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Parametric Stability of Solutions in Models of Economic Equilibrium

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Recent results about strong metric regularity of solution mappings are applied to a challenging situation in basic mathematical economics, a model of market equilibrium in the exchange of goods. The solution mapping goes from the initial endowments of the agents to the goods they end up with and the supporting prices, and the issue is whether, relative to a particular equilibrium, it has a single-valued, Lipschitz continuous localization. A positive answer is obtained when the chosen goods are not too distant from the endowments. A counterexample is furnished to demonstrate that, when the distance is too great, such strong metric regularity can fail, with the equilibrium then being unstable with respect to tiny shifts in the endowment parameters, even bifurcating or, on the other hand, vanishing abruptly.

The approach relies on passing to a variational inequality formulation of equilibrium. This is made possible by taking the utility functions of the agents to be concave and their survival sets to be convex, so that their utility maximization problems are fully open to the methodology of convex analysis. The variational inequality is nonetheless not monotone, at least in the large, and this greatly complicates the existence of a solution. Existence is secured anyway through truncation arguments which take advantage of a further innovation, the explicit introduction of "money" into the classical exchange model, with money-tuned assumptions of survivability of the agents which are unusualy mild. Those assumptions also facilitate application to the model of refinements of Robinson's implicit mapping theorem in variational analysis.

Keywords: Strong metric regularity, solution stability, economic equilibrium, ample survivability, nonmonotone variational inequalities, convex analysis, variational analysis, Robinson's theorem

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1. Introduction

In the most basic mathematical model of economic equilibrium, focused on the exchange of goods in a single time period, there are agents $i = 1, \ldots, r$, who start with goods vectors $x_i^0 \in I\!R^n_+$, and trade them for other goods vectors x_i which have to lie in certain *survival* sets $X_i \subset I\!R^n_+$. The trading is done through a market in which goods can be bought and sold in accordance with a price vector $p \in I\!R^n_+$,

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$p \neq 0$. The optimization problem for agent i is to maximize a *utility* expression $u_i(x_i)$ over $x_i \in X_i$ subject to the *budget constraint* $p{\cdot}x_i \leq p{\cdot}x_i^0$.[1] The vectors x_i^0 that fix the size of the unshared budgets are called the *initial endowments* of the agents.

The main issue, to begin with, is whether there is a price vector p such that, when the agents do this, the total demand will be covered by the total supply, with equality holding except perhaps for goods priced at 0:

$$\sum\nolimits_{i=1}^{r} x_i \leq \sum\nolimits_{i=1}^{r} x_i^0, \qquad p{\cdot}\sum\nolimits_{i=1}^{r} x_i = p{\cdot}\sum\nolimits_{i=1}^{r} x_i^0. \tag{1}$$

If so, p and the agents' demands x_i are said to constitute an *equilibrium*.[2]

Beyond the question of the existence of an equilibrium and its hoped-for uniqueness, at least locally, there is the question of how it may respond to shifts in parameters in the problem. The initial endowments x_i^0 are particularly important in this respect, although other parameters of interest could come up in the specification of the utilities u_i.

The classical theory of existence in Arrow and Debreu [1] requires the sets X_i to be closed and convex, with $X_i + I\!R^n_+ \subset X_i$, and the functions u_i to be continuous, quasi-concave and nondecreasing relative to X_i, as well as insatiable in the sense of never achieving a maximum. An equilibrium is guaranteed then under the additional assumption of *strong survivability*, meaning that x_i^0 lies in the interior of X_i for every agent i. This is an awkward restriction (why should every agent have to start with a positive quantity of every good?), but it has been difficult to replace as a "constraint qualification" without getting into more esoteric assumptions which verge on the unverifiable, cf. [1], [4], [5]. But even with strong survivability, no light is shed on parametric behavior.

In follow-up studies concerned with price adjustment processes that might lead to achieving an equilibrium, much of the economic literature has concentrated on the special situation where the constraints $x_i \in X_i$ are never active (everything takes place in their interiors) and the functions u_i are strictly concave and arbitrarily many times differentiable; cf. [6],[2], and their references. Equilibrium is characterized then by a smooth system of equations, and parametric analysis of solutions can rely on the classical implicit function theorem.

Our aim here is to pass to a different paradigm beyond the implicit function theorem, invoking results instead about solution mappings for parameterized variational inequality problems as in Dontchev and Rockafellar [3]. With that technology we can allow the survival constraints to be active and arrive instead at equilibrium equations that are nonsmooth but possessed of one-sided directional derivatives. We can also reliquish strict concavity of u_i with respect to all goods, thereby per-

[1] $p{\cdot}x_i$ is the dot-product adding for each of the goods $j = 1, \ldots, n$ its price p_j in p times its quantity x_{ij} in x_i.

[2] This is the key idea of decentralization in economics. An equilibrium price vector coordinates supply and demand without an individual agent needing to pay attention to the choices of other agents or to confirm to the dictates of some higher authority forcing them to share their endoments and budgets.

mitting an agent to be personally indifferent to some of the goods, as seems more natural. An equilibrium then no longer may depend smoothly on parameters like the initial endowments, but under a criterion applied from [3] the dependence will be locally Lipschitz continuous with one-sided directional derivatives. In other words, the solution mapping will exhibit *strong metric regularity* combined with *semidifferentiablility.*

A variational inequality formulation of equilibium, as required for our approach, is a relatively new idea in economics, having so far only been given only in a couple of mathematical publications, see [7], [8]. Here, although we forgo incorporating production along side of exchange as in those articles, we take a significant step forward by letting "money" explicitly enter the exchange and have a key role. The variational inequality comes out then to be much more convenient for parametric analysis, and, as a bonus, the assumption of strong survivabiity can be replaced by something simpler and more appealing. This money framework echoes, in the one-period setting, the current development in [9] and [10] of a two-period model of economic equilibrium with financial markets available for hedging against future uncertainty.

2. The equilibrium model with money

At first, we proceed more generally than would be needed for our subsequent parametric analysis. This will help in appreciating the economic scope of the money model we are adopting and support a rigorous proof of existence of equilibrium along the lines of the model in [10], but without all the extra complications. It will also facilitate comparison with the classical exchange model of Arrow and Debreu outlined above.

In addition to being involved with goods vectors x_i and x_i^0 as before, agent i will have an initial amount m_i^0 of money (one can think of dollars for concreteness) and end up, after trading, with an amount m_i. Prices for the other goods will be in dollars, so that the budget constraint now takes the form

$$m_i - m_i^0 + p\cdot[x_i - x_i^0] \leq 0. \tag{2}$$

Subject to this, when presented with a price vector p, agent i will want to maximize a utility expression $u_i(m_i, x_i)$ subject to $(m_i, x_i) \in U_i$, this being the survival set now lying in $\mathbb{R}_+ \times \mathbb{R}^n_+ = \mathbb{R}^{1+n}_+$. We suppose that U_i is nonempty and convex and that u_i is continuous, concave (not just quasi-concave) and nondecreasing over U_i, which implies $U_i + \mathbb{R}^{1+n}_+ \subset U_i$, and moreover that u_i *is increasing always with respect to* m_i. The budget constraint (2) must then hold as an equation in optimality. Employing a device familiar in convex analysis, we let

$$\tilde{u}_i(m_i, x_i) = \begin{cases} u_i(m_i, x_i) & \text{if } (m_i, x_i) \in U_i, \\ -\infty & \text{if } (m_i, x_i) \notin U_i, \end{cases} \tag{3}$$

and identify the maximization of u_i over U_i with the maximization of $\tilde{u}_i$ over $\mathbb{R}^{1+n}$. We suppose that $\tilde{u}_i$ is upper semicontinuous (usc), which is certainly true when U_i

is closed but can also hold in circumstances where u_i goes to $-\infty$ as the boundary of U_i is approached.

Definition 2.1. An *equilibrium* in the exchange model with money is comprised of a price vector $p \geq 0$ and corresponding solutions (m_i, x_i) to the p-dependent optimization problems for the agents i which meet the supply-demand condition (1).

Note that a supply-demand condition for the m_i's is superfluous as a separate stipulation in the definition of an equilibrium because it follows (1) through the budget constraints (2) in their equation form at optimality

In place of strong survivability as an assumption on the initial endowments invoked for the sake of achieving the existence of an equilibrium, we will rely on a much weaker form of such a "constraint qualification".

Ample survivability assumption. The agents have choices $(\hat{m}_i, \hat{x}_i) \in U_i$ available to them for which (a) $\hat{x}_i \leq x_i^0$ but $\hat{m}_i < m_i^0$, and (b) $\sum_{i=1}^r \hat{x}_i < \sum_{i=1}^r x_i^0$.

This postulates that, without the need for any trading at all, the agents could, if they wished, survive with each having some money left over and without exhausting the total supply of any other good. (The strict vector inequality in (b) is to be interpreted component by component.) Observe that the agents in this case must all start out with some money, but they are not required to start with more of any other good than might directly be tied to their own survival.

Another further condition, quite reasonable from the economic perspective, will help us to skip over pointless technical complications later when we come to the perturbation analysis.

