Banach space. Given any $\mu^* \in Z^*$, if g is stable at $\bar{x}$ then:*

(i) $\text{dom}(D_c^w g(\bar{x})) = \text{dom}(D_c(\mu^* \circ g)(\bar{x})) = T(\text{dom}(g), \bar{x})$

(ii) $\mu^*(D_c^w g(\bar{x})(u)) = D_c(\mu^* \circ g)(\bar{x})(u)$, *for every* $u \in T(\mathrm{dom}(g), \bar{x})$.
(iii) $T^* \circ D_c^w g(\bar{x})(u) = D_c(T^* \circ g)(\bar{x})(u)$, *for every* $u \in T(\mathrm{dom}(g), \bar{x})$.

Proof. The proof of (*i*) and (*ii*) is also standard. Taking into account that in reflexive Banach spaces any norm bounded sequence has weakly convergent subsequences, we only need to follow the same steps as in [21, Lemma 4.4] considering in this case the weak contingent cone instead of the contingent cone. By definition of the map T^*, property (*iii*) is a direct consequence of (*i*) and (*ii*) □

Proposition 4.6. *Let* Z *be a reflexive Banach space and let* $\bar{x} \in \mathrm{dom}(g)$. *If* g *is* τ^w*-contingently* K*-convex at* $\bar{x}$, *then* $T^* \circ g$ *is contingently* $\mathbb{K}_n$*-convex at* $\bar{x}$.

Proof. By definition $T^w(\mathrm{epi}(g), (\bar{x}, g(\bar{x}))) = \mathrm{epi}(D_c^w(g+K), (\bar{x}, g(\bar{x})))$. Therefore f is τ^w-contingently K-convex at $\bar{x}$, if and only if $D_c^w(g+K)(x, g(\bar{x}))(u)$ is a convex set for every $u \in L := \mathrm{dom}(D_c^w(g+K), (\bar{x}, g(\bar{x})))$.

On the other hand by [23, Lemma 3.8] we have

$$D_c(T^* \circ g + \mathbb{K}_n)(x, T^*(g(\bar{x})))(u) = D_c(T^* \circ g)(\bar{x})(u) + \mathbb{K}_n,$$
$$D_c^w(g+K)(\bar{x}, g(\bar{x}))(u) = D_c^w g(\bar{x})(u) + K$$

for every $u \in T(\mathrm{dom}(g), \bar{x})$, respectively. From this and Proposition 4.5, we get

$$\begin{aligned} &D_c(T^* \circ g + \mathbb{K}_n)(x, T^*(g(\bar{x})))(u) = T^*\left(D_c^w g(\bar{x})(u) + K\right) \\ &= T^*(D_c^w(g+K)(\bar{x}, g(\bar{x}))(u)) \end{aligned} \tag{17}$$

for every $u \in T(\mathrm{dom}(g), \bar{x})$. By hypothesis $D_c^w(g+K)(x, g(\bar{x}))(u)$ is convex and in the same way its image $T^*(D_c^w(g+K)(\bar{x}, g(\bar{x}))(u))$. By (17) $D_c(T^* \circ g + \mathbb{K}_n)$ $(x, T^*(g(\bar{x})))(u)$ is also convex for every $u \in T(\mathrm{dom}(g), \bar{x})$ and consequently $T^* \circ g$ is contingently $\mathbb{K}_n$-convex at $\bar{x}$. □

From Proposition 4.6, the following corollary of Theorem 4.4 is immediate.

Corollary 4.7. *Assume that* Z *is a reflexive Banach space,* w-int$(K) \neq \emptyset$, $S = \mathrm{dom}(f) = \mathrm{dom}(g)$. *Assume* $\{\mu_1^*, ..., \mu_n^*\} \subset Z^*$ *is any family verifying Theorem* 3.1. *If the map* (f, g) *is* τ^w*-contingently* $C \times K$*-convex and stable at* $\bar{x}$, *then* (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). *Moreover, if* (16) *holds then* $\lambda^* \neq 0$.

Remark 4.8. Since the weak and the norm topology coincide in finite-dimensional spaces, Theorem 4.4 extends Theorem 4.2 from finite-dimensional setting to infinite-dimensional setting. It is known that if one of the conditions of stability, convexity, and regularity are not verified, then Theorem 4.2 does not hold. The same restrictions, therefore, apply to Theorem 4.4.

Furthermore, Example 4.3 also shows that Corollary 4.7, and hence Theorem 4.4, fails if the assumption of nonemptiness of the weak interior is not verified. By a direct computation

$$\begin{aligned} &T^w(\mathrm{graph}((f,g) + C \times K), (0, (0,0))) \\ &= \{(u, (v_1, v_2)) \in \mathbb{R}_+ \times \mathbb{R} \times Z : (v_1, v_2) \in (-u, -ue_1) + C \times K\}, \end{aligned}$$

and therefore, (f, g) is τ^w-contingently $C \times K$-convex at 0. In other words, all the hypotheses of Corollary 4.7, except the non-emptiness of the weak interior, are verified.

5. Concluding Remarks

In this paper, we used the weak-interior of the ordering cone to propose a new scalarization/discretization for a vector optimization problem. We used this condition to derive new multiplier rules in infinite-dimensional setting when the data is stable. The given multiplier rules (see Section 4) provide an extension to the nonsmooth vector case of the multiplier rules given in [10] where a general set-valued problem was considered. In fact, Theorem 4.2 is a direct extension of [10, Theorem 3.1] for a more general class of convex maps and constraint qualification in the finite-dimensional setting, while the equivalence (ii) $\Leftrightarrow$ (iii) in Theorem 4.4 and Corollary 4.7 is an extension of [10, Theorem 3.1] from finite-dimensional to infinite-dimensional spaces without assuming additional differentiability hypotheses on the maps. In this kind of multiplier rules, the multipliers appear as scalarization functions of the maps, instead of the derivatives. If we compare this with the existing literature for this kind of problems, we notice that it has some advantages. Firstly, the derivatives involved are of scalar maps which might be easier to compute and to handle numerically. Furthermore, although these rules are provided for the nonsmooth setting, they are established in terms of a single-valued derivative, instead of set-valued derivatives/subdifferentials as it is more common in the literature (see for example [5, 18, 6, 11]. Moreover, the equivalence (ii) $\Leftrightarrow$ (iii) is a scalarization of the multiplier rule given in [8] which is more general. However, our multipliers rules have the advantage that they provide a way to overcome the difficulty associated to the assumption of the existence of the contingent epiderivarive of (f, g). See [10] for a more detailed discussion of this issue for the finite-dimensional case. To the best of our knowledge, the equivalence (ii) $\Leftrightarrow$ (iii) is a new result in the vector optimization literature, since we provide a multiplier rule of Kuhn-Tucker type for infinite-dimensional problems by means of the new discretization of the cone proposed in Section 3.

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