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A Differential Characterisation of the Minimax Inequality

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We prove the following result: let $K \subseteq \mathbb{R}^N$ be convex with nonempty interior, X a topological space and $f : K \times X \to \mathbb{R}$ be concave and u.s.c. in the first variable and coercive and l.s.c. in the second. Then the (perturbed) strict minimax inequality

$$\sup_{\lambda\in K}\inf_{x\in X} f(\lambda, x) + g(\lambda) < \inf_{x\in X}\sup_{\lambda\in K} f(\lambda, x) + g(\lambda),$$

for some continuous concave $g : K \to \mathbb{R}$, is equivalent to the following condition on superdifferentials: if $F(\lambda) = \inf_X f(\lambda, x)$, for some $\lambda \in \mathring{K}$

$$\partial F(\lambda) \setminus \bigcup_{\substack{x\in X \\ f(\lambda,x)=F(\lambda)}} \partial f(\lambda, x) \neq \emptyset.$$

As an application of this differential characterisation we prove a generalised version of a theorem of Ricceri, a criterion of regularity for marginal functions, and the fact that to check whether some perturbed minimax inequality holds, one can test with affine perturbation only.

Keywords: Minimax inequality, concave functions, marginal functions, multiple solutions to variational problems, nonlinear eigenvalues

1. Introduction

The study of some sort of strict minimax inequalities, inequalities of the form

$$\sup_Y \inf_X f(x, y) < \inf_X \sup_Y f(x, y),$$

has a long history and has proved to be useful in various fields. We recall in particular three theories initiated by Ricceri's works [6], [7] and [8], in which some kind of minimax inequalities are applied to the study of multiplicity of solutions of nonlinear B.V.P. (see [9] for a comprehensive reference on the subject). The first one is given by the following variational principle.

Theorem 1.1 ([6], [1]). *Let X be a reflexive real Banach space, $J : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous functional and $\Phi : X \to \mathbb{R}$ a strongly continuous functional satisfying $\lim_{\|x\|\to\infty} \Phi(x) = +\infty$. Assume also that, for each $\lambda > 0$, $\Phi + \lambda J$ is continuously Gâteaux differentiable, bounded from below and satisfies the Palais Smale condition. For each $r > \inf_X \Phi$, put*

$$\varphi_1(r) = \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{J(x) - \inf_{\Phi^{-1}(]-\infty,r])} J}{r - \Phi(x)},$$

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$$\varphi_2(r) = \sup_{y\in\Phi^{-1}([r,+\infty[)} \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{J(x)-J(y)}{\Phi(y)-\Phi(x)}.$$

Further, assume that there exists $r > \inf_X \Phi$ such that

$$\varphi_1(r) < \varphi_2(r). \tag{1}$$

Then, for each $\lambda \in]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$, $\Phi + \lambda J$ has at least three critical points.

Here, except for technical functional analytic hypotheses, the main assumption is given by the strict inequality (1).

The second tool is given by the following theorem.

Theorem 1.2 ([7]). *Let X be a separable and reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval and $f : I \times X \to \mathbb{R}$ a function satisfying the following conditions:*

1. *For each $x \in X$, $f(\cdot, x)$ is continuous and concave.*
2. *For each $\lambda \in I$ and $t \in \mathbb{R}$, $\{x \in X : f(\lambda, x) \le t\}$ is sequentially weakly compact.*
3. *There exists a continuous concave function $g : I \to \mathbb{R}$ such that*

$$\sup_{\lambda\in I} \inf_{x\in X} (f(\lambda, x) + g(\lambda)) < \inf_{x\in X} \sup_{\lambda\in I} (f(\lambda, x) + g(\lambda)). \tag{2}$$

Then there exists an open interval $I' \subseteq I$ such that for each $\lambda \in I'$, $f(\lambda, \cdot)$ has a local, non global minimum, w.r.t. the weak topology.

Here a strict minimax inequality, (2), explicitly appears as the main requirement which allows, under additional but very general hypotheses, to obtain multiple solutions to $f_x(\lambda, x) = 0$.

Finally, in the recent work [8], Ricceri suggested another way of looking for multiple critical points, namely by looking at multiple global minima for parametrized families of functionals. As a starting step of this program, the following theorem, which we reformulate with a perturbation g, has been obtained.

Theorem 1.3 ([8]). *Let X be a topological space, $I \subseteq \mathbb{R}$ an interval and $f : I \times X \to \mathbb{R}$ satisfying the following conditions:*

1. *For each $\lambda \in I$ and $t \in \mathbb{R}$, $\{x \in X : f(\lambda, x) \le t\}$ is closed and compact.*
2. *There exists a function $g : I \to \mathbb{R}$ such that*

$$\sup_{\lambda\in I} \inf_{x\in X} (f(\lambda, x) + g(\lambda)) < \inf_{x\in X} \sup_{\lambda\in I} (f(\lambda, x) + g(\lambda)),$$

   and that, for each $x \in X$, $f(\cdot, x) + g(\cdot)$ is continuous and quasiconcave (i.e., $\{\lambda \in I : f(\lambda, x) + g(\lambda) \ge t\}$ is an interval for any t).

Then there exists $\lambda_0 \in I$ such that $f(\lambda_0, \cdot)$ has at least two global minima.

As before, with suitable additional hypotheses, the latter theorem provides another source of three critical points results. Notice that the third hypothesis here is a wide generalisation of the quasiconcavity in Theorem 1.2. However, we will mainly (except for the last section) be concerned with the "concave" setting of Theorem 1.2

Let us now briefly discuss the relations between Theorems 1.1, 1.2 and 1.3. Regarding Theorem 1.1 in the form given, assumption (1) is actually equivalent, by a recent work of Faraci and the author, [5], to

$$\sup_{\lambda\in I}\inf_{x\in X}\left(\Phi(x)+\lambda(J(x)-r)\right)<\inf_{x\in X}\sup_{\lambda\in I}\left(\Phi(x)+\lambda(J(x)-r)\right),$$

for some r, thus to a strict minimax inequality for a suitable linear perturbation. Regarding Theorems 1.2 and 1.3, the main point is that, while the function f may not satisfy a strict minimax inequality, one can still hope that a suitable perturbation of it (depending only on λ) will do. Therefore these theorems require almost the same kind of inequality: a linearly perturbed strict minimax inequality for Theorem 1.1, a concave-perturbed strict minimax inequality for Theorem 1.2, and, roughly speaking, a quasi-concave perturbed strict minimax inequality for Theorem 1.3. Clearly, in the last two theorems, the easiest way to check these inequalities for *some* perturbation is to restrict the analysis to the linear ones.

