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Piecewise Smooth Lyapunov Function for a Nonlinear Dynamical System*

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In this paper, stability and attraction for a nonlinear dynamical system with nonsmooth Lyapunov functions are studied. The previous results on stability and attraction with a max-type Lyapunov function are extended to the case where Lyapunov function is piecewise smooth. A condition, under which stability and attraction are guaranteed with a piecewise smooth Lyapunov function, is proposed. Taking two certain classes of piecewise smooth functions as Lyapunov functions, related conditions for stability and attraction are developed.

Keywords: Nonlinear dynamical system, stability, region of attraction, Lyapnov functions, nonsmooth analysis, piecewise smooth function

1. Introduction

Recently, nonsmooth Lyapunov functions are used to study stability and attraction in a dynamical system widely. They are applied to nonlinear systems, switched linear systems and hybrid systems. In the theoretical view, rather general nonsmooth Lyapunov functions are used, by which stability and stabilization are established, see [4, 9] and references therein. In the practical view, nonsmooth Lyapunov functions with specific structure are much more interesting, thus specific results can be obtained. Within this topic, most existing nonsmooth Lyapunov functions are max-type or min-type ones, see for instance [5, 6, 7, 10, 11].

Let us consider a dynamical system:

$$\dot{x}(t) = f(x), \tag{1}$$

where $f : \Re^n \to \Re^n$ is locally Lipschitzian with $f(0) = 0$.

Tan and Parkard [10] studied the system (1) with a nonsmnooth Lyapunov function and proved that the set $\Omega = \{x \mid V(x) \leq 1\}$ is invariant and a region of attraction when Lyapunov function V is a max-type function of the form $\max_{i\in I} V_i(x)$ or a min-type function of the form $\min_{i\in I} V_i(x)$, where each V_i is smooth, I is a finite

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index set. In this paper, we try to use a more broad class of nonsmooth functions, named piecewise smooth function, as a Lyapunov function to study the nonlinear system (1). We first extend some results obtained in [10] to the case where Lyapunov functions are piecewise smooth. Then, we discuss two classes of piecewise smooth Lyapunov functions.

2. Preliminaries

We start with a brief overview of some notions for nonsmooth analysis and viability theory.

Definition 2.1 (see [3]). Let $f : \Re^n \to \Re^m$ be locally Lipschitzian and let D_f denote the set where f is differentiable. The subdifferential in the sense of Clarke of f at x, denoted by $\partial f(x)$, is defined as

$$\partial f(x) = \mathrm{co}\left\{ \lim_{x_n \to x} f'(x_n) \mid x_n \to x,\, x_n \in D_f \right\},$$

where "co" denotes the convex hull.

The Clarke subdifferential is a compact convex set in $\Re^{m\times n}$ and is a generalization of the notion of the classical differential. If $f : \Re^n \to \Re^m$ is continuously differentiable, its Clarke subdifferential happens to be a singleton, i.e., $\partial f(x) = \{\nabla f(x)\}$ when $m = 1$ or $\partial f(x) = \{Jf(x)\}$ when $m \neq 1$.

Based on the Clarke subdifferential, there are a mean-value property and a chain rule for a locally Lipschitzian function, which can be found in [1, 3].

Proposition 2.2 (mean-value property). *Let $f : \Re^n \to \Re$ be locally Lipschitzian. Then, for any $x_1, x_2 \in \Re^n$ there exist $\bar{x}$ on the line-segment with x_1 and x_2 as its end points and $\xi \in \partial f(\bar{x})$ such that*

$$f(x_2) - f(x_1) = \xi^T (x_2 - x_1). \tag{2}$$

Proposition 2.3 (chain rule). *Let both $F : \Re^n \to \Re^m$ and $g : \Re^m \to \Re$ be locally Lipschitzian. Then, the composite function $f(x) = g(F(x))$ is locally Lipschzian and its Clarke subdifferential has the following property*

$$\partial f(x) \subset \mathrm{co}\{\gamma^T \xi \mid \gamma \in \partial g(z) \mid_{z=F(x)},\, \xi \in \partial F(x)\}. \tag{3}$$

We next review the concept of piecewise smooth function and some properties, see [2, 8] for the details.

Definition 2.4. A continuous function $f : \Re^n \to \Re$ is said to be piecewise C^k, where k is a positive integer, if there exist Lebesgue measureable sets S_i and open sets O_i in $\Re^n$ with $\bigcup_{i\in I} S_i = \Re^n, \mathrm{cl}\, S_i \subset O_i$ and C^k functions $f_i : O_i \to \Re$ for $i \in I$, where I is an index set with finitely many numbers, such that $f(x) = f_i(x)$ for any $x \in S_i$. When $k = 1$, the function f is said to be piecewise smooth for short.

