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Céa-Falk's Error Estimate for Strongly Monotone Variational Inequalities of the Second Kind

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In this paper we extend the classical Céa-Falk lemma to variational inequalities of the second kind defined by strongly monotone operators. As an application we derive an asymptotic error estimate for an obstacle problem with a friction.

1. Introduction

As shown in many text books of numerical analysis of partial differential equations, see e.g. [3], Céa's Lemma is a well-known cornerstone in the numerical analysis of elliptic boundary value problems. It reduces the a priori error of a Galerkin approximation to the approximation error of the finite dimensional trial subspace in the solution space. Since for the latter error asymptotic results are available from approximation theory of piecewise polynomial approximation, asymptotic error estimates readily follow for Galerkin finite element approximation. The simple proof of Cea's Lemma for *linear* boundary value problems is based on Galerkin orthogonality and an ε form of the Cauchy inequality (a simple version of the arithmetic geometric mean equality), namely, for any positive numbers a, b, ε, we have

$$ab \leq \varepsilon\, a^2 + \frac{b^2}{4\varepsilon}.$$

If nonlinearities are present, the proof of a Céa's Lemma becomes more involved. If the nonlinear operator is of monotone type with polynomial growth behaviour, instead of Cauchy inequality, Young's inequality comes into play (see e.g. [6, 12]). More importingly, in unilateral variational problems, as e.g. in obstacle problems, also proper subsets have to be approximated. When the obstacle is non-zero or when higher approximation than piecewise linear approximation is used, this leads to so-called *nonconforming* approximation: the approximating set of the given subset is not necessarily a subset of the latter. This more complex situation is treated by Falk's error estimate [6, 7, 8] for variational inequalities of first kind (following the terminology of Glowinski [10]) defined by bilinear forms.
Both extensions of linear variational problems, namely monotone nonlinearity and

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unilateral conditions are covered in Glowinski [10, 11] and further extended and applied to the coupling of finite element and boundary element methods [4].
In this paper we present a novel extension of the Céa-Falk error estimate to non-linear variational inequalities of second kind that applies to *nonsmooth* boundary value problems. Such nonsmooth boundary value problems arise in unilateral contact problems with friction [16]. Here, we deal with strongly monotone variational inequalities. Thus, we include the Céa-Falk error estimates of Ciarlet [6, 7] and Glowinski [10, 11]. Similar to [14], instead of Young's inequlity, we use the concept of the Fenchel conjugate from convex analysis to admit a broader range of nonlinearities. Similarly to not neccessarily piecewise linear approximations that lead to nonconforming approximation, the nonlinear functional appearing in the variational inequality of the second type gives rise to an additional approximation error. As shown in the paper, this approximation error can be further quantified by Kepler's quadrature formula as an example of a quadrature method. Thus, we derive an abstract error estimate for the Galerkin aproximation which generalizes that one in [14]. As seen from the applications given in [14] and the coupled boundary finite element approximation of unilateral boundary value problems [4], such an abstract stability estimate provides an important step towards a priori error estimates for finite element approximations of nonlinear nonsmooth boundary value problems.

2. The abstract error estimate

The abstract setting is the following: let $(V, ||\cdot||_V)$ be a linear normed space, $K \subseteq V$ a nonempty closed convex subset, f a fixed element of the dual space V^* with norm $||\cdot||^*$, $A : V \to V^*$ a generally nonlinear mapping and $j : V \to \mathbb{R} \cup \{+\infty\}$ an extended-real valued functional.
We consider the following variational inequality of the second kind in the general form: find $u \in K$ such that

$$\langle Au, v - u\rangle + j(v) - j(u) \geq \langle f, v - u\rangle \quad \forall v \in K. \tag{1}$$

We assume the following properties for A and j to hold

(i1) A is Lipschitz continuous on bounded sets in the sense that for any ball $B(0, r) = \{z \in V : ||z|| \leq r\}$ there is a positive constant $C(r)$ such that

$$||Av_1 - Av_2||^* \leq C(r)||v_1 - v_2||_V \quad \forall v_1, v_2 \in B(0, r);$$

(i2) A is strongly monotone in the sense that there exists a convex function $\varphi : [0, \infty) \to \mathbb{R}$ satisfying $\varphi(t) = t\chi(t)$, where $\chi : [0, \infty) \to \mathbb{R}$ is a continuous, strictly monotone increasing function such that

$$\chi(0) = 0, \quad \chi(t) \to \infty \text{ as } t \to \infty,$$

and

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq \varphi(||v_1 - v_2||_V) \quad \forall v_1, v_2 \in V. \tag{2}$$

(i3) j is proper, convex and lower semicontinuous.