A natural question arose by Ricceri is thus the following:

Assume X is a topological space, I an interval and $f : I\times X\to\mathbb{R}$ a function such that

1. *For each $x\in X$, $f(\cdot,x)$ is continuous and concave.*
2. *For each $\lambda\in I$ and $t\in\mathbb{R}$, $\{x\in X : f(\lambda,x)\le t\}$ is closed and compact.*

Suppose furthermore that there exists a continuous concave g such that the strict minimax inequality (2) *holds. Is it true that there exists a linear function $\ell(\lambda)$ such that the same inequality holds with ℓ instead of g?*

This question has been answered affirmatively by Cordaro in [3], [4], assuming f was itself linear in the λ variable, i.e. $f(\lambda,x)=\Phi(x)+\lambda J(x)$ for some Φ, $J : X\to\mathbb{R}$; however, the general case remained unsolved. Moreover, one may wonder whether a general statement of this kind holds true for functions depending (in a concave way) on a vectorial parameter $\lambda\in K$, the latter being a general convex subsets of $\mathbb{R}^N$, rather than just an interval.

In this paper we will give a differential characterisation of (2), under the structural condition *1.* and *2.* of Theorem 1.2, with f vectorially parametrized. Therefore, the general setting we will work in is given by a function $f : K\times X\to\mathbb{R}$, where K is a convex subset of $\mathbb{R}^N$ with nonempty interior, X is a topological space, and f is concave and u.s.c. in the first variable and coercive in the second, i.e.

1. For any $\lambda\in K$ and $t\in\mathbb{R}$, $\{x\in X : f(\lambda,x)\le t\}$ is closed and compact.
2. For any $x\in X$, $f(\cdot,x) : K\to\mathbb{R}$ is concave and u.s.c..

Given such an f, we define the concave function

$$F(\lambda):=\inf_{x\in X} f(\lambda,x),$$

often called in the literature *marginal function*.

Denoting with ∂ the superdifferential operator in the λ variable and with (v, w) the standard scalar product in $\mathbb{R}^N$, we will prove the following result.

Theorem 1.4. *Let X be a topological space, K a convex subset of $\mathbb{R}^N$ with nonempty interior and $f : K \times X \to \mathbb{R}$ be such that*

1. *For each $x \in X$, $f(\cdot, x)$ is concave and u.s.c.;*
2. *for each $\lambda \in I$ and $t \in \mathbb{R}$, $\{x \in X : f(\lambda, x) \le t\}$ is closed and compact.*

The strict minimax inequality (2) *holds for some u.s.c., concave g if and only if*

$$S = \bigcup_{\lambda \in \mathring{K}} \left(\partial F(\lambda) \setminus \bigcup_{\substack{x \in X \\ f(\lambda,x)=F(\lambda)}} \partial f(\lambda, x) \right) \neq \emptyset.$$

Moreover if $\alpha \in -S$, then (2) *holds with $g(\lambda) = (\alpha, \lambda)$.*

Implicit in the statement of the theorem is the affirmative answer (in the general, vectorial case) to the question outlined above.

As a further application of this differential characterisation, we give an elementary proof of Theorem 1.3 which on one hand avoids the theory of multifunctions, and on the other holds true in the more general case of convex $K \subseteq \mathbb{R}^N$, instead of just intervals. We remark, however, that the general statement of 1.3 involves functions which are quasiconcave in the real variable, and in this case the whole approach presented here fails, as Example 4.6 at the end of the paper shows.

The plan of the paper is the following. In Section 2 we provide some definitions and recall some elementary results about concave functions. In Section 3 we prove Theorem 1.4, and in the last section we describe some consequence of the latter: a new proof of a generalised version of Theorem 1.3, a regularity theorem for marginal functions and a brief discussion of the quasiconcave case.

2. Preliminary material

In this section we introduce some notations and preliminary propositions.

K will denote an arbitrary convex subset of $\mathbb{R}^N$, with $\mathring{K} \neq \emptyset$. We will denote its boundary with $b(K)$, to avoid unnecessary confusion with the superdifferential operator ∂. Given a concave function $h : K \to \mathbb{R}$, we denote by $\widetilde{h}$ the extension

$$\widetilde{h}(\lambda) = \begin{cases} h(\lambda) & \text{if } \lambda \in K, \\ -\infty & \text{if } \lambda \in \mathbb{R}^N \setminus K, \end{cases}$$

and will say that h is closed if it coincides on K with the closure of $\widetilde{h}$.

Given the topological space X, recall that the function $f : K \times X \to \mathbb{R}$ is such that

1. for any $x \in X$, $f(\cdot, x)$ is u.s.c. and concave function in K;
2. for any $t \in \mathbb{R}$ and $\lambda \in K$, the set $\{x \in X : f(\lambda, x) \le t\}$ is closed and compact.

We will consider the family of functions $\mathcal{F}$ defined as:

$$\mathcal{F} = \{\widetilde{f}(\cdot, x) : x \in X\}, \tag{3}$$

and we will set, for any $\lambda \in \mathbb{R}^N$,

$$F(\lambda) = \inf_{h\in\mathcal{F}} h(\lambda), \qquad D := \mathrm{dom}(F) = \{\lambda \in \mathbb{R}^N : F(\lambda) > -\infty\}.$$

The function F is thus concave and u.s.c., and the coercivity assumption on f implies that $K \subseteq D \subseteq \overline{K}$. In the following we will use, with slight abuse of notation, the same letter F (or $\mathcal{F}$) for the (family of) restrictions on $\overline{K}$ or $\mathring{K}$.