Essential attractiveness assumption. For every good there is at least one agent i such that the utility u_i always increases on U_i when that good component increases.

We already assumed that money has this effect on all agents and are asking here for something complementary: no good in the economy is just "feebly attractive". At least one agent must find it persistently attractive. That way, every good is assured an essential role.

Theorem 2.2 (existence with price positivity). *Under the ample survability assumption and the conditions placed on the sets U_i and functions u_i, an equilibrium in the exchange model with money is sure to exist. The essential attractiveness assumption guarantees in addition that the prices in such an equilibrium will all be positive, so the supply-demand condition* (1) *will hold as an equation.*

The proof of Theorem 2.2 will be provided in Section 6. Our attention for now is devoted rather to developing a variational inequality formulation of equilibrium.

In finite dimensions, a *variational inequality* problem consists of solving a condition of the form

$$-f(z) \in \partial\varphi(z) \tag{4}$$

for a choice of $f : \mathbb{R}^d \to \mathbb{R}^d$ and a lower semicontinuous (lsc) convex function $\varphi : \mathbb{R}^d \to (-\infty, \infty]$, $\varphi \not\equiv \infty$. More specifically, this is a variational inequality of *functional* type; a variational inequality of *geometric* is obtained when φ is the indicator δ_C of a nonempty, closed, convex set, in which case (4) turns into a condition involving the normal cone mapping associated with C,

$$-f(z) \in N_C(z). \tag{5}$$

A basic criterion for the existence of a solution, as set forth in [8, Theorem 1], is having f be continuous relative to $\operatorname{cl}\operatorname{dom}\varphi$, with that set moreover being bounded. (As usual, $\operatorname{dom}\varphi$ denotes the set of points where $\varphi(z) < \infty$.) A major tool in setting up variational inequality problems is *modular construction*: a set of conditions of the form

$$-f_k(z_1, \dots, z_q) \in \partial\varphi_k(z_k) \quad \text{for } k = 1, \dots, q$$

can be posed as a variational inequality (4) by taking

$$f(z_1, \dots, z_q) = (f_1(z_1, \dots, z_q), \dots, f_r(z_1, \dots, z_q)),$$
$$\varphi(z_1, \dots, z_q) = \varphi_1(z_1) + \cdots + \varphi_r(z_q).$$

Making use of this construction technique for the case at hand, we begin by observing that the supply-demand requirement (1) for an equilibrium is a complementary slackness condition which can be written as

$$d \in N_{\mathbb{R}^n_+}(p) \quad \text{for } d = \sum\nolimits_{i=1}^{r} [x_i - x_i^0] \text{ (excess demand)}. \tag{6}$$

Next, optimality in the utility maximization problem of agent i corresponds in terms of a Lagrange multiplier λ_i for the budget constraint (2) to a saddle point of the Lagrangian function

$$L_i(p; m_i, x_i, \lambda_i) = \tilde{u}_i(m_i, x_i) - \lambda_i(m_i - m_i^0 + p{\cdot}[x_i - x_i^0]),$$

which is characterized by the subgradient condition

$$-(\lambda_i, \lambda_i p) \in \partial[-\tilde{u}_i](m_i, x_i), \tag{7}$$

(with $\tilde{u}_i$ flipped from usc concave to lsc convex) together with the complementary slackness condition

$$m_i - m_i^0 + p{\cdot}[x_i - x_i^0] \in N_{R_+}(\lambda_i). \tag{8}$$

Conditions (6)(7)(8) come out as the variational inequality (4) for

$$\begin{aligned} f(p; \dots, m_i, x_i, \lambda_i, \dots) &= \left(-d; \dots, \lambda_i, \lambda_i p, m_i^0 - m_i - p{\cdot}[x_i^0 - x_i], \dots\right), \\ \varphi(p, \dots, m_i, x_i, \lambda_i, \dots) &= \delta_{\mathbb{R}^n_+}(p) + \sum\nolimits_{i=1}^{r} \left([-\tilde{u}_i](m_i, x_i) + \delta_{\mathbb{R}_+}(\lambda_i)\right). \end{aligned} \tag{9}$$

A solution to the variational inequality (9) combines an equilibrium $(p, \dots, m_i, x_i, \dots)$ with multipliers λ_i which themselves have an economic interpretation. Along the familiar lines of convex optimization with its so-called "shadow prices", λ_i gives the marginal utility of money at optimality. Adopting the terminology in an analogous situation in [8], we will refer to the combination $(p, \ldots, m_i, x_i, \lambda_i, \ldots)$ as an *enhanced equilibrium*.

Of course, we already know in advance through Theorem 2.2 that $p > 0$ and $\lambda_i > 0$ in an equilibrium, in which case (6) and (8) merely reduce to *equations*. Why then do we not simply put equations in place of complementary slackness conditions in (6) and (8)? The reason is that the format we have chosen, with its enforcement of nonnegativity of p and λ_i, provides a better foundation for the existence proof, which must cope also with combinations of elements that do not necessarily furnish an equilibrium.

For existence, f clearly meets the condition of being continuous, but φ fails the condition of $\operatorname{dom}\varphi$ being bounded, inasmuch as $\operatorname{dom}\varphi = I\!R^n_+ \times \Pi_{i=1}^r [U_i \times I\!R_+]$ with every component set inherently unbounded! But existence will be established by showing that the given problem can be reduced step by step through truncation to an equivalent one to which the basic existence criterion is applicable.

3. Perturbation analysis

To consolidate notation somewhat, let us take $\lambda = (\lambda_1, \ldots, \lambda_r)$ together with

$$\begin{aligned} m &= (m_1, \ldots, m_r), \qquad & x &= (x_1, \ldots, x_r), \\ m^0 &= (m_1^0, \ldots, m_r^0), \qquad & x^0 &= (x_1^0, \ldots, x_r^0). \end{aligned} \tag{10}$$

An equilibrium corresponds then to some combination (p, m, x, λ), but it also depends on the initial endowment pair (m^0, x^0). Our concern is with understanding the properties of this dependence, or in other words, characteristics of the mapping

$$S : (m^0, x^0) \mapsto \{\text{all corresponding equilibria as solutions } (p, m, x, \lambda) \text{ to } (4)(9)\}. \tag{11}$$

We call this generally set-valued mapping the *equilibrium mapping*.

Our efforts will center on applying results about perturbational stability of variational inequalities that we have laid out in our recent book [3]. However, those results are formulated for variational inequalities (5) of geometric type instead for a variational inequality (4) of functional type as we have here with (9).

There are two ways around this. One is to pass in the case of (4) to the epigraph E of the function φ (this being a nonempty, closed, convex set), rewriting (4) as $-(f(z), 1) \in N_E(z, \alpha)$, where necessarily $\alpha = \varphi(z)$. An alternative strategy is to specialize to the case where φ has the structure $\Phi + \delta_C$ with C being a closed convex set and Φ being a differentiable function on a open set containing C which, in its restriction to C, is convex. In that case we have $\partial\varphi(z) = \nabla\Phi(z) + N_C(z)$ and can rewrite (4) as the geometric variational inequality

$$-g(z) \in N_C(z) \quad \text{with } g(z) = f(z) + \nabla\Phi(z). \tag{12}$$

For purposes of perturbation analysis much of everything is local, and the $\varphi = \Phi + \delta_C$ structure only needs to be available on some neighborhood of the solution that is under scrutiny. The restriction to such structure is therefore often less stringent than it may at first seem. Anyway, the geometric alternative is advantageous here because it will also allow us to exploit second derivatives of the utility functions.

Referring back to the specification of $\tilde{u}_i$ in (3), we suppose now that

$$U_i \text{ is closed and } u_i \text{ is given not just on } U_i, \text{ but on a larger open set where it is twice continuously differentiable.} \tag{13}$$

We rewrite the functional variational inequality (4)(9) as a geometric variational inequality with the endowments as explicit parameters, namely

$$\begin{aligned} &-g(p,m,x,\lambda,m^0,x^0) \in N_C(p,m,x,\lambda) \\ &\qquad \text{where } C = I\!R^n_+ \times [U_1 \times I\!R_+] \times \cdots \times [U_r \times I\!R_+], \\ &g(p,m,x,\lambda,m^0,x^0) \\ &= (-d,\ldots,\lambda_i(1,p) - \nabla u_i(m_i,x_i), m_i^0 - m_i + p{\cdot}[x_i^0 - x_i],\ldots). \end{aligned} \tag{14}$$

Observe that g is continuously differentiable in all variables under our assumptions. That property will be needed later.