For general existence results we refer to Jeggle [15], Zeidler [18] and references therein to the numerious works of Browder, Leray & Lions, Brézis etc. If A is stongly monotone the solution of (1) is unique.
The discrete approximate problem of (1) reads: find $u_h \in K_h$ such that

$$\langle Au_h, v_h - u_h\rangle + j_h(v_h) - j_h(u_h) \geq \langle f, v_h - u_h\rangle \quad \forall v_h \in K_h, \tag{3}$$

where K_h is a nonempty closed convex subset of V_h and V_h is a finite-dimensional subspace of V.
Observe that $\{K_h\}$ approximates the set K in some sense (as we see later in our application), but in general K_h is not a subset of K. Assume also that $K \cap \{\cap_h K_h\} \neq \emptyset$. Further, assume that j_h is proper, convex and lower semicontinuous, and j_h approximates j as explained later. For details we refer to [10, 11, 13]. Moreover, we require that the family

(i4) $\{j_h\}$ is uniformly bounded from below with respect to h in the sense that there exists a positive constant γ independent of h such that

$$j_h(v_h) \geq -\gamma ||v_h||_V \quad \forall v_h \in V_h.$$

Notice also that in this paper we shall apply the general approach of the Fenchel conjugate [14], which as it was mentioned there allows us to study the sum of different monotone operators. For this purpose, we strengthen the definition of the strong monotonicity of A, requiring φ to be, in addition, a convex function. In our application $\varphi(t)$ is defined by $\varphi(t) = ct^p$ with $\chi(t) = ct^{p-1}$, $p > 2$. In what follows c, c_1, $c_2, \ldots$ are generic positive constants.
Further, we extend the domain of φ by setting

$$\varphi(t) = +\infty \quad \text{if } t < 0.$$

Now the Fenchel conjugate $\varphi^* : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ of φ is defined by

$$\varphi^*(s) = \sup_{t \in \mathbb{R}} \{st - \varphi(t)\} = \sup_{t \geq 0} \{st - \varphi(t)\}.$$

Notice that the function φ^* is convex and $\varphi^*(s) = 0$ for $s \leq 0$.
Also, we consider a Hilbert space G with the norm $|| \cdot ||_G$ such that

$$V \hookrightarrow G.$$

Then we can estimate the error $||u - u_h||_V$, where u and u_h are the solutions of problems (1) and (3), respectively. Our result reads as follows.

Theorem 2.1. *Assume that*

$$Au - f \in G^*.$$

Under assumptions (i1) − (i4), *there exist constants* c_1, c_2 *independent of* h *such that*

$$\begin{aligned}\varphi(||u - u_h||_V) \leq c_1 \Big\{ &\inf_{v \in K} \big(||Au - f||_{G^*} ||v - u_h||_G + |j_h(u_h) - j(v)|\big) \\ &+ \inf_{v_h \in K_h} \big(\varphi^*(c_2 ||u - v_h||_V) + ||Au - f||_{G^*} ||u - v_h||_G \\ &+ |j(u) - j_h(v_h)|\big)\Big\}. \end{aligned} \tag{4}$$

Proof. Let $u_0 \in K \cap \{\cap_h K_h\} \neq \emptyset$. Without loss of generality we can assume that $j(u_0) < \infty$. Otherwise, since j is a proper functional, there exists $v^* \in V$ such that $-\infty < j(v^*) < \infty$ and we can define $\tilde{j} : V \to \mathbb{R} \cup \{\infty\}$ by $\tilde{j}(v) = j(v^* - u_0 + v)$. Obviously, $\tilde{j}(u_0) < \infty$.
Moreover, since j is convex, lower semicontinuous and proper, there exist $\lambda \in V^*$ and $\mu \in \mathbb{R}$ such that

$$j(v) \geq \lambda(v) + \mu \quad \forall v \in V$$

(see [9]).
Then, by taking $v = u_0$ in (1) and $v_1 = u$, $v_2 = u_0$ in (2), we have

$$\begin{aligned} ||u - u_0||_V \chi(||u - u_0||_V) &\leq \langle Au - Au_0, u - u_0 \rangle \\ &\leq j(u_0) - j(u) + \langle f, u - u_0 \rangle - \langle Au_0, u - u_0 \rangle \\ &\leq |j(u_0)| + ||\lambda||^* ||u||_V + |\mu| \\ &\qquad + ||f||^* ||u - u_0||_V + ||Au_0||^* ||u - u_0||_V. \end{aligned} \tag{5}$$