With these notations we rewrite (2) as

$$\sup_{\lambda\in K}(F+g)(\lambda) = \sup_{\lambda\in K}\inf_{h\in\mathcal{F}}(h+g)(\lambda) < \inf_{h\in\mathcal{F}}\sup_{\lambda\in K}(h+g)(\lambda), \tag{4}$$

for some u.s.c. and concave function g on $\overline{K}$. Notice that (4) is equivalent to the same statement with K substituted by $\mathring{K}$: indeed, while F is only u.s.c., being also proper and concave, it is continuous along line segments, and thus $\sup_K(F+g) = \sup_{\mathring{K}}(F+g)$.

Finally, being F locally lipschitz continuous in $\mathring{K}$, we will consider $\mathcal{F}\cup\{F\}$ as a subset of $C(\mathring{K})$, where the latter is the usual Frechet Space of continuous functions on $\mathring{K}$, with the topology given by convergence on compact subsets, and metric

$$d(h,k) = \sum_{n=1}^{+\infty}\frac{1}{2^n}\frac{\sup_{K_n}|h-k|}{1+\sup_{K_n}|h-k|},$$

for some increasing chain of convexes $K_n \Subset \mathring{K}$, exhausting $\mathring{K}$. In what follows, we will denote by $\mathcal{F}^*$ the closure in $C(\mathring{K})$ of $\mathcal{F}$.

We point out that while it is not true in $C(\mathring{K})$ with such a topology that from $h_n \to h$ it follows $\sup_{\mathring{K}} h_n \to \sup_{\mathring{K}} h$, the following holds.

Proposition 2.1. *Let $\mathcal{C} \subset C(\mathring{K})$ be the set of concave functions defined in $\mathring{K}$. Then $\mathcal{C}$ is closed and if K is bounded it holds $h_n \to h \Rightarrow \sup_{\mathring{K}} h_n \to \sup_{\mathring{K}} h$.*

Proof. It suffice to prove the last statement. By $h_n \to h$, we have that for any compact $K' \Subset \mathring{K}$, $h_n \to h$ uniformly on K', thus $\sup_{K'} h_n \to \sup_{K'} h$. Therefore for any $K' \Subset \mathring{K}$

$$\varliminf_n \sup_{\mathring{K}} h_n \geq \lim_n \sup_{K'} h_n = \sup_{K'} h,$$

and taking the supremum over K' in the right hand side we have

$$\varliminf_n \sup_{\mathring{K}} h_n \geq \sup_{\mathring{K}} h,$$

which shows the claim in the case $\sup_{\mathring{K}} h = +\infty$. Suppose now that $\sup_{\mathring{K}} h < +\infty$ but

$$\varlimsup_n \sup_{\mathring{K}} h_n - \sup_{\mathring{K}} h = \delta > 0. \tag{5}$$

We choose $\overline{\lambda} \in \mathring{K}$ such that $\sup_{\mathring{K}} h < h(\overline{\lambda}) + \frac{\delta}{2}$, and a sequence $\lambda_n \in \mathring{K}$ such that $h_n(\lambda_n) \to \varlimsup_n \sup_{\mathring{K}} h_n$. Since K is bounded, we can suppose, after reindexing, that $\lambda_n \to \lambda_0 \in \overline{K}$. By concavity

$$h_n\left(\frac{\overline{\lambda}+\lambda_n}{2}\right) \geq \frac{1}{2}\left(h_n(\lambda_n) + h_n(\overline{\lambda})\right),$$

and since $\frac{\overline{\lambda}+\lambda_n}{2} \to \frac{\overline{\lambda}+\lambda_0}{2} \in \mathring{K}$ and we have uniform convergence in compact subsets of $\mathring{K}$, we can safely take the limit in the latter inequality. Rearranging, we obtain

$$\overline{\lim_n} \sup_{\mathring{K}} h_n \le 2h\left(\frac{\overline{\lambda}+\lambda_0}{2}\right) - h(\overline{\lambda}) \le 2\sup_{\mathring{K}} h - h(\overline{\lambda}) \le \sup_{\mathring{K}} h + \frac{\delta}{2},$$

which contradicts (5) □

Notice that boundedness is essential in the previous proposition. Taking for example $K = (0, +\infty)$ and $h_n(\lambda) = \min\{1, \frac{\lambda}{n}\}$ we see that $h_n \to 0$ in $\mathcal{C}$ but $\sup_K h_n \equiv 1$.

Using this proposition we have that, in the case K bounded, (4) is unaltered when $\mathcal{F}$ is replaced by $\mathcal{F}^*$. Indeed the left hand side is unaltered since for any $\lambda \in \mathring{K}$

$$\inf_{h\in\mathcal{F}} (h+g)(\lambda) = \inf_{h\in\mathcal{F}^*} (h+g)(\lambda) = (F+g)(\lambda).$$

Regarding the right hand side, it obviously holds

$$\inf_{h\in\mathcal{F}} \sup_{\mathring{K}} (h+g) \ge \inf_{h\in\mathcal{F}^*} \sup_{\mathring{K}} (h+g)$$

and for any $\varepsilon > 0$, we can pick an element $h_\varepsilon \in \mathcal{F}^*$ such that

$$\sup_{\mathring{K}} (h_\varepsilon + g) < \varepsilon + \inf_{h\in\mathcal{F}^*} \sup_{\mathring{K}} (h+g).$$

Choosing $h_n \to h_\varepsilon$, $h_n \in \mathcal{F}$ and using the previous proposition we have

$$\inf_{h\in\mathcal{F}} \sup_{\mathring{K}} (h+g) \le \lim_n \sup_{\mathring{K}} (h_n+g) = \sup_{\mathring{K}} (h_\varepsilon+g) \le \varepsilon + \inf_{h\in\mathcal{F}^*} \sup_{\mathring{K}} (h+g),$$

which gives the equality taking $\varepsilon \to 0$.

The next proposition will be useful in order to deal with unbounded K, as well as extrema attained on $b(K)$.

Proposition 2.2. *Let $\mathcal{F}$ be constructed via $f : K \times X \to \mathbb{R}$, coercive in the second variable, u.s.c. and concave in the first one. If*

$$\sup_K F < \inf_{h\in\mathcal{F}} \sup_K h$$

then for every chain $\{K_n\}$ of compact subsets of $\mathring{K}$ exhausting $\mathring{K}$, the strict minimax inequality holds on K_n for infinitely many n.