The equilibrium mapping S in (11) can be identified now with the solution mapping associated with the geometric variational inequality (14):

$$S : (m^0,x^0) \mapsto \{\text{all corresponding solutions } (p,m,x,\lambda) \text{ to } (14)\}, \text{ recapturing (11).} \tag{15}$$

We look at a solution $(\bar{p},\bar{m},\bar{x},\bar{\lambda})$ corresponding to a given parameter pair $(\bar{m}^0,\bar{x}^0)$ and investigate how that solution may change under a shift to a different parameter pair. The initial concept to invoke is that of a *single-valued localization*, which by definition is a function s on a neighborhood of $(\bar{m}^0,\bar{x}^0)$ with its graph equal to the intersection of the graph of S with the product of that neighborhood and a neighborhood of $(\bar{p},\bar{m},\bar{x},\bar{\lambda})$, and in particular then having

$$s(\bar{m}^0,\bar{x}^0) = (\bar{p},\bar{m},\bar{x},\bar{\lambda}). \tag{16}$$

In the context of the classical implicit function theorem, we would hope for s to be differentiable and have an expression available for its derivatives, but that is too much to expect from a geometric variational inequality with one-sided constraints built into the set C. The property to fall back on in that case is *strong metric regularity*, which refers to having a single-valued localization which is Lipschitz continuous. There can still be hope then for the existence of *one-sided* directional derivatives. This subject is laid out in great detail in [3], and we recommend that exposition to readers wanting additional explanation and motivation.

The key theorem to apply to get the desired strong metric regularity goes back to Robinson [11], [12]. It revolves about an auxiliary variational inequality obtained by partial linearization. In our case it is parameterized by a vector dual to (p,m,x,λ) which, in the absence of anything better coming to mind, we denote by (p^*,m^*,x^*,λ^*). The auxiliary variational inequality has the form

$$\begin{aligned} &(p^*,m^*,x^*,\lambda^*) - g(\bar{p},\bar{m},\bar{x},\bar{\lambda},\bar{m}^0,\bar{x}^0) \\ &\qquad - \nabla g_{p,m,x,\lambda}(\bar{p},\bar{m},\bar{x},\bar{\lambda},\bar{m}^0,\bar{x}^0)[\,(p,m,x,\lambda) - (\bar{p},\bar{m},\bar{x},\bar{\lambda})\,] \in N_C(p,m,x,\lambda) \end{aligned} \tag{17}$$

with associated solution mapping

$$S^* : (p^*, m^*, x^*, \lambda^*) \mapsto \{\text{all corresponding solutions } (p, m, x, \lambda) \text{ to } (17)\} \qquad (18)$$

having, in particular, $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}) \in S^*(0,0,0,0)$. (Here $\nabla g_{p,x,m,\lambda}$ is the Jacobian of g with respect to (p, x, m, λ).) According to the key part of Robinson's theorem, when applied here,

$$\begin{array}{l}\text{if } S^* \text{ is strongly metrically regular at } (0,0,0,0) \text{ for } (\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}), \\ \text{then } S \text{ is strongly metrically regular at } (\bar{m}^0, \bar{x}^0) \text{ for } (\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}).\end{array} \qquad (19)$$

This may appear on the surface only to replace one difficult issue by another, but that impression can immediately be dispelled by looking at the case where $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$ happens to be in the interior of C rather than on the boundary. Then $N_C(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}) = \{(0,0,0,0)\}$ and we are back in the classical context of equations only. In (14) we are concerned with solving $g(p, m, x, \lambda, m^0, x^0) = (0,0,0,0)$ for (p, m, x, λ) as a function of (m^0, x^0) with reference to $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$ being a solution for $(\bar{m}^0, \bar{x}^0)$, while in (17) we are looking only at solving the linearized equation

$$\begin{aligned}&(p^*, m^*, x^*, \lambda^*)\\ &= g(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}, \bar{m}^0, \bar{x}^0) - \nabla g_{p,m,x,\lambda}(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}, \bar{m}^0, \bar{x}^0)[\,(p, m, x, \lambda) - (\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})\,]\end{aligned}$$

for (p, m, x, λ) as a function of $(p^*, m^*, x^*, \lambda^*)$ with reference to $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$ being a solution for $(p^*, m^*, x^*, \lambda^*) = (0,0,0,0)$. Strong metric regularity in the auxiliary system reverts simply to nonsingularity of the Jacobian matrix

$$A = \nabla g_{p,m,x,\lambda}(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}, \bar{m}^0, \bar{x}^0), \qquad (20)$$

which is the condition in the classical implicit function theorem.

In our territory beyond the classical, we will be aided by further refinements of the Robinson prescription (19) which have been developed in our book [3]. These center on the case where C is *polyhedral* and utilize the *critical cone* to C at $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$ with respect to g as derived from the tangent cone $T_C(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$, namely the polyhedral cone

$$\begin{aligned}K = K(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}) = \big\{ (p', m', x', \lambda') \in T_C(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}) \,\big|\, (p', m', x', \lambda') \perp \bar{g} \,\big\}, \\ \text{where } \bar{g} = g(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}, \bar{m}^0, \bar{x}^0).\end{aligned} \qquad (21)$$

The stronger result in that case allows (17) to be replaced in the Robinson condition (19) by the reduced variational inequality

$$(p^*, m^*, x^*, \lambda^*) - A(p', m', x', \lambda') \in N_K(p', m', x', \lambda') \qquad (22)$$

and its solution mapping

$$\bar{S} : (p^*, m^*, x^*, \lambda^*) \mapsto \{\text{all corresponding solutions } (p', m', x', \lambda') \text{ to } (22)\}, \qquad (23)$$

which then takes over the role of S^*.

The result not only yields strong metric regularity of S but also asserts the existence of directional derivatives $s'(\bar{m}^0, \bar{x}^0; m^{0\prime}, x^{0\prime})$ of the single-valued localization s with respect to shifts from $(\bar{m}^0, \bar{x}^0)$. We bring to its statement the notion of a single-valued mapping being *semidifferentiable* at a given point. This refers to the existence of a so-called first-order approximation in the manner of differentiability but with the approximating function no longer required to be affine; for background see [14, Chapter 7]. When the mapping is locally Lipschitz continuous, semidifferentiability is automatic from the existence of one-sided directional derivatives.

Theorem 3.1 (strong metric regularity with semidifferentiability). *In the context of assumption* (13) *and C being polyhedral, if the reduced mapping $\bar{S}$ is single-valued, then the equilibrium mapping S is strongly metrically regular at $(\bar{m}^0, \bar{x}^0)$ for $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$ and the corresponding single-valued localization s is not only Lipschitz continuous but also semidifferentiable with its one-sided directional derivatives given in terms of the parameter Jacobian*

$$B = \nabla g_{m^0,x^0}(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda}, \bar{m}^0, \bar{x}^0) \tag{24}$$

by the variational inequality formula

$$s'(\bar{m}^0, \bar{x}^0 m^{0\prime}, x^{0\prime}) = \bar{S}(-B(m^{0\prime}, x^{0\prime})\,). \tag{25}$$

Furthermore, the following condition is sufficient for $\bar{S}$ to be single-valued and thus for those conclusions to hold:

$$\begin{aligned} (p', m', x', \lambda') \in K - K, \qquad A(p', m', x', \lambda') \perp K \cap (-K), \\ (p', m', x', \lambda')\cdot A(p', m', x', \lambda') \leq 0 \\ \Longrightarrow \quad (p', m', x', \lambda') = (0, 0, 0, 0), \end{aligned} \tag{26}$$

where in fact A can be replaced by a matrix

$$\begin{aligned} &\tilde{A} = MA \text{ for any nonsingular } M \text{ having the property that} \\ &MN_K(p', m', x', \lambda') = N_K(p', m', x', \lambda') \text{ for all } (p', m', x', \lambda'). \end{aligned} \tag{27}$$

Proof. This specializes most of [3, Theorem 2E.8] on the basis of g being strictly differentiable through our assumption of twice continuous differentiability of the utility functions u_i. The final claim is an innovation based on noting that (26) is a condition applicable to problems of the "affine-polyhedral" form (22) regardless of the economic setting. Multiplying (22) on by both sides by M under (26) would preserve this problem except for replacing the parameter element $(p^*, m^*, x^*, \lambda^*)$ by $(\tilde{p}^*, \tilde{m}^*, \tilde{x}^*, \tilde{\lambda}^*) = M(p^*, m^*, x^*, \lambda^*)$. That would not affect the single-valuedness being sought, hence (26) with $\tilde{A}$ substituting for A is just as good a criterion as (26) itself. □

We speak of (25) as a "variational inequality formula" because it says that the directional derivative of the localized equilibrium solution mapping s, with respect to shifting the initial supply in the direction of the vector $(m^{0\prime}, x^{0\prime})$, is the vector (p', m', x', λ') calculated by solving a certain "derivative variational inequality problem" parameterized by $(m^{0\prime}, x^{0\prime})$. The variational inequality is generated by putting $-B(m^{0\prime}, x^{0\prime})$ in place of $(p^*, m^*, x^*, \lambda^*)$ in (22).

Note from convex analysis that $K - K$ is the smallest subspace containing the polyhedral cone K, whereas $K \cap (-K)$ is the largest subspace contained within K. Both agree with K when K is itself a subspace. In order to get a better understanding the consequences of Theorem 3.1, we will need to examine the structure of K and the matrix A with an eye to applying the condition in (26). This is the object of the next section. The flexibility at the end of the Theorem 3.1 will make it possible to take better advantage of the structure we are presented with in the economic model.