If $||u - u_0|| \geq 1$ then

$$\chi(||u - u_0||_V) \leq ||f||^* + ||Au_0||^* + ||\lambda||^* + |\mu| + |j(u_0)| + ||\lambda||^* ||u_0||_V.$$

The last inequality follows from (5) by using the triangle inequality $||u||_V \leq ||u - u_0||_V + ||u_0||_V$ and division by $||u - u_0||_V$.
Therefore, since χ is strictly increasing with $\chi(0) = 0$ and $\lim_{t\to\infty} \chi(t) = \infty$, we have

$$\begin{aligned} ||u|| &\leq ||u_0||_V + \max\{\chi^{-1}\left(||f||^* + ||Au_0||^* + ||\lambda||^*(1 + ||u_0||_V) + |\mu| + |j(u_0)|\right), 1\} \\ &=: r_1. \end{aligned}$$

A similar inequality can be obtained for u_h. Namely,

$$||u_h|| \leq ||u_0||_V + \max\{\chi^{-1}\left(||f||^* + ||Au_0||^* + \gamma(1 + ||u_0||_V) + |j(u_0)|\right), 1\} =: r_2.$$

So, both solutions u and u_h have the same a priori bound $r = \min\{r_1, r_2\}$.
Further, using inequalities (1) and (3) and grouping terms, we get

$$\begin{aligned} \langle Au - Au_h, u - u_h \rangle &\leq \langle Au - Au_h, u - u_h \rangle \\ &\qquad + \langle Au, v - u \rangle + j(v) - j(u) - \langle f, v - u \rangle \\ &\qquad + \langle Au_h, v_h - u_h \rangle + j_h(v_h) - j_h(u_h) - \langle f, v_h - u_h \rangle \\ &= \langle Au - f, v - u_h \rangle + \langle Au - f, v_h - u \rangle \\ &\qquad + \langle Au_h - Au, v_h - u \rangle + j(v) - j_h(u_h) + j_h(v_h) - j(u). \end{aligned}$$

Since by assumption $Au - f \in G^*$, we have, using the strong monotonicity and the Lipschitz continuity of A on bounded sets, that

$$\begin{aligned} \varphi(||u - u_h||_V) &\leq ||Au - f||_{G^*} ||v - u_h||_G + ||Au - f||_{G^*} ||v_h - u||_G \\ &\qquad + C(r) ||u_h - u||_V \, ||v_h - u||_V \\ &\qquad + |j(v) - j_h(u_h)| + |j_h(v_h) - j(u)|. \end{aligned} \tag{6}$$

Applying the Fenchel inequality

$$ts \leq \varphi_\tau(t) + \varphi_\tau^*(s) \tag{7}$$

where φ_τ and φ_τ^* are defined, respectively, by

$$\varphi_\tau(t) = \tau\varphi(t), \quad \varphi_\tau^*(s) = \tau\varphi^*\left(\frac{s}{\tau}\right) \quad \text{with } \tau = \frac{1}{2C(r)},$$

we get the estimate

$$C(r)||u_h - u||_V||v_h - u||_V \leq \frac{1}{2}\varphi(||u_h - u||_V) + \frac{1}{2}\varphi^*\left(\frac{1}{\tau}||v_h - u||_V\right).$$

