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Perturbation Method for Variational Problems*

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We provide a general method for proving existence of solutions of suitable perturbations of certain variational problems. A novel variational principle enables perturbing only the integrand, thus preserving the form of the problem.

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1. Introduction

We prove the following existence theorem.

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a Banach space. Let $f : X \to \mathbb{R} \cup \{\infty\}$ be closed, convex and such that $f \geq 0$, $f(0) = 0$ and $f \geq k\|\cdot\|$ for some $k > 0$. Let $a \in \operatorname{dom} f \setminus \{0\}$ be fixed.*

Consider the optimisation problem

$$(V_{\|\cdot\|}) \begin{cases} \int_0^\infty (\|v(t)\|^2 + f(u(t))\; dt \to \min \\ u(t) = a + \int_0^t v(s)\, ds, \qquad\qquad v \in L^2([0,\infty), X)\,. \end{cases} \tag{1}$$

For each $\varepsilon > 0$ there is equivalent norm $|\cdot|$ on X such that

$$\|\cdot\| \leq |\cdot| \leq (1+\varepsilon)\|\cdot\|$$

and the problem $(V_{|\cdot|})$ has a solution.

From the proof it is clear that the statement holds for f with strong minimum at 0, that is, if $x_n \to 0$ whenever $f(x_n) \to 0$.

The above result does not follow directly from Ekeland Variational Principle.

The key to obtaining it is the abstract Theorem 3.2.

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2. Preliminaries

2.1. Annotations

Even though some of the results can be – in an obvious manner – extended to complete metric spaces, we prefer to work on closed subsets of Banach spaces for the sake of brevity.

For a Banach space $(X, \|\cdot\|)$ the unit ball, resp. sphere, are denoted by $B_X = \{x : \|x\| \le 1\}$, resp. $S_X = \{x : \ \|x\| = 1\}$. For a nonempty set $A \subset X$ the distance from x to A is denoted by $d(x, A) = \inf\{\|y - x\| : \ y \in A\}$. The indicator function $\boldsymbol{\delta}_A$ of a set $A \subset X$ equals 0 on A and ∞ outside A. We denote $\boldsymbol{\delta}_n = \boldsymbol{\delta}_{nB_X}$ for $n \in \mathbb{N}$.

The function $f : X \to \mathbb{R} \cup \{\infty\}$, where X is a Banach space, is called closed if its *epigraph*

$$\operatorname{epi} f = \{(x, t); \ f(x) \le t\}$$

is closed, and proper if its *domain*

$$\operatorname{dom} f = \{x; \ f(x) < \infty\}$$

is non-empty.

The topic of epigraph convergence is vast, e.g. [1, 2, 8], and we do not consider full generality. Instead, we fix a metric, ρ_e, which is sufficient for the applications we pursue. For two proper functions $f, g : X \to \mathbb{R} \cup \{\infty\}$ we define

$$\rho_\infty(f, g) = \sup\{|f(x) - g(x)|; \ x \in \operatorname{dom} f \cup \operatorname{dom} g\},$$

$$\rho_{\infty,n}(f, g) = \sup\{|f(x) - g(x)|; \ x \in (\operatorname{dom} f \cup \operatorname{dom} g) \cap nB_X\},$$

and

$$\rho_e(f, g) = \sum_{n=1}^{\infty} 2^{-n} \rho_{\infty,n}(f, g). \tag{2}$$

Obviously, $\rho_\infty(f, g) = \rho_e(f, g) = \infty$ if $\operatorname{dom} f \ne \operatorname{dom} g$. The meaning of ρ_e is that it implies a kind of uniform on bounded sets convergence, if we adopt the rule $\infty - \infty = 0$. Note also that if $\rho_e(f_n, 0) \to 0$ as $n \to \infty$ (the zero in the brackets stands for zero function) then $\rho_e(f + f_n, f) \to 0$.