Proof. It suffice to show that for every chain there is at least one K_n for which the strict minimax inequality holds on it. Since F and every $h \in \mathcal{F}$ are u.s.c. and concave, they are all continuous on line segments, and thus the minimax inequality on K is equivalent to the one on $\mathring{K}$. We choose t such that

$$\sup_{\lambda\in\mathring{K}} \inf_{h\in\mathcal{F}} h < t < \inf_{h\in\mathcal{F}} \sup_{\lambda\in\mathring{K}} h, \tag{6}$$

and consider $C_n = \{x \in X : \sup_{K_n} f(\cdot, x) \leq t\}$. Suppose, by contradiction, that

$$\sup_{\lambda \in K_n} \inf_{h \in \mathcal{F}} h = \inf_{h \in \mathcal{F}} \sup_{\lambda \in K_n} h$$

for every n: then C_n is never empty, and it is closed being the intersection of closed compact sets. Since $\{C_n\}$ obviously has the finite intersection property, and is contained in the closed compact set C_1, there exists $\overline{x} \in \bigcap_{n \geq 1} C_n$ and therefore, $\sup_{\mathring{K}} f(\cdot, \overline{x}) \leq t$, which contradicts (6), since letting $\overline{h} = f(\cdot, \overline{x}) \in \mathcal{F}$, one has

$$\inf_{\mathcal{F}} \sup_{\mathring{K}} h \leq \sup_{\mathring{K}} \overline{h} \leq t.$$

□

We remark that given an arbitrary family $\mathcal{F}$ of u.s.c., concave functions, (i.e. not necessarily a "coercive" one), an argument similar to the one used in the proof of Proposition 2.1 gives the same statement in the case K bounded.

We finally recall some classical results on concave functions defined (and finite) in a convex set $K \subseteq \mathbb{R}^N$ with nonempty interior.

Proposition 2.3.

1. Superdifferential. *The* superdifferential *of a concave function $h : K \to \mathbb{R}$ at a point λ_0 of its domain is the set $\partial h(\lambda_0) := \{\alpha \in \mathbb{R}^N : h(\lambda) \leq h(\lambda_0) + (\alpha, \lambda - \lambda_0)\}$. A concave function is said to be* superdifferentiable *at λ_0 iff $\partial f(\lambda_0) \neq \emptyset$. The superdifferential is a closed convex subset of $\mathbb{R}^N$ and it is nonempty and bounded iff $\lambda_0 \in \mathring{K}$.*

2. Differentiability properties. *A concave function is differentiable almost everywhere in its domain, and λ_0 is a differentiability point iff the superdifferential consists of precisely one vector (the gradient).*

3. Extrema. *A concave function $h : K \to \mathbb{R}$ is continuous in $\mathring{K}$, and it attains a maximum at $\lambda_0 \in K$ iff $0 \in \partial h(\lambda_0)$.*

4. Sum of concave functions. *If $h, g : K \to \mathbb{R}$ are concave functions, then for any $\lambda_0 \in K$,*

$$\partial(h + g)(\lambda_0) = \partial h(\lambda_0) + \partial g(\lambda_0),$$

with the right hand side being the set of the sums $\alpha + \beta$ with $\alpha \in \partial h(\lambda_0)$ and $\beta \in \partial g(\lambda_0)$.

5. Infimum of concave functions. *If $\mathcal{F}$ is a family of concave functions and $F = \inf_{h \in \mathcal{F}} h$, then F is concave, and at any point λ_0 of its domain it holds*

$$\partial F(\lambda_0) \supseteq \bigcup_{\substack{h \in \mathcal{F} \\ h(\lambda_0) = F(\lambda_0)}} \partial h(\lambda_0).$$

Indeed from $h(\lambda) \leq h(\lambda_0) + (\alpha, \lambda - \lambda_0)$ it follows $F(\lambda) \leq h(\lambda_0) + (\alpha, \lambda - \lambda_0)$ and taking the infimum over $h \in \mathcal{F}$ shows the claim.

3. The differential characterisation

We now want to characterise the marginal functions F for which there exists an affine perturbation ℓ such that (4) holds. We will always consider families $\mathcal{F}$ constructed from functions $f : K \times X \to \mathbb{R}$, which satisfy the hypotheses of Theorem 1.4, i.e., u.s.c. and concave in the first variable and coercive in the second. We will sometimes call such a family of concave functions, a *coercive* family, due to Lemma 3.2 below.

Definition 3.1. Let F be the infimum of a family $\mathcal{F}$ of concave functions on K. For any $\lambda_0 \in K$, we set

$$\partial_{\mathcal{F}} F(\lambda_0) = \bigcup_{\substack{h \in \mathcal{F}^* \\ h(\lambda_0) = F(\lambda_0)}} \partial h(\lambda_0).$$

Notice that it holds $\partial F(\lambda_0) \supseteq \partial_{\mathcal{F}} F(\lambda_0)$ for every $\lambda_0 \in K$. Indeed if $\alpha \in \partial h(\lambda_0)$ for some $h \in \mathcal{F}^*$ such that $h(\lambda_0) = F(\lambda_0)$, then

$$F(\lambda_0) + (\alpha, \lambda - \lambda_0) = h(\lambda_0) + (\alpha, \lambda - \lambda_0) \geq h(\lambda) \geq F(\lambda), \quad \forall \lambda \in \mathring{K}$$

since $F(\lambda) = \inf_{h \in \mathcal{F}} h(\lambda) = \inf_{h \in \mathcal{F}^*} h(\lambda)$ for $\lambda \in \mathring{K}$. Using the continuity along line segments we obtain the same inequality on the whole K, and thus $\alpha \in \partial F$ as claimed.

Suppose now that $g : K \to \mathbb{R}$ is any u.s.c. and concave function, and define the family $\mathcal{F} + g$ as $\{h + g : h \in \mathcal{F}\}$. Then, by the additivity of the superdifferential, it is easy to see that

$$\partial_{\mathcal{F}+g}(F + g) = \partial_{\mathcal{F}} F + \partial g. \tag{7}$$

While the use of the family $\mathcal{F}^*$ seems artificial, for coercive families we can equivalently use $\mathcal{F}$ instead.