Usually, f_i and S_i, mentioned in Definition 2.4, are called the i-th piece function and the i-th piece region, respectively.

The piecewise smooth functions play an important role in nonsmooth analysis, optimization and control. Many functions are contained in this family, for instance a maximum of finitely many smooth functions, a smooth compositions of max-type functions and a minmax-type function. It was shown that a piecewise smooth function is locally Lipschitzian.

For a piecewise smooth function f given as in Definition 2.4, define two active index sets of f at x as follows:

$$I_f(x) = \{i \in I \mid f_i(x) = f(x)\}$$

and

$$\bar{I}_f(x) = \left\{ i \in I \mid \exists \delta > 0, \text{s.t.} \operatorname{meas}\left(B(x,\delta) \bigcap O_i\right) > 0,\ f_i(x) = f(x) \right\},$$

where $B(x,\delta)$ denotes the ball with x as its center and δ as its radius, respectively, and "meas" denotes Lebesgue measure. According to [2], the Clarke subdifferential of f at x is formulated as

$$\partial f(x) \subset \operatorname{co}\{\nabla f_i(x) \mid i \in I_f(x)\} \tag{4}$$

and

$$\partial f(x) = \operatorname{co}\{\nabla f_i(x) \mid i \in \bar{I}_f(x)\}. \tag{5}$$

We next present the notion of viability and region of attraction for a dynamical system, which can be found in [1, 3, 10]

Definition 2.5. The set $K \subset \Re^n$ is said to be viable or invariant under the system (1) if for any initial point $x_0 \in K$, the solution $x(t)$ of (1) remains in K for ever, in other words, $x(t) \in K, \forall t > 0$. Moreover, K is said to be a region of attraction if $\lim_{t\to\infty} x(t) = 0$.

3. Stability and Attraction

In this section, we proposed a theorem on stability and attraction for the system (1), which is an extension of the related work in [10]

Theorem 3.1. *Let $V : \Re^n \to \Re$ be piecewise smooth with piece functions $V_i, i \in I$ and piece regions $S_i, i \in I$, where I is a finite index set. If V is positive definite, i.e., $V(x) > 0$ for any $x \neq 0$ and $V(0) = 0$, the set $\Omega = \{x \mid V(x) \leq 1\}$ is bounded and*

$$\nabla V_i(x)^T f(x) < 0, \quad \forall\, x \in S_i \setminus \{0\},\, i \in I, \tag{6}$$

then the set Ω is invariant and a region of attraction for the system (1)*, namely for any initial point $x(0) \in \Omega$, the solution $x(t)$ of* (1) *satisfies $x(t) \in \Omega$ and* $\lim_{t\to\infty} x(t) = 0$.

Proof. By Proposition 2.3 and the formula (4), we have

$$\begin{aligned}\partial V(x(t)) &\subset \operatorname{co}\{\xi^T \dot{x}(t) \mid \xi \in \partial V(x) \mid_{x=x(t)}\} \\ &\subset \operatorname{co}\{\nabla V_i(x)^T f(x) \mid i \in I_V(x)\}.\end{aligned}$$

According to (6), $\nabla V_i(x)^T f(x) < 0, \forall i \in I_V(x)$. Notice the fact that for an univariate function g, if $\xi < 0$ for all $\xi \in \partial g(t), t \geq 0$, then g is decreasing on $[0, \infty)$. Thus, $x(0) \in \Omega$ implies that $V(x(t)) \leq V(x(0)) \leq 1$, in other words, if $x(0) \in \Omega$, then $V(x(t)) \in \Omega$.

Given a number $\epsilon > 0$, denote

$$S^\epsilon = \left\{ x \in \Re^n \mid \frac{1}{2}\epsilon \leq V(x) \leq 1 \right\}.$$

According to (6) and the definition of the set S^ϵ, we have

$$S^\epsilon \bigcap S_i \subset S_i \setminus \{0\} \subset \{x \in S_i \mid \nabla V_i(x)^T f(x) < 0\}, \quad i \in I.$$

Noticing that $S^\epsilon \bigcap S_i$, $i \in I$ are compact as well as f and V_i, $i \in I$ are continuous, there exist numbers $\delta_i > 0, i \in I$ such that

$$\nabla V_i(x)^T f(x) < -\delta_i < 0, \quad \forall x \in S^\epsilon \bigcap S_i, \, i \in I.$$