2.2. Cantor-Kuratowski-De Blasi Lemma

Kuratowski, e.g. [7], proved a generalisation of Cantor Lemma in terms of the measure of non-compactness he defined. This line was extended by De Blasi [3]. He defined the *measure of weak non-compactness* of a subset A of a Banach space X as

$$\beta(A) = \inf\{\varepsilon; \text{ there is weakly compact } B \text{ s.t. } A \subset B + \varepsilon B_X\}.$$

It is obvious that $\beta(A) = 0$ if and only if the weak closure of A is compact; $\beta(A) = \beta(\overline{\mathrm{co}} A)$ and $\beta(A_1) \le \beta(A_2)$ if $A_1 \subset A_2$.

Lemma 2.1 ([3]). *Let $\{A_n\}_{n\in\mathbb{N}}$ be a nested ($A_{n+1} \subset A_n$, $\forall n$) family of weakly closed subsets of a Banach space such that*

$$\lim_{n\to\infty} \beta(A_n) = 0.$$

Then $A = \cap A_n$ is nonempty.

2.3. Optimisation

Let S be a closed convex subset of the Banach space X and let $f : S \to \mathbb{R} \cup \{\infty\}$ be closed, convex, proper and bounded below. Consider the optimisation problem

$$(f, S) \begin{cases} f(x) \to \min; \\ x \in S. \end{cases}$$

Following [8] we define

$$\varepsilon\text{-argmin}_S f = \{x \in S; f(x) \le \inf f + \varepsilon\}.$$

Of course, $\text{argmin}_S f = 0\text{-argmin}_S f$ is the usual set of minima of f on S.

Definition 2.2. We say that the problem (f, S) is *weakly well-posed* if

$$\lim_{\varepsilon \to 0} \beta\left(\varepsilon\text{-argmin}_S f\right) = 0. \tag{3}$$

From Hahn-Banach Theorem and Lemma 2.1 it follows that a weakly well-posed problem has solutions (that is, $\text{argmin}_S f \neq \emptyset$).

We need few simple facts. For similar results see [8].

Lemma 2.3. *If $a, b, c > 0$ and a-$\text{argmin}_S f \cap b$-$\text{argmin}_S g \neq \emptyset$ then*

$$c\text{-argmin}_S (f + g) \subset (a + b + c)\text{-argmin}_S f.$$

Proof. Let $x_0 \in S$ be such that $f(x_0) \le \inf_S f + a$ and $g(x_0) \le \inf_S g + b$. If $x \in S$ is such that $f(x) + g(x) \le \inf_S (f + g) + c$ then $f(x) + g(x) \le f(x_0) + g(x_0) + c$ and we can write $-b \le \inf_S g - g(x_0) \le g(x) - g(x_0) \le f(x_0) - f(x) + c$. Therefore, $f(x) \le f(x_0) + b + c \le \inf_S f + a + b + c$. □

Lemma 2.4. *Assume that S is bounded. If $\rho_e(f_n, f) \to 0$ then for $\delta \le \varepsilon/3$ and all n large enough*

$$\beta\left(\delta\text{-argmin}_S f_n\right) \le \beta\left(\varepsilon\text{-argmin}_S f\right).$$

Proof. Since S is bounded, we may assume that $\rho_\infty(f_n, f) < \delta$ for all n large enough. Let $g_n(x) = f_n(x) - f(x)$ if $x \in \text{dom}\, f = \text{dom}\, f_n$ and $g_n(x) = 0$ otherwise. Obviously, δ-$\text{argmin}_S g_n = S$. Since $f_n = f + g_n$, by Lemma 2.3 with $a = b = c = \delta$

$$\delta\text{-argmin}_S f_n \subset \varepsilon\text{-argmin}_S f.$$ □

2.4. Curves on Banach spaces

For detailed presentation of Bochner integral, see [6].