Lemma 3.2. *Suppose that the family $\mathcal{F}$ is coercive. Then, for any $h \in \mathcal{F}^*$, there exists $\overline{h} \in \mathcal{F}$ such that $\overline{h} \leq h$. Moreover, for any $\lambda_0 \in \mathring{K}$,*

$$\bigcup_{\substack{h \in \mathcal{F}^* \\ h(\lambda_0) = F(\lambda_0)}} \partial h(\lambda_0) = \bigcup_{\substack{h \in \mathcal{F} \\ h(\lambda_0) = F(\lambda_0)}} \partial h(\lambda_0). \tag{8}$$

Proof. Let us prove the first statement. Suppose $h_k \to h$ uniformly on compact subsets, with $h_k \in \mathcal{F}$. Let $\{x_k\}$ be such that $h_k(\cdot) = f(\cdot, x_k)$, and $\{\lambda_m\}$ be a dense subset of K. The family of subsets of X

$$C_n = \left\{ x \in X : \forall m \leq n,\ f(\lambda_m, x) \leq h(\lambda_m) + \frac{1}{n} \right\}$$

is a chain, and each C_n is not empty, since it contains some x_k for sufficiently large k. Therefore it has the finite intersection property and, by coerciveness, every C_n is closed and compact. Therefore we can pick $\overline{x} \in \cap_{n \geq 1} C_n$, and setting $\overline{h}(\lambda) = f(\lambda, \overline{x})$, for every m it holds $f(\lambda_m, \overline{x}) \leq h(\lambda_m)$ and thus it holds for any $\lambda \in K$.

To prove (8), it suffices to prove the inclusion $\subseteq$. Given $h \in \mathcal{F}^*$ such that $h(\lambda_0) = F(\lambda_0)$, consider $\alpha \in \partial h(\lambda_0)$. If $\overline{h} \in \mathcal{F}$ is such that $\overline{h} \leq h$, one has $F(\lambda_0) \leq \overline{h}(\lambda_0) \leq h(\lambda_0) = F(\lambda_0)$ and therefore $\overline{h}(\lambda_0) = F(\lambda_0)$. Moreover, being $(\alpha, \lambda - \lambda_0) + F(\lambda_0)$ a support hyperplane for h, it is so for $\overline{h}$ too, being $\overline{h} \leq h$. Therefore $\alpha \in \partial\overline{h}$, and the inclusion is proved. $\square$

Theorem 3.3. *Suppose there exists $\lambda_0 \in \mathring{K}$ such that*

$$\partial F(\lambda_0) \supsetneq \partial_{\mathcal{F}} F(\lambda_0). \tag{9}$$

Then for any $\alpha \in \partial F(\lambda_0) \setminus \partial_{\mathcal{F}} F(\lambda_0)$, it holds

$$\sup_K \big(F(\lambda) + (\alpha, \lambda)\big) < \inf_{h \in \mathcal{F}} \sup_K \big(h(\lambda) + (\alpha, \lambda)\big).$$

Proof. As stated in the previous section, it suffice to prove the strict minimax inequality for the family $\mathcal{F}^*$, and we will prove it in the equivalent form

$$\sup_K \big(F(\lambda) + (\alpha, \lambda - \lambda_0)\big) < \inf_{h \in \mathcal{F}^*} \sup_K \big(h(\lambda) + (\alpha, \lambda - \lambda_0)\big).$$

Let

$$\alpha \in \partial F(\lambda_0) \setminus \partial_{\mathcal{F}} F(\lambda_0),$$

and consider $\ell(\lambda) = -(\alpha, \lambda - \lambda_0)$. We have $0 \in \partial(F + \ell)(\lambda_0)$ and therefore $F + \ell$ attains its maximum in λ_0. Moreover for any $h \in \mathcal{F}^*$, we have either $h(\lambda_0) > F(\lambda_0)$ or, if $h(\lambda_0) = F(\lambda_0)$, $0 \notin \partial(h + \ell)$. Therefore for any $h \in \mathcal{F}^*$ it holds $\sup_K(h + \ell) > F(\lambda_0)$. Consider now a closed convex $K' \Subset \mathring{K}$ with $\lambda_0 \in \mathring{K}'$ and suppose we have a sequence $h_n \in \mathcal{F}^*$ such that

$$F(\lambda_0) + 1 \geq \sup_K (h_n + \ell) \to F(\lambda_0).$$

In the whole K', $h_n + \ell$ is bounded above by $F(\lambda_0) + 1$ and below by $\inf_{K'}(F + \ell)$. Therefore h_n is equicontinuous and equibounded and by Ascoli-Arzelá's theorem we can suppose, by renaming an appropriate subsequence, that $h_n \to \overline{h} \in \mathcal{F}^*$ uniformly on K'. But then it holds

$$F(\lambda_0) \leq \overline{h}(\lambda_0) \leq \sup_{K'} (\overline{h} + \ell) \leq \lim_n \sup_{K'} (h_n + \ell) = F(\lambda_0).$$

Therefore $\overline{h} + \ell$ attains its maximum at $\lambda_0 \in \mathring{K}'$, thus giving $0 \in \partial(\overline{h} + \ell)(\lambda_0)$. This is a contradiction, since then we would obtain $\alpha \in \partial\overline{h}(\lambda_0)$. $\square$

The converse is now proved.

Theorem 3.4. *Suppose that for every $\lambda_0 \in \mathring{K}$ it holds*

$$\partial F(\lambda_0) = \partial_{\mathcal{F}} F(\lambda_0) \tag{10}$$

then for every u.s.c. and concave g it holds

$$\sup_K (F + g) = \inf_{h \in \mathcal{F}} \sup_K (h + g).$$

Proof. We argue by contradiction and apply Proposition 2.2 to obtain that the strict minimax inequality holds for some convex compact $K_1 \Subset \mathring{K}$. Notice further that the u.s.c. and concave perturbation g is inessential, since by (7),

$$\partial F(\lambda_0) = \partial_{\mathcal{F}} F(\lambda_0) \qquad \Rightarrow \qquad \partial(F+g)(\lambda_0) = \partial_{\mathcal{F}+g}(F+g)(\lambda_0).$$

Therefore we can, without loss of generality, suppose $g = 0$, and that (10) holds for every $\lambda_0 \in K_1$ (considering the superdifferential as the one defined by concave functions on K, i.e., neglecting the normal cones on boundary points of K_1). Suppose then that (4) holds. Since F is continuous on K_1, it attains its maximum at $\lambda_0 \in K_1$, and this is equivalent to the existence $\alpha \in \partial F(\lambda_0)$ with $-\alpha \in N(\lambda_0)$, the latter being the normal cone to K_1 at λ_0. Then, by (10), and Lemma 3.2, for some $\overline{h} \in \mathcal{F}$ with $\overline{h}(\lambda_0) = F(\lambda_0)$ we have $\alpha \in \partial\overline{h}(\lambda_0)$. But then $\overline{h}$ attains its maximum at λ_0, being α in the normal cone to K_1 at λ_0, and this contradicts the strict minimax inequality since its maximum is $F(\lambda_0)$. □

Theorems 3.3 and 3.4 thus gives our main result.