4. Further development of the application

We continue with the study of a particular enhanced equilibrium $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$, keeping the same notation and recalling that equilibrium under our assumptions entails

$$\bar{\lambda}_i > 0 \quad \text{for } i = 1, \ldots, r, \text{ and } \bar{p} > 0. \tag{28}$$

The modular structure in (14) will serve prominently. The tangent cone to C for the given equilibrium is the product of the tangent cones at the component sets, and accordingly, the critical cone K will be the product of critical cones to the components. To articulate this conveniently, let

$$\bar{g}_i = (\bar{\lambda}_i, \bar{\lambda}_i \bar{p}) - \nabla u_i(\bar{m}_i, \bar{x}_i) \tag{29}$$

and define K_i to be the critical cone to U_i at $\bar{g}_i$, namely

$$K_i = \left\{ (m_i', x_i') \in T_{U_i}(\bar{m}_i, \bar{x}_i) \,\middle|\, (m_i', x_i') \perp \bar{g}_i \right\}. \tag{30}$$

The equilibrium has price vector $\bar{p} > 0$ and excess demand

$$\bar{d} = \sum\nolimits_{i=1}^{r} (\bar{x}_i - x_i^0) = 0 \tag{31}$$

by Theorem 2.2, so the critical cone to the price orthant $\mathbb{R}^n_+$ at $\bar{p}$ for $-\bar{d}$ is all of $\mathbb{R}^n$. Similarly, because $\bar{\lambda}_i > 0$, the critical cone to $\mathbb{R}_+$ at $\bar{\lambda}_i$ for the net budget expenditure $\bar{m}_i - m_0^0 + \bar{p}{\cdot}(\bar{x}_i - x_i^0)$, necessarily 0, coincides with $\mathbb{R}$. Therefore

$$\begin{aligned} (p', m', x', \lambda') \in K &\iff (m_i', x_i') \in K_i \quad \text{for } i = 1, \ldots, r, \\ (p', m', x', \lambda') \in K - K &\iff (m_i', x_i') \in K_i - K_i \quad \text{for } i = 1, \ldots, r, \\ (p', m', x', \lambda') \in K \cap [-K] &\iff (m_i', x_i') \in K_i \cap [-K_i] \quad \text{for } i = 1, \ldots, r. \end{aligned} \tag{32}$$

In all cases, p' can be anything in $\mathbb{R}^n$ and λ_i' can be anything in $\mathbb{R}$, in consequence of (28).

The next step concerns the structure of the Jacobian matrix A in (17), which is a matter of calculating the partial derivatives of g with respect to the various components. Let

$$H_i = \text{the Hessian of } u_i \text{ at } (\bar{m}_i, \bar{x}_i), \tag{33}$$

which from the concavity of u_i will be *negative semidefinite*. Then

$$A(p',m',x',\lambda') = -(d',\dots,v_i,t_i,\dots) \text{ for } \begin{cases} d' = \sum_{i=1}^r x_i', \\ v_i = -\lambda_i'(1,\bar{p}) - \bar{\lambda}_i(0,p') + H_i(m_i',x_i'), \\ t_i = m_i' + \bar{p}\cdot x_i' + p'\cdot(\bar{x}_i - x_i^0). \end{cases} \tag{34}$$

While we are at it, we can also record that the parameter Jacobian has

$$B(m^{0\prime},x^{0\prime}) = \left(\sum_{i=1}^r x_i^{0\prime},\dots,0,0,m_i^{0\prime} + \bar{p}\cdot x_i^{0\prime},\dots\right). \tag{35}$$

The variational inequality (22) emerges then as the combination of conditions

$$\begin{aligned} &\textstyle\sum_{i=1}^r x_i' = 0, \qquad m_i' + \bar{p}\cdot x_i' + p'\cdot(\bar{x}_i - x_i^0) = 0, \\ &-\lambda_i'(1,\bar{p}) - \bar{\lambda}_i(0,p') + H_i(m_i',x_i') \in N_{K_i}(m_i',x_i'). \end{aligned} \tag{36}$$

However, we wish here to employ the modification of A allowed at the end of Theorem 3.1, specifically through "normalizing" the middle set of conditions in (36) by dividing by the positive factors $\bar{\lambda}_i$, which has no effect on the cones $N_{K_i}(m_i',x_i')$. In other words, we wish to replace A in condition (26) by the matrix $\tilde{A}$ having

$$\begin{aligned} &\tilde{A}(p',m',x',\lambda') = -(d',\dots,\tilde{v}_i,t_i,\dots) \\ &\text{for } \begin{cases} d' = \sum_{i=1}^r x_i', \\ \tilde{v}_i = -\bar{\lambda}_i^{-1}\lambda_i'(1,\bar{p}) - (0,p') + \bar{\lambda}_i^{-1}H_i(m_i',x_i'), \\ t_i = m_i' + \bar{p}\cdot x_i' + p'\cdot(\bar{x}_i - x_i^0). \end{cases} \end{aligned} \tag{37}$$

This makes no difference in the initial pair of conditions in (26) but has an important influence on the quadratic form there. As seen from (31), the version of (26) with $\tilde{A}$ asks us to verify that

$$\begin{aligned} &(p',m',x',\lambda') = (0,0,0,0) \text{ holds whenever} \\ &\quad \textstyle\sum_{i=1}^r x_i' = 0, \qquad m_i' + \bar{p}\cdot x_i' + p'\cdot(\bar{x}_i - x_i^0) = 0, \\ &\quad (m_i',x_i') \in K_i - K_i, \qquad -\lambda_i'(1,\bar{p}) - \bar{\lambda}_i(0,p') + H_i(m_i',x_i') \perp K_i \cap [-K_i], \\ &\quad 0 \geq -p'\cdot d' + \textstyle\sum_{i=1}^r (m_i',x_i')\cdot[\bar{\lambda}_i^{-1}\lambda_i'(1,\bar{p}) + (0,p') - \bar{\lambda}_i^{-1}H_i(m_i',x_i')] \\ &\qquad\qquad + \textstyle\sum_{i=1}^r \lambda_i'[m_i' + \bar{p}\cdot x_i' + p'(\bar{x}_i - x_i^0)] \\ &\quad\ = -p'\cdot\textstyle\sum_{i=1}^r x_i' + \sum_{i=1}^r[\bar{\lambda}_i^{-1}\lambda_i'(m_i' + \bar{p}\cdot x_i') + p'\cdot x_i'] - \bar{\lambda}_i^{-1}(m_i',x_i')\cdot H_i(m_i',x_i')] \\ &\quad\ = p'\cdot\textstyle\sum_{i=1}^r \bar{\lambda}_i^{-1}\lambda_i'(x_i^0 - \bar{x}_i) - \sum_{i=1}^r \bar{\lambda}_i^{-1}(m_i',x_i')\cdot H_i(m_i',x_i'). \end{aligned} \tag{38}$$

Here many terms in the quadratic expression have dropped out because of the equations at the beginning of (38), the second of which makes it possible to identify $m_i' + \bar{p}\cdot x_i'$ with $p'\cdot(x_i^0 - \bar{x}_i)$.

To make further progress in applying the criterion (38) we have extracted from (26)–(27), we must scrutinize the critical cones K_i. For this purpose we now introduce a further simplication in the economic structure. We suppose that

$$\text{the survival sets } U_i \text{ merely enforce some given lower bounds,} \quad m_i \geq m_i^-, \qquad x_i \geq x_i^-, \tag{39}$$

and the Hessians of u_i on U_i have a common null space which is spanned by the axes of $\mathbb{R}^n$ for the indices of goods not of interest to agent i, in the sense that u_i does not at all depend on them.

Inasmuch as the Hessians of u_i are already known to be negative semidefinite, this assumption guarantees that they are in fact negative definite with respect to the goods that really are of interest to agent i. Then u_i is *strongly* concave locally for those goods.

We are close to formulating our chief result about the equilibrium mapping. A norm on the goods space appears in the statement, and it could be any norm, but $||\cdot||_\infty$ seems most appropriate for our context.