In short, $L^2([0, \infty), X)$ is the closure of the *stepwise* functions

$$\sum_{i=1}^{n} \chi_{(a_i, b_i)}(t) x_i, \quad x_i \in X; \qquad \chi_{(a_i, b_i)}(t) = \begin{cases} 1, & t \in (a_i, b_i), \\ 0, & \text{otherwise} \end{cases}$$

in the norm

$$\|v\|_2 = \left(\int_0^\infty \|v(t)\|^2 \, dt \right)^{1/2}, \quad v : [0, \infty) \to X.$$

For $v \in L^2([0, \infty), X)$

$$u(t) = \int_0^t v(s) \, ds$$

is well defined function from $[0, \infty)$ to X. Moreover, it satisfies

$$\|u(t_1) - u(t_2)\| \le \|v\|_2 \, |t_1 - t_2|^{1/2}, \quad \forall t_{1,2} \ge 0.$$

We denote $Y = L^2([0, \infty), X)$.

Lemma 2.5. *Let X_1 be a finite-dimensional subspace of X and $Y_1 = L^2([0, \infty), X_1)$. Then for any $v \in Y$*

$$d^2(v, Y_1) = \int_0^\infty d^2(v(t), X_1) \, dt.$$

Proof. It is immediate that $d^2(v, Y_1) \ge \int_0^\infty d^2(v(t), X_1) \, dt$.

Assume first that X is separable and $\| \cdot \|$ is strictly convex. Then $d(v(t), X_1) = \|v(t) - w(t)\|$ where $w(t) \in X_1$ is unique. By construction $w(t)$ is measurable: if v is almost everywhere limit of stepwise functions then $w(t)$ is almost everywhere limit of their metric projections over X_1 which will be also stepwise. Since $0 \in X_1$ we have $\|w(t) - v(t)\| \le \|v(t)\|$ and therefore $\|w(t)\| \le 2\|v(t)\|$, so $w \in Y_1$.

If X is separable one can approximate the norm by strictly convex norms, see e.g. [5]. The result follows by passing to the limit.

If X is arbitrary then it is known that v has essentially countably many values, so we may restrict our considerations to the *separable* closed linear span of $\{v(t) : t \in U\} \cup X_1$ where $U \subset [0, \infty)$ is of full measure. □

2.5. One dimensional Lemma

Lebesgue proved that his integral can be approximated by Riemann sums. However, we need approximation by trapeze formula and – being unable to provide suitable reference – we prove the partial case we use.

Lemma 2.6. *Let $f : [0, 1] \to \mathbb{R}^+$ be Lebesgue integrable. Then for each $\varepsilon, \delta > 0$ there is a partition Δ of $[0, 1]$ with diameter $< \delta$ (that is, $t_0 = 0 < t_1 < \ldots < t_{n+1} = 1$, $t_{i+1} - t_i < \delta$), such that*

$$\sum_{i=0}^{n} \frac{f(t_i) + f(t_{i+1})}{2} (t_{i+1} - t_i) < \int_0^1 f(t) \, dt + \varepsilon. \tag{4}$$

Proof. Since adding a constant to f changes nothing as far as (4) is concerned, we may assume that $f \ge 1$.

Let m be the Lebesgue measure. Since $\sum_{n=1}^{\infty} nm\left(\{t:\ n < f(t) \le n+1\}\right) < \infty$, for all $N \in \mathbb{N}$ large enough $Nm\left(\{t:\ N < f(t)\}\right) < \min\{\varepsilon, \delta\}$. We fix a N like this which also satisfies $N > \max\{f(0), f(1)\}$.

Let $A = \{t \in (0,1):\ f(t) \le N\}$ and B be the set of those $t \in A$ which are Lebesgue points of f. Then $m(A \setminus B) = 0$.

Moreover, for each $t \in B$ there is $\bar{t} \in (t, 1)$ such that $\bar{t} - t < \delta$, $f(\bar{t}) < N + 1$ and

$$\frac{f(t) + f(\bar{t})}{2}(\bar{t} - t) \le (1 + \varepsilon) \int_t^{\bar{t}} f(t)\, dt.$$

We consider $(t, \bar{t})$, $t \in B$, in the context of Sierpinski Lemma, e.g. [9, p. 356], and pick non-intersecting finite system of intervals $(t_j, \bar{t_j})$, $j = 1, \dots k$ such that the part of B not covered by their union has measure less than $\min\{\delta, \varepsilon/N\}$.