Denoting $\delta = \min_{i \in I} \delta_i$, since the index set I is finite, such δ exists, then

$$\nabla V_i(x)^T f(x) < -\delta < 0, \quad \forall x \in S^\epsilon \bigcap S_i, \, i \in I. \tag{7}$$

Given an interval $[t_a, t_b]$, applying Proposition 2.2 to the one-dimensional function $\bar{V}(t) = V(x(t))$ on the interval $[t_a, t_b]$, there exist $\bar{t} \in [t_a, t_b]$ and $\bar{\xi} \in \partial \bar{V}(\bar{t})$ such that

$$\bar{V}(t_b) - \bar{V}(t_a) = V(x(t_b)) - V(x(t_a)) = \bar{\xi}(t_b - t_a). \tag{8}$$

By virtue of Proposition 2.3 and the formula (4), we get that

$$\begin{aligned} \partial \bar{V}(t) &\subset \text{co}\left\{ \gamma^T \dot{x}(t) \mid \gamma \in \partial V(z), \, z = x(t) \right\} \\ &= \text{co}\left\{ \nabla V_i(x(t))^T \dot{x}(t) \mid i \in I_V(x(t)) \right\} \\ &= \text{co}\left\{ \nabla V_i(x(t))^T f(x(t)) \mid i \in I_V(x(t)) \right\}. \end{aligned} \tag{9}$$

According to (7), (9) and the definition of $I_V(x)$, all elements in $\partial \bar{V}(t)$ for $t \in [t_a, t_b]$ are less than $-\delta$, thus $\bar{\xi} < -\delta$. By virtue of (8), we have that

$$V(x(t_b)) < V(x(t_a)) - \delta(t_b - t_a).$$

Furthermore, $\delta > 0$ implies that there exists $\bar{t} > 0$ such that $V(x(t)) \leq \epsilon$ for all $t > \bar{t}$, this means that $\lim_{t \to \infty} V(x(t)) = 0$ if $x(0) \in \Omega$.

Let $\epsilon > 0$ and let

$$\Omega_\epsilon = \{x \in \Re^n \mid \epsilon \leq \|x\|, \, V(x) \leq 1\}.$$

Evidently, the set Ω_ϵ is compact with $0 \notin \Omega_\epsilon$. Since V is continuous and positive definite, there exists $\gamma \in (0, 1)$ such that $0 < \gamma \leq V(x), x \in \Omega_\epsilon$. On the other hand, $\lim_{t \to \infty} V(x(t)) = 0$ ensures that there exists t_1 such that $V(x(t)) < \gamma$ for all $t > t_1$. This yields that $x(t) \notin \Omega_\epsilon$, that is $\|x(t)\| < \epsilon$, therefore $\lim_{t \to \infty} x(t) = 0$. We thus complete the proof of the theorem. □

4. Two Classes of Piecewise Smooth Lyapunov Functions

In this section, we consider two classes of piecewise smooth Lyapunov functions. One is the sum of a max-type function and a min-type function, the other is minmax-type function. These two classes of piecewise smooth functions are widely used in nonsmooth analysis, optimization and control, their piecewise smoothness is well-known. In what follows we try to find their specific pieces and regions, then develop conditions of stability and attraction.

4.1. The Sum of Max-Type Function and Min-Type Function

Let us consider the sum of a max-type function and a min-type function of the form:

$$V(x) = \max_{i\in I} U_i(x) + \min_{j\in J} W_j(x), \tag{10}$$

where $U_i : \Re^n \to \Re, i \in I$ and $W_j : \Re^n \to \Re, j \in J$ are continuously differentiable, both I and J are finite index sets. We first try to find specific pieces for the function V given in (10). Given a pair of indices $(s,t) \in I \times J$, define the set S_{st} as the following:

$$S_{st} = \{x \in \Re^n \mid U_s(x) \geq U_i(x), \forall i \in I\} \bigcap \{x \in \Re^n \mid W_t(x) \leq W_j(x), \forall j \in J\}. \tag{11}$$

For a fixed $x \in \Re^n$, denote the index set

$$(I \times J)(x) = \{(s,t) \in I \times J \mid U_s(x) \geq U_i(x), \forall i \in I, W_t(x) \leq W_j(x), \forall j \in J\}.$$

Evidently, $\bigcup_{s\in I, t\in J} S_{st} = \Re^n$ and for any $x \in \Re^n$, the index set $(I \times J)(x)$ is nonempty. For any $x \in \Re^n$, there exist $s_1 \in I, t_1 \in J$ such that $x \in S_{s_1t_1}$ and $V(x) = U_{s_1}(x) + W_{t_1}(x)$, moreover $V(x) = U_s(x) + W_t(x)$ for any $(s,t) \in (I \times J)(x)$. Choose $O_{st} = \Re^n, s \in I, t \in J$, thus V is piecewise smooth function with S_{st} as piece regions and $S_s(x) + V_t(x)$ as piece functions. Evidently, V has no more than $\text{card}\, I \times \text{card}\, J$ pieces, where "card" denotes cardinality. By the definition of S_{st} and $(I \times J)(x)$, we obtain the following proposition immediately.