Theorem 4.1 (parametric stability of equilibrium). *Let the conditions on the survival sets and utility functions hold with the strengthened features in* (13) *and* (39). *Consider an enhanced equilibrium* $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$ *associated with an initial endowment* $(\bar{m}^0, \bar{x}^0)$ *for which ample survivability holds, and suppose that no agent ends up tight with money, or in other words,*

$$\bar{m}_i > m_i^- \quad \textit{for } i = 1, \ldots, r. \tag{40}$$

Then there is an $\varepsilon > 0$ *such that if*

$$\sum\nolimits_{i=1}^r ||\bar{x}_i^0 - \bar{x}_i||_\infty < \varepsilon, \tag{41}$$

the equilibrium mapping S *in* (11) *is strongly metrically regular at* $(\bar{m}^0, \bar{x}^0)$ *for* $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$, *and the corresponding single-valued localization* s *is not only Lipschitz continuous but also semidifferentiable, with its directional derivatives given by*

$$\begin{aligned} &s'(\bar{m}^0, \bar{x}^0; m^{0\prime}, x^{0\prime}) \\ &= \textit{the unique solution } (p', m', x', \lambda') \textit{ to the variational inequality} \\ &\qquad - A(p', m', x', \lambda') - B(m^{0\prime}, x^{0\prime}) \in N_K(p', m', x', \lambda') \end{aligned} \tag{42}$$

for the Jacobians A *and* B *in* (20) *and* (24). *This variational inequality is comprised of the conditions*

$$\begin{gathered} \sum\nolimits_{i=1}^r x_i' = \sum\nolimits_{i=1}^r x_i^{0\prime}, \qquad m_i' - m_i^{0\prime} + \bar{p}\cdot(x_i' - x_i^{0\prime}) + p'\cdot(\bar{x}_i - \bar{x}_i^0) = 0, \\ -\lambda_i'(1, \bar{p}) - \bar{\lambda}_i(0, p') + H_i(m_i', x_i') \in N_{K_i}(m_i', x_i'). \end{gathered} \tag{43}$$

Proof. The part of the theorem concerned with directional derivatives just puts together the assertions in Theorem 3.1 in light of what has been learned about the matrices A and B from (34) and (35). The condition in (38) is sufficient for all this, and our plan is to proceed from it in utilizing the extra structure in (39). We aim at demonstrating that with this structure the stipulations in (38) do ensure that $(p', m', x', \lambda') = (0, 0, 0, 0)$.

Expanding the terminology in the theorem's statement, let us say that a good is tight for agent i in the equilibrium at hand if it is at its lower bound in (39), i.e., if that component of $\bar{x}_i$ equals the corresponding component of x_i^-. Surely goods not of interest to agent i must be tight, because they have no effect on utility and yet have positive price, cf. (28), so that holding on to them beyond bare necessity is never optimal.

For goods that are not tight at the optimal solution $(\bar{m}_i, \bar{x}_i)$ to the utility problem of agent i, the corresponding axes of the goods space must lie in the critical cone K_i, hence also in $K_i - K_i$ and $K_i \cap [-K_i]$. This applies to the money axis in particular through assumption (40). For those goods, the condition involving $K_i \cap [-K_i]$ in (38) requires therefore that the corresponding components of $-\lambda_i'(1,\bar{p}) - \bar{\lambda}_i(0,p') + H_i(m_i', x_i')$ be 0.

We will put this to work in the following way. The conditions in (38) before the quadratic inequality give a system of homogeneous linear equations in (p', m', x', λ') which define a linear subspace of the space of these elements. We claim that

$$\text{this subspace is the graph of a linear transformation } \; T : (m', x') \mapsto (p', \lambda'). \tag{44}$$

For proof it is enough to show that $(p', 0, 0, \lambda')$ cannot lie in this subspace unless $(p', \lambda') = (0, 0)$. If $(p', 0, 0, \lambda')$ lies in the subspace, we know from the observation at the end of the preceding paragraph that the components of $-\lambda_i'(1,\bar{p}) - \bar{\lambda}_i(0,p')$ corresponding to goods that are not tight in equilibrium must be 0, and moreover this must hold for every agent i.

Because money is not tight for any agent, we immediately see that $\lambda_i' = 0$ for all i. The case of p' is more subtle, but because all agents are in the picture, we only need to know that every good can be associated with at least one agent i for which it is not tight in the equilibrium. But that follows from part (b) of our assumption of ample survivability in Theorem 2.2. If a good were tight for every agent, there would be no freedom for the strict inequality in that condition to hold. Hence $p' = 0$ as well.

The linear transformation in (44) furnishes us with the existence of constants $\alpha > 0$ and $\beta > 0$ such that

$$\begin{array}{l} \sum_{i=1}^r x_i' = 0, \qquad m_i' + \bar{p}{\cdot}x_i' + p'{\cdot}(\bar{x}_i - \bar{x}_i^0) = 0, \qquad (m_i', x_i') \in K_i - K_i \\ \qquad \Longrightarrow \quad |\lambda_i'| \le \alpha ||(m_i', x_i')||_\infty, \qquad ||p'||_\infty \le \beta ||(m_i', x_i')||_\infty. \end{array} \tag{45}$$

Then for the final version of the quadratic form in (38) we have the existence of a constant $\gamma > 0$ such that, under (41),

$$\left| p'{\cdot} \sum_{i=1}^r \bar{\lambda}_i^{-1} \lambda_i' (\bar{x}_i^0 - \bar{x}_i) \right| \le \varepsilon\gamma ||(m', x')||_\infty^2. \tag{46}$$

However, we also have, through the negative definiteness secured in (39), a constant $\rho > 0$ such that

$$\sum_{i=1}^r \bar{\lambda}_i^{-1} (m_i', x_i'){\cdot}H_i(m_i', x_i') \le -\rho ||(m', x')||_\infty^2. \tag{47}$$

Combining (46) and (47), we see that

$$p'\cdot\sum_{i=1}^{r}\bar\lambda_i^{-1}\lambda_i'[\bar x_i^0-\bar x_i]-\sum_{i=1}^{r}\bar\lambda_i^{-1}(m_i',x_i')\cdot H_i(m_i',x_i')\geq(\rho-\varepsilon\gamma)||(m',x')||_\infty^2. \quad (48)$$

Therefore, if we add to the other conditions set forth in (39) the assumption that the left side of (48) is ≤ 0, we get the right side to be ≤ 0, too. Because $\rho>0$, that implies, for ε sufficiently small, that $(m',x')=(0,0)$. But then, via the linear transformation in (44), we must have $(p',\lambda')=(0,0)$ as well. □

Extrapolation. The conclusions of Theorem 4.1 can be amplified in the light of strong metric regularity being a property that holds on a neighborhood. We are provided with a localization s of the equilibrium mapping S which is single-valued and Lipschitz continuous on a neighborhood of $(\bar m^0,\bar x^0)$ and has $(\bar p,\bar m,\bar x,\bar\lambda)=s(\bar m^0,\bar x^0)$. For any $(\tilde m^0,\tilde x^0)$ in that neighborhood and the associated equilibrium $(\tilde p,\tilde m,\tilde x,\tilde\lambda)=s(\tilde m^0,\tilde x^0)$, the same properties will then hold, and if it is close enough, the bound in (41) will be preserved. (The same coefficients α, β, γ, will continue to work because the elements from which they derive change only gradually in the presence of the Lipschitz continuity.) The semidifferentiability will therefore persist as well, with the directional derivatives given by the same formulas except for passing to the new elements and the corresponding new versions of the Jacobians A and B. Thus, everything in Theorem 4.1 has "regional" meaning for S and is not limited to the particular equilibrium under investigation.

5. Interpretation and an example of instability

The import of Theorem 4.1 may be somewhat surprising and perhaps even mysterious. Parametric stability with respect to the initial endowments m_i^0 and x_i^0 is assured as long at these are near enough to the eventual holdings $\bar m_i$ and $\bar x_i$, but why should that distance have any role? After all, these endowments enter utility optimization constraints of the agents only by fixing the total amount of money available in the budgets through the equations.

$$m_i+p\cdot x_i=m_i^0+p\cdot x_i^0 \quad \text{for } i=1,\ldots,r. \quad (48)$$

What seems odd is the following. There can be other initial endowments than (m^0,x^0) that support the same equilibrium $(\bar p,\bar m,\bar x,\bar\lambda)$, but if they require too great a move to reach it, they might, according to Theorem 4.1, create parametric instability.

Proposition. *With respect to an equilibrium* $(\bar p,\bar m,\bar x,\bar\lambda)$ *associated with* $(\bar m^0,\bar x^0)$, *let*

$$E(\bar p)=\left\{(m^0,x^0)\,\middle|\,m_i^0+\bar p\cdot x_i^0=\bar m_i^0+\bar p\cdot\bar x_i^0 \text{ for } i=1,\ldots,r\right\}, \quad (49)$$

noting that $(\bar m^0,\bar x^0)\in E(\bar p)$ *in particular. Then for every* $(m^0,x^0)\in E(\bar p)$ *the equilibrium mapping* S *in* (11) *will have* $(\bar p,\bar m,\bar x,\bar\lambda)\in S(m^0,x^0)$, *and yet the criterion in Theorem* 4.1 *will not guarantee that* S *is strongly metrically regular at* (m^0,x^0) *for* $(\bar p,\bar m,\bar x,\bar\lambda)$ *unless* (m^0,x^0) *is sufficiently close to* $(\bar m^0,\bar x^0)$

Proof. This is a simple consequence of the observation that, with the price vector fixed and the total money available in the budget thereby fixed at the same level, the agents will arrive at the same optimal solutions $(\bar{m}_i, \bar{x}_i)$ and multipliers $\bar{\lambda}_i$. □

An analogy of sorts can be made with nonconvex optimization. Although in convex optimization global results about solutions prevail, in nonconvex optimization the difficulty of reaching a solution may well depend on the starting point. Starting too far away may even introduce uncertainty in the particular solution might be reached (if any), since there can be multiple solutions isolated from each other. In our equilibrium framework, each agent's utility maximization belongs to the category of convex optimization, but nonconvexity enters the overall picture through interactions between the agents.