This combined with (6) yields

$$\frac{1}{2}\varphi(||u - u_h||_V) \leq ||Au - f||_{G^*}||v - u_h||_G + ||Au - f||_{G^*}||v_h - u||_G$$
$$+ |j(v) - j_h(u_h)| + |j_h(v_h) - j(u)| + \frac{1}{2}\varphi^*\left(\frac{1}{\tau}||v_h - u||_V\right),$$

from which inequality (4) follows immediately. □

3. Application to the finite element discretization of an obstacle problem with friction

Now we apply the abstract error estimate of Lemma 1 to the finite element approximation of the problem with the following data

$$Au := -\nabla_p u + \rho\,|u|^{p-2}u, \quad \rho \geq 0,\, p > 2$$

where $\nabla_p u$ is the $p-$Laplacian defined by

$$\nabla_p u = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u),$$

j is the real-valued friction functional defined by

$$j(v) = \int_{\Gamma_c} g|v|\,ds, \quad g \in L^\infty(\Gamma_c),\, g \geq 0 \text{ a.e. on } \Gamma_c,$$

and f is the linear form

$$\langle f, v\rangle = \int_{\Gamma_F} f\,v\,ds.$$

All data A, j and f are defined on the function space

$$V = \{v \in W^{1,p}(\Omega) : v = 0 \text{ on } \Gamma_u\}, \quad \Omega \subset \mathbb{R}^2$$

and the convex closed set

$$K = \{v \in V : v \geq \chi \text{ on } \Gamma_c\}, \quad \text{with given } \chi \in H^2(\Omega),\ \chi \leq 0 \text{ on } \Gamma_c.$$

Here Ω is a polygonal domain with Lipschitz boundary Γ and $\Gamma = \overline{\Gamma}_u \cup \overline{\Gamma}_c \cup \overline{\Gamma}_F$, where the open parts Γ_u, Γ_c and Γ_F are nonempty and mutually disjoint. Moreover, $|\cdot|$ stands here for the standard Euclidean norm.
Introducing

$$\langle A_1 u, \varphi\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx \quad \forall\, \varphi \in W^{1,p}(\Omega)$$

and

$$\langle A_2 u, \varphi\rangle = \rho \int_\Omega |u|^{p-2} u\, \varphi \, dx \quad \forall\, \varphi \in W^{1,p}(\Omega),$$

the operator A has the form

$$Au = A_1 u + A_2 u.$$

First we show that j is Lipschitz continuous on $W^{1,p}(\Omega)$. Indeed, with γ denoting the trace mapping we have from the trace theorem and since $p > 2$ the following estimate

$$\begin{aligned} |j(v_1) - j(v_2)| &\leq \int_{\Gamma_c} g\big||v_1| - |v_2|\big| \, ds \leq \int_{\Gamma_c} g|v_1 - v_2| \, ds \\ &\leq ||g||_{L^2(\Gamma_c)} ||\gamma(v_1 - v_2)||_{L^2(\Gamma_c)} \qquad (8) \\ &\overset{p>2}{\leq} c_1 ||g||_{L^2(\Gamma_c)} ||\gamma(v_1 - v_2)||_{W^{1-\frac{1}{p},p}(\Gamma)} \\ &\leq c_2 ||g||_{L^2(\Gamma_c)} ||v_1 - v_2||_{W^{1,p}(\Omega)} \quad \forall v_1, v_2 \in W^{1,p}. \qquad (9) \end{aligned}$$

Now with a triangulation $\mathscr{T}_h$ of the set $\overline{\Omega}$ we associate piecewise linear finite-element approximations of V and K, respectively, i.e.

$$V_h = \{v_h \in C(\overline{\Omega}) : v_{h|T} \in P_1, \forall T \in \mathscr{T}_h, v_h = 0 \text{ on } \Gamma_u\}$$

and

$$K_h = \{v_h \in V_h : v_h(b) \geq \chi(b) \; \forall b \in \mathscr{N}_h\}.$$

Here $\mathscr{N}_h$ denotes the set of all nodes of the triangulation lying on Γ_c and P_1 is the space of all polynomials in two variables of degree less than or equal to one. Note that the set K_h is not in general contained in the set K.
The discretization of the friction functional j using different quadrature rules has been already investigated in [10, 11] and then extended in [13] for the polynomials of degree less than or equal to three. In this work, for simplicity, we assume that g is piecewise constant and we approximate j, using Kepler's trapezoidal rule, by

$$j_h(v_h) = \frac{1}{2} \sum_i g_i |P_i P_{i+1}| \Big(|v_h(P_i)| + |v_h(P_{i+1})| \Big),$$

where

$$g\big|_{(P_i, P_{i+1})} = g_i = \text{const} > 0.$$

Here we are summing over all sides (P_i, P_{i+1}) of triangles of $\mathscr{T}_h$ whose union gives Γ_c. Moreover, let $q_h : C(\Gamma) \to L^\infty(\Gamma)$ be a piecewise constant function defined by