In an obvious manner this finite system can be completed to form a finite partition Δ of $[0,1]$. Since the total measure of $[0,1] \setminus \cup_1^k (t_j, \bar{t_j})$ is smaller than 2δ, any interval of Δ not belonging to the above finite system will have length less than 2δ. Those in the finite system are shorter than δ by construction. So, the diameter of Δ is smaller than 2δ.

In order to verify (4) we split the sum in the left hand side in two:

The sum over all intervals in the finite system is less than $(1 + \varepsilon) \int_0^1 f(t)\, dt$ by construction.

The sum over remaining intervals is smaller than the sum of their lengths, ergo $< 2\varepsilon/N$, times $\max_j\{f(t_j), f(\bar{t_j})\} \le N + 1$ and this product is smaller than 3ε. □

2.6. Graphical density of stepwise functions

Let $f : X \to \mathbb{R} \cup \{\infty\}$ and $a \in \operatorname{dom} f$ satisfy the conditions of Theorem 1.1, that is, f is closed, convex, $f \ge k\|\cdot\|$ and such that $f(0) = 0$; and $a \ne 0$. We can define

$$F_{\|\cdot\|} : L^2([0,\infty), X) \to \mathbb{R} \cup \{\infty\}$$

by

$$F_{\|\cdot\|}(v) = \int_0^{\infty} \left(\|v(t)\|^2 + f(u(t)\right)\, dt, \qquad u(t) = a + \int_0^t v(s)\, ds. \tag{5}$$

Obviously, the problem $(V_{\|\cdot\|})$, see (1), is equivalent to minimisation of $F_{\|\cdot\|}$ over $L^2([0,\infty), X)$.

Proposition 2.7. *Under the assumptions of Theorem* 1.1 *the above constructed* $F = F_{\|\cdot\|}$ *is proper, closed, convex and positive.*

Moreover, the stepwise functions are graphically dense in $\operatorname{dom} F$, *that is, for any* $v \in \operatorname{dom} F$ *there exists a sequence of stepwise* v_k *such that* $v_k \to v$ *and* $F(v_k) \to F(v)$.

Proof. It is clear that F is convex and positive. Also, considering $v = -a\chi_{(0,1)}$ and using the convexity of f, we see that F is proper.

If $\|v_k - v\|_2 \to 0$ then the corresponding u_k tend pointwise (in fact, uniformly on bounded intervals) to u, so Fatou Lemma gives

$$\liminf \int_0^\infty f(u_k(t))\,dt \geq \int_0^\infty \liminf f(u_k(t))\,dt \geq \int_0^\infty f(u(t))\,dt, \qquad (6)$$

using for the latter the closedness of f.

For the graphical density, fix $\varepsilon \in (0,1)$ and let $v \in \operatorname{dom} F$. We may assume that the corresponding u eventually vanishes. Indeed, there is $t_1 > 0$ such that $f(u(t_1)) < k\varepsilon/2$ and $\int_{t_1}^\infty (\|v(t)\|^2 + f(u(t)))\,dt < \varepsilon/2$. Let $v_1 \equiv v$ on $[0,t_1]$. If $u(t_1) = 0$ then $v_1(t) = 0$ for $t \geq t_1$, otherwise $v_1(t) = (t - t_1)h$ for $t \in (t_1, t_1 + \|u(t_1)\|)$, where $h = -u(t_1)/\|u(t_1)\|$, and $v_1(t) = 0$ for $t \geq t_1 + \|u(t_1)\|$. It is clear that the corresponding u_1 eventually vanishes and – using $\|u(t_1)\| < \varepsilon/2$ and the convexity of f – one easily estimates $\|v - v_1\|_2 < \sqrt{\varepsilon}$ and $|F(v) - F(v_1)| < \varepsilon$.

So, let us assume that $u(t) = v(t) = 0$ for $t > T$. Consider a partition $\Delta = \{t_0 = 0 < t_1 < \ldots < t_n = T\}$ of $[0,T]$ and define

$$v_\Delta(t) = (t_{i+1} - t_i)^{-1} \int_{t_i}^{t_{i+1}} v(s)\,ds, \quad t \in (t_i, t_{i+1}).$$

The corresponding u_Δ is piece-wise linear and such that $u_\Delta(t_i) = u(t_i)$.