To confirm that the threat of instability is real, we offer a small example in which the initial endowments lead to two separate equilibria, a "good" one which exhibits strong regularity with respect to the endowments as parameters and a "bad" one which does not.

Example overview. There are only two agents, $i = 1, 2$, and besides money, in "dollars", only one other good, "cookies". The agents' variables are m_i for dollars and x_i for cookies, with initial endowments m_i^0 and x_i^0. The dollar price per cookie is p. There are 24 dollars and 24 cookies in total, distributed initially as

$$m_1^0 = 1, \quad x_1^0 = 23, \quad m_2^0 = 23, \quad x_2^0 = 1. \tag{50}$$

The survival sets are $U_i = I\!R^2_+$, but they will not actually come into play; everything will take place in their interiors.

The utility functions will be chosen in such a manner that a bad equilibrium (without parametric stability) occurs with

$$\bar{p} = 1, \quad \bar{m}_1 = \bar{x}_1 = 12, \quad \bar{m}_2 = \bar{x}_2 = 12, \tag{51}$$

whereas a good equilibrium (with parametric stability) occurs with

$$\tilde{p} = 5/6, \quad \tilde{m}_1 = \tilde{x}_1 = 11, \quad \tilde{m}_2 = \tilde{x}_2 = 13. \tag{52}$$

The slightest shift of initial money from agent 1 to agent 2 will destroy the bad equilibrium entirely, whereas an opposite shift of initial money from agent 2 to agent one bifurcates the bad equilibrium into two good ones, but with the cookie price changing "infinitely fast" at first. Under such shifts of initial money the second equilibrium behaves smoothly.

Counterexamples are often constructed backwards from the desired outcome, which in this case is to make the cubic equation in p that will characterize equilibrium have $\bar{p} = 1$ as a double root. Although this background can be disguised, the algebra is simpler if the exposition takes advantage of the structure by working with the relative variables

$$\begin{aligned} n_1 = m_1 - 12,\ n_2 = m_2 - 12,\ y_1 = x_1 - 12,\ y_2 = x_2 - 12, \\ n_1^0 = m_1^0 - 12,\ n_2^0 = m_2^0 - 12,\ y_1^0 = x_1^0 - 12,\ y_2^0 = x_2^0 - 12. \end{aligned} \tag{53}$$

The equilibrium values in (51) and (52) are thereby shifted accordingly. The value 12 plays no real role (and has no effect on prices). It merely has to be big enough that the goods values in (51) and (52) come out positive relative to the shifts of size 11 corresponding to (53). that amount 11 indeed being a consequence of the specification of the utility functions.

We take the concave utility functions to be separable and strictly *quadratic*. They fail then to be nondecreasing globally over the survival sets $U_i = I\!R^2_+$, but this will not matter as long as they are increasing in a region that encompasses the endowments and the values in (51) and (52). To make them have the desired monotonicity, they could then be modified smoothly in a way that would not change the equilibrium results.

To prepare the way for the discussion of parametric instability, a parameter μ corresponding to a shift of intitial money to Agent 1 from Agent 2 is incorporated. Thus, we work with

$$n_1^0 = -11 + \mu, \quad n_2^0 = 11 - \mu, \quad y_1^0 = 11, \quad y_2^0 = -11. \tag{54}$$

Optimization for agent 1. Maximize $4n_1 + 4y_1 - n_1^2 - \frac{1}{2}y_1^2$ subject to the budget constraint

$$0 = [n_1 + 11 - \mu] + p[y_1 - 11], \quad \text{so that } n_1 = -11 + \mu - py_1 + 11p. \tag{a1}$$

With λ_1 as the Lagrange multiplier, the optimality conditions (neglecting nonnegativity, which will be automatic) are

$$0 = 4 - 2n_1 - \lambda_1 \quad \text{so that } \lambda_1 = 26 - 2\mu - 22p + 2py_1, \tag{b1}$$

together with

$$0 = 4 - y_1 - \lambda_1 p = 4 - y_1 - 26p + 2\mu p - 2p^2 y_1 + 22p^2,$$

which can be organized as

$$0 = 2[(p-1)(11p-2) + \mu p] - y_1(2p^2 + 1). \tag{c1}$$

This results in the demand formula

$$y_1 = 2[(p-1)(11p-2) + \mu p]/[2p^2 + 1]. \tag{d1}$$

The quadratic utility for this agent increases in the region where $n_1 < 2$ and $y_1 < 4$.

In converting from the relative n, y, variables back to the original m, x, variables, a constant could of course be added to the utility to make it vanish at the origin. This would have no affect on the equilibrium calculations.

Optimization for agent 2. Maximize $8n_2 + 8y_2 - n_2^2 - \frac{3}{2}y_2^2$ subject to the budget constraint

$$0 = [n_2 - 11 + \mu] + p[y_2 + 11], \quad \text{so that } n_2 = 11 - \mu - py_2 - 11p. \tag{a2}$$

With λ_2 as the Lagrange multiplier, the optimality conditions are

$$0 = 8 - 2n_2 - \lambda_2 \quad \text{so that } \lambda_2 = -14 + 2\mu + 2py_2 + 22p, \tag{b2}$$

together with

$$0 = 8 - 3y_2 - \lambda_2 p = 8 - 3y_2 + 14p - 2\mu p - 2p^2 y_2 - 22p^2,$$

which can be organized as

$$0 = -2[(p-1)(11p+4) + \mu p] - y_2(2p^2+3). \tag{c2}$$

This results in the demand formula

$$y_2 = -2[(p-1)(11p+4) + \mu p]/[2p^2+3]. \tag{d2}$$

The quadratic utility for this agent increases in the region where $n_2 < 4$ and $y_2 < 8/3$.

An equilibrium price is signaled by inducing $y_1 + y_2 = 0$, which then implies $n_1 + n_2 = 0$ through (a1) and (a2). We can exploit this in more than one way. It's convenient first to add (c1) and (c2) with $y_2 = -y_1$, getting

$$\begin{aligned} 0 &= 2[(p-1)(11p-2) + \mu p] - y_1(2p^2+1) \\ &\qquad - 2[(p-1)(11p+4) + \mu p] + y_1(2p^2+3) \\ &= 2(p-1)[(11p-2) - (11p+4)] + y_1[(2p^2+3) - (2p^2+1)] \\ &= -12(p-1) + 2y_1, \end{aligned}$$

from which it follows that

$$y_1 = 6(p-1) = -y_2, \qquad n_1 = (1-p)(6p-1) + \mu = -n_2. \tag{55}$$

Substituting this expression for y_1 in (c1) and dividing by -2, yields the equation

$$\begin{aligned} 0 &= -(p-1)(11p-2) - \mu p + 3(p-1)(2p^2+1) \\ &= (p-1)[-11p + 2 + 6p^2 + 3] - \mu p, \end{aligned}$$

which works out to

$$0 = h_\mu(p) =: (p-1)^2(6p-5) - \mu p. \tag{56}$$

In combination with (55) this yields, for $\mu = 0$, the equilibria

$$\bar{p} = 1, \quad \bar{n}_1 = \bar{y}_1 = 0, \quad \bar{n}_2 = \bar{y}_2 = 0, \tag{57}$$

$$\tilde{p} = 5/6, \quad \tilde{n}_1 = \tilde{y}_1 = -1, \quad \tilde{n}_2 = \tilde{y}_2 = +1, \tag{58}$$

both of which do fall in the region where the respective utility functions are increasing. With the translation back to the original variables, they give the equilibria in (51) and (52).

The cubic polynomial h_0 has a local minimum at the double root 1. In passing to h_μ with $\mu > 0$, it is tilted by subtracting a linear function, and this has the initial effect of making the local minimum drop below the p-axis. The double root thereby splits infinitely fast, with respect to μ, into a close pair of roots; the associated values of y_1 and y_2 likewise, through (55), shift at an initially infinite rate. However, in passing to h_μ with $\mu < 0$, the cubic graph tilts up and region of the local minimum no longer touches the p-axis. There is then only one equilibrium, corresponding to the perturbed value of $\tilde{p}$, and it behaves nicely.

Tying in with Theorem 4.1. It is instructive to return now the observations in the Proposition earlier in this section. The trouble with the equilibrium (51) has arisen not because of anything inherent in this equilibrium, but only because the initial endowments were too far away. If these endowments were replaced by others sufficiently near and yielding the same budgets with respect to $\bar{p} = 1$, everything would be all right.

6. Proof of Theorem 2.2

We return now to the unfinished task of proving Theorem 2.2 of Section 2, which resides in a broader framework of utility than the one we moved into subsequently. The associated variational inequality (9), of functional instead of geometric type, puts together (6), (7) and (8). The price positivity assertion of Theorem 2.2 comes right out of (7) and the assumption that for each good there is at least one agent i for which u_i increases with respect to that good.