Theorem 3.5. *The strict minimax inequality* (2) *holds for some u.s.c. and concave g if and only if*

$$S = \bigcup_{\lambda \in \mathring{K}} (\partial F(\lambda) \setminus \partial_{\mathcal{F}} F(\lambda)) \neq \emptyset. \tag{11}$$

Moreover if $\alpha \in -S$, then (2) *holds with $g(\lambda) = (\alpha, \lambda)$.*

One may wonder if $-S$ is actually the set of all the slopes α such that (2) holds with $g(\lambda) = (\alpha, \lambda)$. This is true in dimension one.

Theorem 3.6. *Suppose $K \subset \mathbb{R}$. If* (2) *holds for $g(\lambda) = (\alpha, \lambda)$, then $\alpha \in -S$.*

Proof. Using Proposition 2.2, we can suppose that K is a closed bounded interval. Let $\ell(\lambda) = \alpha\lambda$ and suppose that $F + \ell$ attains its maximum at $\lambda_0 \in \overline{K}$. If $\lambda_0 \in \mathring{K}$ then $0 \in \partial(F+\ell)(\lambda_0) = \partial F(\lambda_0) + \alpha$, and thus $-\alpha \in \partial F(\lambda_0)$. It cannot be $-\alpha \in \partial_{\mathcal{F}} F(\lambda_0)$ for otherwise there would exists $h \in \mathcal{F}$ such that $h(\lambda_0) = F(\lambda_0)$ and $0 \in \partial(h+\ell)(\lambda_0)$: indeed this would imply that $h + \ell$ has the same maximum as $F + \ell$, contradicting (2).

Therefore $\lambda_0 \in b(K)$ and $F + \ell$ is monotone. Suppose, without loss of generality, $K = [\lambda_0, b]$ and $F + \ell$ nonincreasing. If $\partial(F+\ell)(\lambda_0) = [a, +\infty[$, then $a \leq 0$ and $\partial F(\lambda_0) = [a - \alpha, +\infty[$.

Using Proposition 2.3, point *2.* and *5.*, we can pick now a sequence $\{\lambda_n\}$ of differentiability points for F, $\lambda_n \downarrow \lambda_0$, and h_n such that $h_n(\lambda_n) = F(\lambda_n) \to F(\lambda_0)$, $h_n'(\lambda_n) = F'(\lambda_n) \uparrow a - \alpha$. Clearly $\{h_n\}$ is equibounded and thus equilipschitz on compact subintervals of $\mathring{K}$, and we can suppose that $h_n \to h \in \mathcal{F}^*$ uniformly on compact subsets. By Lemma 3.2 we can choose $\overline{h} \in \mathcal{F}$ such that $\overline{h} \leq h$. It then holds

$$F(\lambda) \leq h_n(\lambda) \leq F(\lambda_n) + F'(\lambda_n)(\lambda - \lambda_n)$$

for every $\lambda \in K$ and passing to the limit we infer

$$F(\lambda_0) \leq \overline{h}(\lambda_0) \leq h(\lambda_0) \leq F(\lambda_0)$$

and

$$\overline{h}(\lambda) \leq h(\lambda) \leq h(\lambda_0) + (a-\alpha)(\lambda-\lambda_0) = \overline{h}(\lambda_0) + (a-\alpha)(\lambda-\lambda_0).$$

Therefore $(a-\alpha) \in \partial\overline{h}(\lambda_0)$ and thus $0 \geq a \in \partial(\overline{h}+\ell)$. This shows that $\overline{h}+\ell$ is nonincreasing, and therefore

$$\max_K(\overline{h}+\ell) = (\overline{h}+\ell)(\lambda_0) = (F+\ell)(\lambda_0) = \max_K(F+\ell),$$

contradicting (2). □

In higher dimension this characterisation of the set S fails, as the following example shows.

Example 3.7. Let $K = [0,1] \times [-1,1] \subset \mathbb{R}^2$, and let $\mathcal{F} = \{h_1, h_2\}$, with

$$h_1(x,y) = -x-y, \qquad h_2(x,y) = y-x.$$

If $F = \min\{h_1, h_2\}$ then F attains its maximum at the origin and

$$0 = \max_K F < \min\left\{\max_K h_1, \max_K h_2\right\} = 1$$

thus the strict minimax inequality holds (with the zero affine perturbation). However for any $\lambda_0 \in \mathring{K}$,

$$\partial F(\lambda) = \{-1\} \times [-1,1] \quad \Rightarrow \quad S = \{-1\} \times (-1,1)$$

and $(0,0) \notin -S$.

4. Some consequences of the differential characterisation

We can use the characterisation given to obtain an elementary proof of Theorem 1.3, without using the theory of multifunctions. To this end we will need a lemma stating that $\partial_{\mathcal{F}}F$ is actually big enough. Recall that an *exposed point* of a closed convex set K is a point $e \in b(K)$ such that there exists a supporting hyperplane intersecting K only in e. More explicitly, there exists $\alpha \in \mathbb{R}^N$ such that

$$(\alpha, e-z) < 0, \quad \forall z \in K \setminus \{e\}.$$

The set of all exposed points of K will be denoted by $Ex(K)$. Straszewicz's theorem states that $Ex(K)$ is dense in the set of extreme point, and thus if K is a compact convex set, it coincides with the closed convex envelope of $Ex(K)$.