For any partition Δ with $\operatorname{diam}\Delta = \max_i (t_{i+1} - t_i)$ small enough we have that $\|v - v_\Delta\|_2 < \varepsilon$. This is because $\|v_\Delta\|_2 \leq \|v\|_2$, as follows from Cauchy inequality, and the map $v \to v_\Delta$ is linear. So, if w is continuous $[0,T]$, vanishing on $[T,\infty)$ and such that $\|v - w\|_2 < \varepsilon/3$, then

$$\|v - v_\Delta\|_2 \leq \|v - w\|_2 + \|w - w_\Delta\|_2 + \|w_\Delta - v_\Delta\|_2 \leq \|w - w_\Delta\|_2 + 2\varepsilon/3.$$

But w is uniformly continuous on $[0,T]$ and therefore $\|w - w_\Delta\|_2 < \varepsilon/3$ for $\operatorname{diam}\Delta$ small enough.

We can now complete the proof. By convexity of f

$$\int_{t_i}^{t_{i+1}} f(u_\Delta(t))\,dt \leq \frac{f(u(t_i)) + f(u(t_{i+1}))}{2}(t_{i+1} - t_i).$$

Using this and Lemma 2.6 we can find partitions Δ_k such that $\operatorname{diam}\Delta_k \to 0$ as $k \to \infty$ and for $v_k = v_{\Delta_k}$ and the respective u_k

$$\limsup \int_0^T f(u_k(t))\,dt \leq \int_0^T f(u(t))\,dt.$$

From (6) it follows that $\int_0^T f(u_k(t))\,dt \to \int_0^T f(u(t))\,dt$ as $k \to \infty$. This and $\|v_k - v\|_2 \to 0$ imply $F(v_k) \to F(v)$. $\square$

3. Variational principle

We elaborate on the method of Deville, Godefroy and Zizler [4], see also [5].

Definition 3.1. Let S be a closed convex subset of a Banach space $(X, \|\cdot\|)$. Let $f : S \to \mathbb{R} \cup \{\infty\}$ be closed, convex, proper and bounded below.

A complete metric space (P, ρ) of positive convex continuous functions from X to $\mathbb{R}$ is called perturbation space relative to (f, S) if:

(i) P is a convex cone, that is, $g_i \in P$, $i = 1, 2 \Rightarrow g_1 + g_2 \in P$, and $\forall g \in P,\ c \geq 0 \Rightarrow cg \in P$;

(ii) if $g, g_k \in P$ and $\rho(g_k, 0) \to 0$ then $\rho(g, g + g_k) \to 0$;

(iii) there is $c > 0$ such that $\rho_e \leq c\rho$ on P;

(iv) for any $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that: for any $x \in \operatorname{dom} f \cap S$ and any $\delta > 0$ there is $y \in \operatorname{dom} f \cap S$ such that

$$\|y - x\| < \delta, \qquad |f(x) - f(y)| < \delta,$$

and $g \in P$ such that $\rho(g, 0) < \varepsilon$, $g(y) < \delta$ and

$$\beta\,(t_\varepsilon\text{-argmin}_S\, g) < \varepsilon.$$

Theorem 3.2. *Let S be a closed convex and bounded subset of a Banach space X. Let $f : S \to \mathbb{R} \cup \{\infty\}$ be closed, convex, proper and bounded below. If (P, ρ) is a perturbation space relative to (f, S) then for any $\varepsilon > 0$ there is $g \in P$ such that $\rho(g, 0) < \varepsilon$ and $(f + g, S)$ is weakly well posed.*

Proof. Consider for $n \in \mathbb{N}$ the subset A_n of P defined by

$$g \in A_n \iff \exists t > 0 :\ \beta(t\text{-argmin}_S\,(f + g)) < \frac{1}{n}. \tag{7}$$

We will show that A_n is dense and open in P. Then by Baire Category Theorem there will be $g \in \cap A_n$ such that $\rho(g, 0) < \varepsilon$ and by Definition 2.2 $(f + g, S)$ will be weakly well posed.