$$q_h(\mu) = \sum_{P_i \in \Gamma \cap \Sigma_h} \mu(P_i)\, \chi_i, \quad \forall \mu \in C(\Gamma),$$

where χ_i is the characteristic function of $\widehat{P_{i-\frac{1}{2}} P_i P_{i+\frac{1}{2}}}$. Here $P_{i-\frac{1}{2}}$ and $P_{i+\frac{1}{2}}$ are the midpoints of the edges (P_{i-1}, P_i) and (P_i, P_{i+1}), respectively, and Σ_h denotes the set of all nodes of triangulation.
Then, according to [11], p. 258, the following estimate holds

$$||q_h(\gamma v_h)||_{L^2(\Gamma)} \leq 2||\gamma v_h||_{L^2(\Gamma)}. \tag{10}$$

Moreover, it can be easily seen, see [11] again, that

$$j_h(v_h) = \int_{\Gamma_C} g|q_h(\gamma v_h)|\, ds.$$

Hence, for any $u_h, v_h \in V_h$, we can estimate

$$\begin{aligned} |j_h(v_h)| \;&\leq\; \int_{\Gamma_C} g|q_h(\gamma v_h)|\, ds \leq ||g||_{L^2(\Gamma_c)} ||q_h(\gamma v_h)||_{L^2(\Gamma_c)} \\ &\overset{(10)}{\leq}\; 2||g||_{L^2(\Gamma_c)} ||\gamma v_h||_{L^2(\Gamma)} \overset{p>2}{\leq} 2c_2 ||g||_{L^2(\Gamma_c)} ||v_h||_{W^{1,p}(\Omega)}, \end{aligned}$$

which shows that the family $\{j_h\}$ is uniformly bounded with respect to h.
Later on we shall use the following inequality (see [11], Lemma 1.2) as well

$$|j_h(v_h) - j(v_h)| \leq c_3 ||g||_{L^2(\Gamma)} h^s ||\gamma v_h||_{H^s(\Gamma)}, \quad s \in [0, 1]. \tag{11}$$

Now we assume that the solution u of the problem (1) belongs to the space $W^{2,p}(\Omega)$ for some $2 < p$. First, we show that $Au \in L^2(\Omega)$. According to Lemma 2 in [14], $A_1 u \in L^2(\Omega)$ and

$$||A_1 u|| \leq c(u).$$

Now we prove that $A_2 u \in L^2(\Omega)$. With $q = \frac{p}{p-1}$, we can estimate that

$$\int_\Omega |u|^{2(p-1)}\, dx \leq \left\{ \int_\Omega |u|^{2(p-1)q}\, dx \right\}^{\frac{1}{q}} \left\{ \int_\Omega 1^p\, dx \right\}^{\frac{1}{p}} = ||u||^{2(p-1)}_{L^{2p}} \operatorname{mes}(\Omega)^{\frac{1}{p}}.$$

Since $u \in W^{1,p} \subset L^{2p}$ it follows that $A_2 u \in L^2(\Omega)$ and consequently $Au = A_1 u + A_2 u \in L^2(\Omega)$. Moreover,

$$||Au - f||_{L^2(\Omega)} \leq c(u, f).$$

Secondly, we show that A is Lipschitz continuous on bounded sets and strongly monotone. According to the proof of Theorem 2 in [14], A_1 is Lipschitz continuous on bounded sets with respect to the norm $|| \cdot ||_{1,p}$, i.e.

$$|\langle A_1 u_1 - A_1 u_2, \varphi \rangle| \leq \hat{c}_4 ||u_1 - u_2||_{1,p} (||u_1||_{1,p} + ||u_2||_{1,p})^{p-2} ||\varphi||_{1,p}.$$

Analogously, since $p > 2$, using Hölder's inequality, we can estimate

$$\begin{aligned}
&|\langle A_2u_1 - A_2u_2, \varphi\rangle| \\
&= \rho \, |\int_\Omega (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)\varphi \, dx| \\
&\leq \rho\tilde{c}_4 \int_\Omega |u_1 - u_2|(|u_1| + |u_2|)^{p-2}|\varphi| \, dx \\
&\leq \rho\tilde{c}_4 \left(\int_\Omega |u_1 - u_2|^p \, dx\right)^{\frac{1}{p}} \left(\int_\Omega \{||u_1| + |u_2|\}^p\right)^{\frac{p-2}{p}} \left(\int_\Omega |\varphi|^p \, |dx\right)^{\frac{1}{p}} \\
&\leq \rho\tilde{c}_4 ||u_1 - u_2||_{1,p}(||u_1||_{1,p} + ||u_2||_{1,p})^{p-2}||\varphi||_{1,p}.
\end{aligned}$$