Lemma 4.1. *For any $\lambda_0 \in \mathring{K}$,*

$$\partial_{\mathcal{F}}F(\lambda_0) \supseteq Ex\left(\partial F(\lambda_0)\right).$$

Proof. Consider first the case $\partial F(\lambda_0) = \{\nabla F(\lambda_0)\}$, i.e. λ_0 is a differentiability point for F. Let

$$C_n := \left\{ x \in X : f(\lambda_0, x) \le F(\lambda_0) + \frac{1}{n} \right\},$$

and, by the usual argument, pick $\overline{x} \in \bigcap_n C_n$. Then clearly $\overline{h} := f(\cdot, \overline{x}) \in \mathcal{F}$ satisfies $\overline{h}(\lambda_0) = F(\lambda_0)$. Moreover $\partial\overline{h}(\lambda_0)$ is not empty being $\lambda_0 \in \overset{\circ}{K}$ and thus by Proposition 2.3, point *5.*, $\partial\overline{h}(\lambda_0) = \{\nabla F(\lambda_0)\}$.

Suppose now that λ_0 is arbitrary, and let e be an exposed point of $\partial F(\lambda_0)$. By Theorem 25.6 of [10], there exists a sequence $\lambda_n \to \lambda_0$ of differentiability points for F such that $\nabla F(\lambda_n) \to e$. By what has just been proved, we can pick $h_n \in \mathcal{F}$ such that $h_n(\lambda_n) = F(\lambda_n)$ and $\partial h_n(\lambda_n) = \nabla F(\lambda_n)$. On arbitrary $K' \Subset K$ containing λ_0, h_n is definitely bounded below (by F) and above (by, e.g., $|(e, \lambda - \lambda_0)| + 1$), thus we can suppose $h_n \to h \in \mathcal{F}^*$. Clearly $h(\lambda_0) = F(\lambda_0)$ and passing to the limit in

$$h_n(\lambda) \le F(\lambda_n) + (\nabla F(\lambda_n), \lambda - \lambda_n)$$

we obtain that $e \in \partial h(\lambda_0)$. It suffice now to apply Lemma 3.2 to obtain $\overline{h} \in \mathcal{F}$ such that $F \le \overline{h} \le h$, and then $\overline{h}(\lambda_0) = F(\lambda_0)$ and $e \in \partial\overline{h}(\lambda)$. □

Theorem 4.2. *Let X be a topological space, K a convex subset of $\mathbb{R}^N$ with $\overset{\circ}{K} \ne \emptyset$, and $f : K \times X \to \mathbb{R}$ a function such that*

1. *For any $\lambda \in K$ and $t \in \mathbb{R}$, $\{x \in X : f(\lambda, x) \le t\}$ is closed and compact.*
2. *For any $x \in X$, $f(\cdot, x) : K \to \mathbb{R}$ is u.s.c. and concave.*

If for some u.s.c. and concave $g : K \to \mathbb{R}$ it holds

$$\sup_{\lambda \in K} \inf_{x \in X} f(\lambda, x) + g(\lambda) < \inf_{x \in X} \sup_{\lambda \in K} f(\lambda, x) + g(\lambda),$$

then there exists $\lambda \in \overset{\circ}{K}$ such that $f(\lambda, \cdot) : X \to \mathbb{R}$ has at least two global minima.

Proof. By Theorem 3.4, there is $\lambda_0 \in \overset{\circ}{K}$ such that

$$\partial F(\lambda_0) \supsetneq \partial_{\mathcal{F}} F(\lambda_0). \tag{12}$$

By the previous lemma, for any exposed point $e \in Ex\,(\partial F(\lambda_0))$, we can pick $x_e \in X$ such that

$$f(\lambda_0, x_e) = h_e(\lambda_0) = F(\lambda_0) = \inf_{x \in X} f(\lambda_0, x), \quad e \in \partial f(\lambda_0, x_e).$$

Now it cannot be $x_e \equiv \overline{x}$ for every $e \in Ex\,(\partial F(\lambda_0))$, for otherwise $\overline{h} := f(\cdot, \overline{x}) \in \mathcal{F}$ would satisfy $\partial\overline{h}(\lambda_0) \supset Ex\,(\partial F(\lambda_0))$ which implies $\partial\overline{h}(\lambda_0) = \partial F(\lambda_0)$, contradicting (12). □

Remark 4.3. It is easy to check that this theorem holds true if the first hypothesis on f is replaced by

1. For every $\lambda \in K$ $f(\lambda, \cdot)$ is sequentially l.s.c., and its sublevel sets are sequentially compact,

which is more handy in applications.

Remark 4.4. In the scalar case $K \subseteq \mathbb{R}$, Theorem 3.6 shows that the affine perturbations for which (2) holds are precisely the ones whose slope is in the set

$$-\bigcup_{\lambda \in \mathring{K}} \Big(\partial F(\lambda) \setminus \partial_{\mathcal{F}} F(\lambda)\Big).$$

Obviously such a slope cannot exists if F is differentiable in the whole $\mathring{K}$, since in this case for any $\lambda \in \mathring{K}$, $\partial h(\lambda) = \{F'(\lambda)\}$ for any $h \in \mathcal{F}^*$ such that $h(\lambda) = F(\lambda)$. Therefore (2) can hold for some g only if F is not differentiable at some interior point. Moreover, the set of "multiple global minima" parameters $\lambda \in K$ detected by Theorem 1.3 is at most denumerable, since it is contained in the set of non differentiability of F.

One useful corollary links the uniqueness of global minima with the regularity of the marginal function F. Recall that one says that a function $h : K \to \mathbb{R}$ is C-semiconcave if $h - C|\lambda|^2$ is a concave function. Here is a variation of a well known regularity criterion for marginal functions (see e.g. Theorem 3.4.4 and Proposition 3.3.4 d) in [2]), which one can apply, for example, to smooth, C-semiconcave and coercive families of functions.