To this end, fix $n \in \mathbb{N}$.

If A_n were not open there would have been $g, g_k \in P$ such that $g \in A_n$, $g_k \notin A_n$ and $\lim \rho(g_k, g) = 0$. But if g satisfies (7) with t then by Lemma 2.4 g_k will eventually satisfy (7) with $t/3$. Contradiction.

Let now $h \in P$ be arbitrary. Fix $\nu > 0$. We will find $g \in P$ such that $h + g \in A_n$ and $\rho(h, h + g) < \nu$.

By Definition 3.1(ii) there is $\nu_1 > 0$ such that for any $g \in P$ with $\rho(g, 0) < \nu_1$ it follows that $\rho(h + g, h) < \nu$.

Let in Definition 3.1(iv) $\varepsilon = \min\{1/n, \nu_1\}$ and let $\bar{t} = t_\varepsilon/3$.

Pick $x \in (\bar{t}/2)\text{-argmin}_S\,(f + h)$. Let $\delta \in (0, \bar{t}/4)$ be such that $\|y - x\| < \delta \Rightarrow |h(y) - h(x)| < \bar{t}/4$. By Definition 3.1(iv) there is $y \in S$ and $g \in P$ such that $\|x - y\| \leq \delta$ and $|f(x) - f(y)| \leq \delta$, in particular

$$y \in \bar{t}\text{-argmin}_S\,(f + h),$$

$\rho(g,0) < \varepsilon \le \nu_1$, $y \in \delta$-$\mathrm{argmin}_S\, g \subset \bar{t}$-$\mathrm{argmin}_S\, g$ and

$$\beta\,(t_\varepsilon\text{-}\mathrm{argmin}_S\, g) < \varepsilon.$$

By Lemma 2.3 with $(f+h), g$ and $a = b = c = \bar{t}$ we get

$$\bar{t}\text{-}\mathrm{argmin}_S\,(f+h+g) \subset t_\varepsilon\text{-}\mathrm{argmin}_S\, g,$$

so $\beta(\bar{t}\text{-}\mathrm{argmin}_S\,(f+h+g)) < \varepsilon \le 1/n$. Therefore, $h + g \in A_n$. □

In a standard way we can drop boundedness assumption on the set in most cases.

Corollary 3.3. *Let S be a closed convex subset of the Banach space X. Let $f : S \to \mathbb{R}\cup\{\infty\}$ be closed, convex, proper and bounded below. If (P,ρ) is a perturbation space relative to (f,S) such that there are $g_k \in P$ with $\rho(g_k,0) \to 0$ and*

$$\lim_{\|x\|\to\infty} g_k(x) = \infty, \quad \forall k \in \mathbb{N},$$

then for any $\varepsilon > 0$ there is $g \in P$ such that $\rho(g,0) < \varepsilon$ and $(f+g,S)$ is weakly well posed.

Proof. Fix $\varepsilon > 0$. Let $g_1 \in P$ be such that $\rho(g_1,0) < \varepsilon/2$ and $g_1 \to \infty$ as $\|x\| \to \infty$.

Consider for $r > 0$ the optimisation problem $(f+g_1, S\cap rB_X)$. From Theorem 3.2 there is $g_2 \in P$ with $\rho(g_2) < \varepsilon/2$ such that $(f+g_1+g_2, S\cap rB_X)$ is weakly well posed. But since $f+g_1+g_2 \to \infty$ as $\|x\| \to \infty$, the latter optimisation problem is equivalent to $(f+g_1+g_2, S)$ if r is large enough. □

4. Existence of solutions to (1)

We can now present

Proof of Theorem 1.1. Recall that in terms of (5) the problem (1) may be reformulated simply as

$$F_{\|\cdot\|}(v) \to \min, \quad v \in Y = L^2([0,\infty), X)$$

for the proper, closed, convex and positive F (see Proposition 2.7).