Altogether, with $c_4 := \max\{\hat{c}_4, \rho\tilde{c}_4\}$, we obtain

$$|\langle Au_1 - Au_2, \varphi\rangle| \leq c_4||u_1 - u_2||_{1,p}(||u_1||_{1,p} + ||u_2||_{1,p})^{p-2}||\varphi||_{1,p}$$

which implies the Lipschitz continuity of A on bounded subsets.
On the other hand, (see again the proof of Theorem 2 in [14])

$$\langle A_1u_1 - A_1u_2, u_1 - u_2\rangle \geq \hat{c}_5||\nabla(u_1 - u_2)||^p_{L^p}$$

and analogously

$$\begin{aligned}
\langle A_2u_1 - A_2u_2, u_1 - u_2\rangle &= \rho \int_\Omega (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2) \, dx \\
&\geq \rho\tilde{c}_5 \int_\Omega |u_1 - u_2|^p \, dx = \rho\tilde{c}_5||u_1 - u_2||^p_{L^p}.
\end{aligned}$$

Hence,

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \geq c_5||u_1 - u_2||^p_{1,p},$$

with $c_5 := \min\{\hat{c}_5, \rho\tilde{c}_5\}$, i.e., A is strongly monotone with $\varphi(t) = c_5t^p$ and $\chi(t) = c_5t^{p-1}$.

Now we are ready to derive an asymptotic error bound for the disscretization error in the norm $|| \cdot ||_{1,p}$.

Theorem 3.1. *Let $2 < p$, $\rho \geq 0$, $u \in W^{2,p}(\Omega)$, $f \in L^2(\Omega)$ and $\chi \in H^2(\Omega)$. Then, there exists a positive constant $c = c(u, f, \chi)$ independent of h such that*

$$||u - u_h||_{1,p} \leq ch^{\frac{1}{2p}}.$$

Proof. Applying Theorem 2.1 with

$$\varphi(t) = c_5t^p \ (t \geq 0), \qquad \varphi^*(s) = c_5^*s^q \ (s \geq 0), \quad q = \frac{p}{p-1} \quad \text{and} \quad G = L^2(\Omega)$$

we have

$$\begin{aligned}
&c_5||u - u_h||^p_{1,p} \\
&\leq C_1\Big\{ \inf_{v_h \in K_h} \left(||Au - f||_{L^2(\Omega)}||v_h - u||_{L^2(\Omega)} + \tilde{c}||u - v_h||^{\frac{p}{p-1}}_{1,p} + |j_h(v_h) - j(u)|\right) \\
&\qquad + \inf_{v \in K} \left(||Au - f||_{L^2(\Omega)}||v - u_h||_{L^2(\Omega)} + |j(v) - j_h(u_h)|\right)\Big\}, \qquad (12)
\end{aligned}$$

for some positive constants C_1 and $\tilde{c}$.
For the latter inf in (12) we use $u_h^* = \max\{u_h, \chi\} \in K$ and moreover, the following estimation holds true (see the proof of Theorem 23.1 in [7])

$$||u_h - u_h^*||_{L^2(\Omega)} \leq ||\chi - \pi_h\chi||_{L^2(\Omega)} \leq C||\chi||_{H^2(\Omega)}h^2.$$

Here $\pi_h\chi$ is the linear interpolate of χ with respect to $\mathscr{T}_h$.
By the triangle inequality

$$|j(u_h^*) - j_h(u_h)| \leq |j(u_h^*) - j(u_h)| + |j(u_h) - j_h(u_h)|,$$

and then by (8)

$$|j(u_h^*) - j(u_h)| \leq ||g||_{L^2(\Gamma_c)}||\gamma u_h^* - \gamma u_h||_{L^2(\Gamma_c)}.$$