Proving the rest of Theorem 2.2 is harder. The challenge is that the basic existence criterion for a functional variational inequality, namely the continuity of f relative to the closure of the effective domain of φ, with that domain being *bounded*, fails through the absence of boundedness. Our tactic will be to systematically replace the targeted variational inequality by others through step by step truncation, all the while preserving the same set of solutions. By arriving ultimately at a version in which the effective domain is indeed bounded, we will have secured the existence. In this process we can work with successively modified conditions displayed as in (6), (7) and (8), rather in the format of (9).

Some further notation will help us along the path. In taking $j = 1, \dots, n$, as the index for the goods beyond money, the corresponding components of x_i and p can be denoted by x_{ij} and p_j. Then, in terms of

$$\Gamma = N_{\mathbb{R}_+}, \quad \text{with } \beta \in \Gamma(\alpha) \iff \begin{cases} \beta \le 0 & \text{if } \alpha = 0, \\ \beta = 0 & \text{if } \alpha > 0, \end{cases} \tag{59}$$

we can reconstitute (6) as

$$\sum\nolimits_{i=1}^{r} [x_{ij} - x_{ij}^0] \in \Gamma(p_j) \quad \text{for } j = 1, \dots, n, \tag{6_0}$$

and write (8) as

$$m_i - m_i^0 + p \cdot [x_i - x_i^0] \in \Gamma(\lambda_i) \quad \text{for } i = 1, \dots, r. \tag{8_0}$$

The original variational inequality, now viewed as (6_0), (7) and (8_0), will be referred to as $\mathcal{V}_0$.

A device in our truncation approach will be to replace Γ in some cases by

$$\Gamma_\eta = N_{[0,\eta]}, \quad \text{with } \beta \in \Gamma_\eta(\alpha) \iff \begin{cases} \beta \le 0 & \text{if } \alpha = 0 \\ \beta = 0 & \text{if } 0 < \alpha < \eta \\ \beta \ge 0 & \text{if } \alpha = \eta \end{cases} \iff \alpha\beta = \eta \max\{0, \beta\}. \tag{60}$$

Let $\mathcal{V}_1(\eta)$ denote the variational inequality obtained this way from $\mathcal{V}_0$ in replacing (6_0) by

$$\sum\nolimits_{i=1}^{r} [x_{ij} - x_{ij}^0] \in \Gamma_\eta(p_j) \quad \text{for } j = 1, \ldots, n, \tag{6(η)}$$

which implies that

$$p_j \sum\nolimits_{i=1}^{r} [x_{ij} - x_{ij}^0] = \eta \max\Big\{0, \sum\nolimits_{i=1}^{r} [x_{ij} - x_{ij}^0]\Big\}. \tag{61}$$

This change does not effect conditions (7) and (8_0), which characterize saddle point optimality for the Lagrangians in the maximization problems of the agents on the space of goods.

Next, though, we do wish to look at truncations of goods. Let

$$G_\mu = \big\{\text{the vectors in } \mathbb{R}^{1+n} \text{ having all components} \le \mu\big\}. \tag{62}$$

We fix $\eta \in (0, \infty)$ for now and call on stage the vectors $(\hat{m}_i, \hat{x}_i)$ in the ample survivablity assumption in Section 2. We choose

$$\bar{\mu} \text{ high enough that } (\hat{m}_i, \hat{x}_i) \in G_\mu \quad \text{for } i = 1, \ldots, r. \tag{63}$$

For $\mu \in [\bar{\mu}, \infty)$ we define truncations of the extended-real-valued utilities $\tilde{u}_i$ in (7) by

$$\tilde{u}_i^\mu(m_i, x_i) = \begin{cases} u_i(m_i, x_i) & \text{if } (m_i, x_i) \in U_i^\mu = U_i \cap G_\mu \text{ and } u_i(m_i, x_i) \ge u_i(\hat{m}_i, \hat{x}_i) - 1, \\ -\infty & \text{otherwise.} \end{cases} \tag{64}$$

These functions are still concave and usc, and their essential domains U_i^μ are nonempty, convex and *bounded.* The modified subgradient condition

$$-(\lambda_i, \lambda_i p) \in \partial[-\tilde{u}_i^\mu](m_i, x_i) \tag{7(μ)}$$

can therefore serve as a replacement for (7) and will be equivalent to it when applied to pairs (m_i, x_i) not hitting the upper bound dictated by μ. The variational inequality comprised of the conditions $(6(\eta))$, $(7(\mu))$ and (8_0) will be denoted by $\mathcal{V}_2(\eta, \mu)$.

For easy reference, we symbolize the original price-driven utility maximization of agent i by $\mathcal{P}_i(p)$ and let $\mathcal{P}_i^\mu(p)$ stand for the modified version in which the functions $\tilde{u}_i^\mu$ have replaced the functions $\tilde{u}_i$. We then obviously have the truth of the following.

Lemma 6.1. *For the truncated problems $\mathcal{P}_i^\mu(p)$, conditions (7(μ)) and (8_0) characterize optimality in terms of a saddle point of the corresponding Lagrangian function*

$$L_i^\mu(p; m_i, x_i, \lambda_i) = \tilde{u}_i^\mu(m_i, x_i) - \lambda_i(m_i - m_i^0 + p\cdot[x_i - x_i^0]), \tag{65}$$

just as (7) and (8_0) do for the problem $\mathcal{P}_i(p)$.

The variational inequality $\mathcal{V}_2(\eta, \mu)$ thus stands for a version of the original variational inequality $\mathcal{V}_0$ in which, along with a truncation of the goods available to the agents, a kind of η-relaxation has been introduced in the supply-demand requirement.

Lemma 6.2. *There exist $\bar\eta \in (0, \infty)$ and $\bar\mu \in (0, \infty)$ such that, when $\eta \in [\bar\eta, \infty)$ and $\mu \in [\bar\mu, \infty)$, the solutions of the variational inequality $\mathcal{V}_1(\eta)$ (if any) are the same as those of the variational inequality $\mathcal{V}_2(\eta, \mu)$.*

The argument is that a solution to $\mathcal{V}_2(\eta, \mu)$ will also solve $\mathcal{V}_1(\eta)$ if the additional bounds in the truncated problems $\mathcal{P}_i^\mu(p)$ do not come into play. We have this at least for the utility bound incorporated in (64), because (63) makes $(\hat m_i, \hat x_i)$ be feasible for agent i by (63), ensuring that any optimal solution (m_i, x_i) must have $\tilde u_i(m_i, x_i) \geq \tilde u_i(\hat m_i, \hat x_i) > \tilde u_i(\hat m_i, \hat x_i) - 1$. The remaining issue about bounds in Lemma 6.2 will be resolved by showing that condition (6(η)), common to both of the variational inequalities, entails bounds which make the μ bounds superfluous when μ is high enough.

To ascertain this, we add the budgets constraints for the agents, which we know can be treated as equation in optimality, getting

$$0 = \sum\nolimits_{i=1}^r [m_i - m_i^0 + p\cdot(x_i - x_i^0)] = \sum\nolimits_{i=1}^r (m_i - m_i^0) + \sum\nolimits_{j=1}^n p_j \sum\nolimits_{i=1}^r (x_{ij} - x_{ij}^0). \tag{66}$$

and consequently from (61) that

$$\sum\nolimits_{i=1}^r m_i + \eta \sum\nolimits_{j=1}^n \max\Big\{0, \sum\nolimits_{i=1}^r [x_{ij} - x_{ij}^0]\Big\} = \sum\nolimits_{i=1}^r m_i^0. \tag{67}$$

By lowering η to $\bar\eta$ in this inequality, we derive upper bounds on the (nonnegative) m_i's and x_{ij}'s which hold for any $\eta \in [\bar\eta, \infty)$ and are independent of p (as long as $p_j \in [0, \bar\eta]$, a circumstance which we must make sure about later). We then merely need to take μ high enough to keep it out of the way.