Corollary 4.5. *Let X be a topological space, K a convex subset of $\mathbb{R}^N$ with $\mathring{K} \neq \emptyset$, and $f : K \times X \to \mathbb{R}$ a function such that*

1. *For any $\lambda \in K$ and $t \in \mathbb{R}$, $\{x \in X : f(\lambda, x) \leq t\}$ is closed and compact.*
2. *For some continuous differentiable $g : \overline{K} \to \mathbb{R}$, the functions $f(\cdot, x) + g(\cdot)$ are concave and C^1 for each $x \in X$.*

If $f(\lambda, \cdot)$ has a unique global minimum for every $\lambda \in \mathring{K}$, then the marginal function $F(\lambda) = \inf_X f(\lambda, x)$ is $C^1(\mathring{K})$.

Proof. Notice that F is C^1 iff $G(\lambda) := F(\lambda) + g(\lambda)$ is differentiable at every point, since the differential of a concave function is always continuous in the set of differentiability points. Call $\mathcal{G}$ the family of concave functions $\{f(x, \cdot) + g(\cdot) : x \in X\}$; clearly its marginal function is G. Suppose $\lambda_0 \in \mathring{K}$ is such that $\partial G(\lambda_0)$ contains more than one point. By the previous theorem, one must have $\partial_{\mathcal{G}} G(\lambda_0) = \partial G(\lambda_0)$, since otherwise $f(\lambda_0, \cdot)$ would have at least two global minima. We can thus pick α_1, α_2 in $\partial_{\mathcal{G}} G(\lambda_0)$ with $\alpha_1 \neq \alpha_2$, and therefore x_1 and x_2 in X such that, having cancelled the terms $g(\lambda_0)$,

$$f(\lambda_0, x_1) = f(\lambda_0, x_2) = \inf_X f(\lambda_0, x) = F(\lambda_0)$$

and, since $\partial\big(f(\lambda_0, x_i) + g(\lambda_0)\big) = \partial f(\lambda_0, x_i) + g'(\lambda_0)$,

$$\alpha_1 - g'(\lambda_0) \in \partial f(\lambda_0, x_1), \qquad \alpha_2 - g'(\lambda_0) \in \partial f(\lambda_0, x_2).$$

Since $f(\lambda_0, x_i)$ is differentiable, the latter conditions imply that $x_1 \neq x_2$, contradicting the well posedness of $\inf_X f(\lambda_0, x)$. $\square$

Theorem 4.2 also shows that the class of affine perturbation is large enough, in the sense described in the introduction. Indeed if (2) holds for some concave g, then by Theorem 3.4, there exists $\lambda_0 \in \mathring{K}$ such that $\partial F(\lambda_0) \supsetneq \partial_{\mathcal{F}} F(\lambda_0)$. But then Theorem 3.3 applies, giving a positive answer to the question outlined in the introduction.

However, we stress that Theorem 1.3, as proved by Ricceri using the theory of multifunctions, holds (in one dimension) under the much weaker assumption that $f(\cdot, x) + g(\cdot)$ is continuous and quasiconcave for every $x \in X$.

One may then wonder if the analogous of the previous statement holds in this more general setting, or, at least, whether there exists a linear perturbation giving the strict minimax inequality if one already has it for a quasiconcave perturbation. The following example shows that this is not the case.

Example 4.6. Let $K = [-1, 1]$, and $X = \{0, 1\}$. Set, for $\varepsilon \in (0, 1/2)$,

$$f(\lambda, 1) = \begin{cases} \lambda & \text{if } \lambda \leq 0, \\ -\varepsilon\lambda & \text{if } \lambda \geq 0; \end{cases} \qquad f(\lambda, 2) = \begin{cases} -\lambda & \text{if } \lambda \geq 0, \\ -\varepsilon\lambda & \text{if } \lambda \leq 0. \end{cases}$$

One has $F(\lambda) = \min\{f(\lambda, 1), f(\lambda, 2)\} = -|\lambda|$. Moreover

$$\partial F(0) = \partial f(0, 1) \cup \partial f(0, 2), \qquad \partial F(\lambda) = \begin{cases} \partial f(\lambda, 1) & \text{if } \lambda < 0, \\ \partial f(\lambda, 2) & \text{if } \lambda > 0. \end{cases}$$

We then have

1. f satisfies *1.* and *2.* of Theorem 1.2;
2. by Theorem 3.5, for every concave g (and thus, *a fortiori*, for every affine ℓ) it holds
$$\sup_{\lambda \in K} \inf_{x \in X} f(\lambda, x) + g(\lambda) = \inf_{x \in X} \sup_{\lambda \in K} f(\lambda, x) + g(\lambda).$$

Let now
$$g(\lambda) = \begin{cases} 0 & \text{if } \lambda \leq 0, \\ 2\varepsilon\lambda & \text{if } \lambda \geq 0. \end{cases}$$

Then

1. g is quasiconcave;
2. $f(\cdot, i) + g(\cdot)$ is concave and $\sup_{\lambda \in K} f(\lambda, i) + g(\lambda) = \varepsilon > 0$, for $i = 1, 2$;
3. $F + g$ is concave and $\sup_K (F + g) = 0$.

Therefore the strict minimax inequality holds with respect to the perturbation g.

References

[1] D. Averna, G. Bonanno: A three critical points theorem and its applications to the ordinary Dirichlet problem, Topol. Methods Nonlinear Anal. 22 (2003) 93–103.

[2] P. Cannarsa, C. Sinestrari: Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control, Progress in Nonlinear Partial Differential Equations and their Applications 58, Birkhäuser, Boston (2004).

[3] G. Cordaro: On a minimax problem of Ricceri, J. Inequal. Appl. 6 (2001) 261–285.

[4] G. Cordaro: Further results related to a minimax problem of Ricceri, J. Inequal. Appl. (2005) 523–555.

[5] F. Faraci, S. Mosconi: On the relationship between two three critical points theorems, Nonlinear Anal., to appear.

[6] B. Ricceri: A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000) 401–410.

[7] B. Ricceri: On a three critical points theorem, Arch. Math. 75 (2000) 220–226.

[8] B. Ricceri: Multiplicity of global minima for parametrized functions, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. 21 (2010) 47–57.

[9] B. Ricceri: Nonlinear eigenvalue problems, in: Handbook of Nonconvex Analysis and Applications, D. Y. Gao, D. Montreanu (eds.), International Press, Somerville (2010) 543–595.

[10] R. T. Rockafellar: Convex Analysis, Princeton Mathematical Series 28, Princeton University Press, Princeton (1970).