Denote $\Lambda_h = \{x \in \Gamma_c : u_h < \chi\}$. Since the linear interpolate $\pi_h\chi$ of χ with respect to $\mathscr{T}_h$ satisfies

$$u_h(b) \geq \chi(b) = \pi_h\chi(b) \quad \forall b \in \mathscr{N}_h,$$

it follows that

$$u_h(x) \geq \pi_h\chi(x) \quad \forall x \in \Gamma_c.$$

Further, since on Λ_h

$$0 < (\chi - u_h)(x) \leq (\chi - \pi_h\chi)(x) = |\chi - \pi_h\chi|.$$

we have

$$\begin{aligned} ||\gamma u_h^* - \gamma u_h||^2_{L^2(\Gamma_c)} &= \int_{\Gamma_c} |u_h^* - u_h|^2 dx = \int_{\Lambda_h} |\chi - u_h|^2 dx = \int_{\Lambda_h} (\chi - u_h)^2 dx \\ &\leq \int_{\Lambda_h} |\chi - \pi_h\chi|^2 dx \leq ||\gamma\chi - \gamma\pi_h\chi||^2_{L^2(\Gamma_c)}. \end{aligned}$$

Since $\gamma\pi_h\chi = \pi_h\gamma\chi$ and $\gamma\chi \in H^{\frac{3}{2}}(\Gamma) \subset C^0(\Gamma)$ we can use (see [6], Theorem 3.1.6) the following estimate

$$||\gamma\chi - \pi_h\gamma\chi||_{L^2(\Gamma)} \leq c_6 h^{\frac{3}{2}}||\gamma\chi||_{H^{\frac{3}{2}}(\Gamma)} \leq c_7 h^{\frac{3}{2}}||\chi||_{H^2(\Omega)}. \qquad (13)$$

Furthermore, by (11)

$$|j(u_h) - j_h(u_h)| \leq c_8 h^{\frac{1}{2}}||\gamma u_h||_{H^{\frac{1}{2}}(\Gamma)} \leq c_9 h^{\frac{1}{2}}||u_h||_{H^1(\Omega)}. \qquad (14)$$

For the former inf in (12) we consider the linear interpolate of u with respect to $\mathscr{T}_h$ i.e. $v_h = \pi_h u$. Since by imbedding $u \in W^{2,p}(\Omega) \subset H^2(\Omega) \subset C^0(\overline{\Omega})$ with $p > 2$, we can use (see [6], Theorem 3.1.6)

$$||u - \pi_h u||_{1,p} \leq c_{10} h||u||_{2,p}, \qquad ||u - \pi_h u||_{0,2} \leq c_{11} h^2||u||_{2,2}.$$

By the triangle inequality

$$|j_h(\pi_h u) - j(u)| \leq |j_h(\pi_h u) - j(\pi_h u)| + |j(\pi_h u) - j(u)|.$$

Using (8) and then (13) applied to $\gamma u \in H^{3/2}(\Gamma) \subset C^0(\Gamma)$ we can estimate

$$|j(\pi_h u) - j(u)| \leq c_{12} h^{3/2} ||\gamma u||_{H^{3/2}(\Gamma)} \leq c_{13} h^{3/2} ||u||_{H^2(\Omega)}.$$

Then, since $H^1(\Gamma) \subset C^0(\Gamma)$ we can apply ([6], Theorem 3.1.6)

$$||\gamma u - \pi_h \gamma u||_{H^1(\Gamma)} \leq c ||\gamma u||_{H^1(\Gamma)} \tag{15}$$

which implies from (11) that

$$|j_h(\pi_h u) - j(\pi_h u)| \leq c_{14} h ||\gamma \pi_h u||_{H^1(\Gamma)} \overset{(15)}{\leq} h c_{15} ||\gamma u||_{H^1(\Gamma)}.$$

Summing all up we arrive at

$$||u - u_h||_{1,p}^p \leq c(u, f, \chi)(h^2 + h^{\frac{p}{p-1}} + h + h^{3/2} + h^2 + h^{3/2} + h^{1/2}) = \mathscr{O}(h^{1/2}),$$

since $\frac{p}{p-1} > \frac{1}{2}$.
Consequently

$$||u - u_h||_{1,p} \leq c h^{\frac{1}{2p}}$$

and the lemma is proved. □

Note that because of nonconforming approximation of the convex set K ($K_h \not\subset K$) the consistency error (14) (namely $h^{\frac{1}{2}}$) makes the order of convergence suboptimal.

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