Lemma 6.3. *For $\bar\eta$ and $\bar\mu$ as in Lemma 6.2, there further exists $\bar\zeta \in (0, \infty)$ large enough that, for any $\eta \in [\bar\eta, \infty)$ and $\mu \in [\bar\mu, \infty)$, solutions to $\mathcal{V}_2(\eta, \mu)$ will have*

$$\tilde u_i(m_i, x_i) \leq \zeta \quad \text{and} \quad \lambda_i < \zeta. \tag{68}$$

The existence of first bound in (68) comes from the bounds in Lemma 6.2 relative to $\bar\mu$ and the upper semicontinuity of $\tilde u_i$. In terms of ζ_i^* being the max of $\tilde u_i$ over a closed set associated with those earlier bounds, we can take $\zeta^* = \max\{\zeta_1^*, \ldots, \zeta_r^*\}$ to get a bound independent of any particular agent:

$$\tilde u_i(m_i, x_i) \leq \zeta^*. \tag{69}$$

A bound on λ_i in Lemma 6.2 will be derived from this and the saddle point condition for optimality in the maximization problem $\mathcal{P}_i^\mu(p)$. The maximization part of this saddle point condition for the Lagrangian L_i^μ in (65) at $(m_i, x_i; \lambda_i)$ tells us that

$$L_i^\mu(m_i, x_i; \lambda_i) \geq L_i^\mu(\hat{m}_i, \hat{x}_i; \lambda_i). \tag{70}$$

Because the budget constraint must hold as an equation in optimality, we know that

$$L_i^\mu(m_i, x_i; \lambda_i) = \tilde{u}_i(m_i, x_i). \tag{71}$$

The combination of (69), (70) and (71) reveals through the expression (65) for the Lagrangian that

$$\zeta^* \geq \tilde{u}_i(\hat{m}_i, \hat{x}_i) - \lambda_i(\hat{m}_i - m_i^0 + p{\cdot}[\hat{x}_i - x_i^0]). \tag{72}$$

From part (a) of the ample survivability assumption we have $\hat{m}_i - m_i^0 < 0$ and $\hat{x}_i - x_i^0 \leq 0$, hence $p{\cdot}[\hat{x}_i - x_i^0] \leq 0$, and this leads us to the upper bound

$$\lambda_i \leq [\zeta^* - \tilde{u}_i(\hat{m}_i, \hat{x}_i)]/[m_i^0 - \hat{m}_i]. \tag{73}$$

By taking $\bar{\zeta}$ higher than this bound and ζ^*, we finish the proof of Lemma 6.3.

Having the bounds in Lemma 6.3 we can move toward truncating the last untruncated part the variational inequality, which is in the budget condition (8_0). We replace it now by

$$m_i - m_i^0 + p{\cdot}[x_i - x_i^0] \in \Gamma_\zeta(\lambda_i) \quad \text{for } i = 1, \dots, r. \tag{8(ζ)}$$

Conditions $(6(\eta))$, $(7(\mu))$ and now $(8(\zeta))$ constitute a variational inequality which will be denoted by $\mathcal{V}_3(\eta, \mu, \zeta)$.

Lemma 6.4. *For $\bar{\eta}$, $\bar{\mu}$ and $\bar{\zeta}$ as in Lemmas* 6.2 *and* 6.3*, and the variational inequality $\mathcal{V}_3(\eta, \mu \zeta)$ with respect to any choice of $\eta \in [\bar{\eta}, \infty)$, $\mu \in [\bar{\mu}, \infty)$ and $\zeta \in [\bar{\zeta}, \infty)$,*

(*a*) *solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$ coincide with solutions to $\mathcal{V}_2(\eta, \mu)$,*

(*b*) *a solution to $\mathcal{V}_3(\eta, \mu, \zeta)$ exists.*

In (*a*) we have put together the properties already developed in Lemma 6.3, while in (*b*) we have merely recorded the consequence of the boundedness of the truncated domains in $\mathcal{V}_3(\eta, \mu, \zeta)$ for applicabiity of the basic existence criterion.

However, this is not the end of the story. We are obliged still to demonstrate that by taking η large enough the upper price bound of η built into $(6(\eta))$ can surely be made inactive. Then we will be able to conclude that solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$ are the same as solutions to the original variational inequality, presented as $\mathcal{V}_0$, which are guaranteed to exist.

The key at this stage is developing a positive lower bound on the multipliers λ_i to combine with the upper bound in Lemma 6.3.

Lemma 6.5. *For $\bar{\eta}$, $\bar{\mu}$ and $\bar{\zeta}$ as in Lemmas* 6.2 *and* 6.3*, there exists $\varepsilon > 0$ such that, as long as $\mu \in [\bar{\mu} + 1, \infty)$ and $\zeta \in [\bar{\zeta}, \infty)$, together with $\eta \in [\bar{\eta}, \infty)$, solutions to $\mathcal{V}_3(\eta, \mu, \zeta)$ will have*

$$\lambda_i \geq \varepsilon \quad \text{for } i = 1, \dots, r. \tag{74}$$

In proving this Lemma we look to condition (7(μ), noting that it implies for solutions to $\mathcal{V}_3(\eta,\mu,\zeta)$ that

$$\tilde{u}_i^\mu(m_i+1,x_i) \le \tilde{u}_i^\mu(m_i,x_i) + \lambda_i. \tag{75}$$

In Lemma 6.2 we have also $(m_i,x_i) \in G_{\bar\mu}$, and in that case $(m_i+1,x_i) \in G_\mu$ because $\mu \ge \bar\mu + 1$. Then, by the definition of $\tilde{u}_i^\mu$ in (64), we have $\tilde{u}_i^\mu(m_i,x_i) = \tilde{u}_i^\mu(m_i,x_i)$ and $\tilde{u}_i^\mu(m_i+1,x_i) = \tilde{u}_i(m_i+1,x_i)$, moreover $\tilde{u}_i(\hat{m}_i, x\hat{x}_i) - 1 \le \tilde{u}_i^\mu(m_i+1,x_i)$. That gives us through (75) that

$$\tilde{u}_i(\hat{m}_i,\hat{x}_i) - 1 \le \tilde{u}_i(m_i+1,x_i) \le \tilde{u}_i(m_i,x_i) + \lambda_i. \tag{76}$$

We claim that for pairs $(m_i,x_i) \in G_{\bar\mu}$, even if not part of a solution to $\mathcal{V}_3(\eta,\mu,\zeta)$, there is a positive lower bound to the values of λ_i occurring in (75). The argument is by contradiction. If there is no such lower bound, we can produce with $k = 1,2,\ldots,$ a sequence of pairs $(m_i^k,x_i^k) \in G_{\bar\mu}$ and multipliers λ_i^k having

$$\tilde{u}_i(\hat{m}_i,\hat{x}_i) - 1 \le \tilde{u}_i(m_i^k+1,x_i^k) \le \tilde{u}_i(m_i^k,x_i^k) + \lambda_i^k \quad \text{and} \quad \lim_{k\to\infty} \lambda_i^k = 0. \tag{77}$$

The sequence is bounded, so we can suppose that (m_i^k,x_i^k) converges to some pair (m_i^∞,x_i^∞). Then (m_i^k+1,x_i^k) converges to $(m_i^\infty+1,x_i^\infty)$. Because of our assumption that $\tilde{u}_i$ is continuous on U_i and usc in the large, so that the set $\{(m_i,x_i) \,|\, \tilde{u}_i(m_i,x_i) \ge \tilde{u}_i(\hat{m}_i,\hat{x}_i) - 1\}$ is closed, we get from (77) in the limit as $k \to \infty$ that $\tilde{u}_i(m_i^\infty+1,x_i^\infty) \le \tilde{u}_i(m_i^\infty,x_i^\infty)$. This is impossible because utility is supposed always to increase when money increases.

Lemma 6.6. *For $\bar\eta$, $\bar\mu$ and $\bar\zeta$ as in Lemmas 6.2 and 6.3, there is a bound ψ such that, in any solution to the variational inequality $\mathcal{V}_3(\eta,\mu,\zeta)$ with $\eta \in [\bar\eta,\infty)$, $\mu \in [\bar\mu+1,\infty)$ and $\zeta \in [\bar\zeta,\infty)$, the prices will satisfy $p_j < \psi$ for $j = 1,\ldots,n$.*

This comes from the lower bounds in Lemma 6.5 and the inequality (72), where ζ^* can be supplanted by $\bar\zeta$, which was chosen to be higher. Recalling that $\hat{m}_i < m_i^0$ and $\hat{x}_i < x_i^0$ from part (a) of the assumption of ample survivability, we see that

$$\bar\zeta - \tilde{u}_i(\hat{m}_i,\hat{x}_i) \ge \lambda_i(m_i^0 - \hat{m}_i + p{\cdot}[x_i^0 - \hat{x}_i]) \ge \varepsilon p{\cdot}[x_i^0 - \hat{x}_i].$$

Adding over the agents, we get

$$r\bar\zeta - \sum_{i=1}^r \tilde{u}_i(\hat{m}_i,\hat{x}_i) \;\ge\; \varepsilon p \cdot \sum_{i=1}^r [x_i^0 - \hat{x}_i].$$

Here $\sum_{i=1}^r [x_i^0 - \hat{x}_i] > 0$ from part (b) of the assumption of ample survivability, so this inequality induces upper bounds on all the price components of p.

Final argument. Lemma 6.4 assured us that, when μ and ζ are large enough, the solutions to the variational inequality $\mathcal{V}_3(\eta,\mu,\zeta)$, which we do know to exist, agree with the solutions to the variational inequality $\mathcal{V}_1(\eta)$ for all $\eta \in [\bar\eta,\infty)$. When η exceeds the ψ in Lemma 6.6, we are guaranteed that the remaining constraints introduced artificially through truncation will likewise be inactive in solutions to $\mathcal{V}_3(\eta,\mu,\zeta)$, hence also in solutions to $\mathcal{V}_1(\eta)$. In those circumstances the solutions to $\mathcal{V}_1(\eta)$ are identical to the solutions to $\mathcal{V}_0$, and we may conclude that a solution to $\mathcal{V}_0$ exists.

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