Proposition 4.1. *Let $x \in \Re^n$ and $(i,j) \in I \times J$. Then, $x \in S_{ij}$ given in* (11) *if and only if $(i,j) \in (I \times J)(x)$.*

Let us consider the system (1) and corresponding Lyapunov function V given in (10). Suppose that V is positive definite, the set $\Omega = \{x \in \Re^n \mid V(x) \leq 1\}$ is bounded and the following condition holds:

$$(\nabla U_s(x) + \nabla W_t(x))^T f(x) < 0, \quad \forall (s,t) \in (I \times J)(x). \tag{12}$$

By virtue of Theorem 3.1 and Proposition 4.1, the set Ω is invariant and a region of attraction for the system (1), namely, initial point $x(0) \in \Omega$ guarantees that $x(t) \in \Omega$ and $\lim_{t\to\infty} x(t) = 0$.

We can extend the above results to a smooth composition of max-type functions of the form

$$V(x) = g(\max_{j\in J_1} V_{1j}(x), \ldots, \max_{j\in J_m} V_{mj}(x)), \tag{13}$$

where $g : \Re^m \to \Re$ and $V_{ij} : \Re^n \to \Re$ are continuously differentiable and $J_j, j = 1, \ldots, m$ are finite index sets.

4.2. The Minmax-Type Function

We next consider a minmax-type Lyapunov function of the form:

$$V(x) = \min_{i \in I} \max_{j \in J} V_{ij}(x), \tag{14}$$

where $V_{ij} : \Re^n \to \Re, i \in I, j \in J$ are continuously differentiable, both I and J are finite index sets.

The minmax-type function (14) is widely used in nonsmooth analysis. It was shown that under some conditions, a piecewise smooth function can be reformulated as a minmax-type function, see [8].

Given a pair of indices $(s,t) \in I \times J$, define the set

$$S_{st} = \{x \in \Re^n \mid V_{st}(x) \geq V_{sj}(x), \forall j \in J, V_{st}(x) \leq \max_{j \in J} V_{ij}(x), \forall i \in I\}. \tag{15}$$

Evidently, $\bigcup_{s \in I, t \in J} S_{st} = \Re^n$. It can be verified that $V(x) = V_{st}(x), \forall x \in S_{st}$, we choose $O_{st} = \Re^n$, thus V is piecewise smooth with piece functions V_{st} and piece regions S_{st}.

Given a fixed $x \in \Re^n$, define index sets as follows:

$$J_s(x) = \{j \in J \mid V_{sj}(x) = \max_{t \in J} V_{st}(x)\}, \quad s \in I$$

and

$$I(x) = \{i \in I \mid \max_{t \in J_i(x)} V_{it}(x) = \min_{i \in I} \max_{t \in J_i(x)} V_{it}(x)\}.$$

By the definition of $J_s(x)$ and $I(x)$, we obtain the following proposition immediately.

Proposition 4.2. *Let $x \in \Re^n$ and $(i,j) \in I \times J$. Then, $x \in S_{ij}$ given in* (15) *if and only if there exists $s \in I$ such that $(i,j) \in I(x) \times J_s(x)$.*

Let us consider the system (1) and Lyapunov function V given in (14). Suppose that V is positive definite, the set $\Omega = \{x \mid V(x) \leq 1\}$ is bounded and

$$\nabla V_{st}(x)^T f(x) < 0, \quad \forall\, s \in I(x),\, t \in J_s(x). \tag{16}$$

By virtue of Theorem 3.1 and Proposition 4.2, the set Ω is invariant and a region of attraction for the system (1), in other words, for any initial point $x(0) \in \Omega$, the solution $x(t)$ of (1) is such that $x(t) \in \Omega$ and $\lim_{t \to \infty} x(t) = 0$.

5. Conclusions

In this paper, we first generalize the existing viability condition and attraction region condition to the case where Lyapunov functions are piecewise smooth. Then, we discuss two widely used classes of piecewise smooth functions, take them as Lyapunov functions, viability condition and attraction region condition are proposed.

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