In order to apply Theorem 3.2, consider the cone P over Y consisting of all functions of the form

$$v \to \int_0^\infty |v(t)|^2\, dt,$$

where $|\cdot|$ is some equivalent norm on X. We equip P with the metric ρ inherited from ρ_e on X: for two functions $g_i \in P$, $i = 1,2$ obtained from the norms $|\cdot|_1$ and $|\cdot|_2$, respectively, i.e. $g_i(v) = \int_0^\infty |v(t)|_i^2\, dt$, $i = 1,2$,

$$\rho(g_1,g_2) = \rho_e(|\cdot|_1^2, |\cdot|_2^2).$$

From Corollary 3.3 it is clear that we only need to prove that the so defined (P,ρ) is a perturbation space relative to (F,Y).

The first three axioms of Definition 3.1 are immediately fulfilled.

Set

$$M = \sum_{n=1}^{\infty} \frac{n^2}{2^{n-1}}. \tag{8}$$

Let $\varepsilon \in (0,1)$. We will show that $t_\varepsilon = \varepsilon^2/(2M)$ does the job.

Fix $v_1 \in \operatorname{dom} F$ and $\delta > 0$. From Proposition 2.7 there is stepwise $w_1 \in \operatorname{dom} F$ such that $\|v_1 - w_1\|_2 < \delta$ and $|F(v_1) - F(w_1)| < \delta$. It is easy to see that $0 \notin \operatorname{dom} F$ and thus $w_1 \neq 0$. Let $X_1 = \operatorname{span} w_1([0,\infty))$. Then $\dim X_1 < \infty$.

Take $r > 0$ such that $r < \min\{1, \delta/(\|w_1\|_2^2 + 1)\}$ and define

$$|x|_1^2 = d^2(x, X_1) + r\|x\|^2.$$

Obviously, $|\cdot|_1$ is an equivalent norm on X. Let $g_1 \in P$ be the function $g_1(v) = \int_0^\infty |v|_1^2\, dt$. Then

$$g_1(w_1) = r\|w_1\|_2^2 < \delta, \tag{9}$$

and, having in mind (8),

$$\rho(g_1, 0) = \rho_e(|\cdot|_1^2, 0) \le (1+r)\sum_{n=1}^{\infty} \frac{n^2}{2^n} < M, \tag{10}$$

since $r < 1$.

Finally, let $g = (\varepsilon/M)g_1$. Obviously, $g(w_1) < \delta$ by (9) and, moreover,

$$\rho(g, 0) = (\varepsilon/M)\rho(g_1, 0) < \varepsilon$$

by (10).

Consider the ball $A = r^{-1}B_{Y_1}$, where $Y_1 = L^2([0,\infty), X_1)$. Note that Y_1 is reflexive and therefore A is weakly compact. We will use it to estimate $\beta(t_\varepsilon\text{-argmin}_S\, g)$. By Lemma 2.5

$$\begin{aligned} g_1(v) = \int_0^\infty |v(t)|_1^2\, dt &= \int_0^\infty (d^2(v(t), X_1) + r\|v(t)\|^2)\, dt \\ &= d^2(v, Y_1) + r\|v\|_2^2, \quad \forall v \in Y. \end{aligned}$$

Obviously $\min g = 0$. If $g(v) < t_\varepsilon$ then $g_1(v) = (\varepsilon/M)g(v) < (\varepsilon t_\varepsilon)/M = \varepsilon^3/(2M^2) < \varepsilon^2$ and therefore $d(v, Y_1) < \varepsilon$ and $v \in r^{-1}B_Y$. So, $d(v, A) < \varepsilon$. Thus, $\beta(t_\varepsilon\text{-argmin}_S\, g) < \varepsilon$. □

References

[1] G. Beer, P. Kenderov: On the argmin multifunction for lower semicontinuous functions, Proc. Amer. Math. Soc. 102 (1988) 107–113.

[2] G. Beer, R. T. Rockafellar, R. J.-B. Wets: A characterisation of epi-convergence in terms of convergence of level sets, Proc. Amer. Math. Soc. 116 (1992) 753–761.

[3] F. S. De Blasi: On a property of the unit sphere in a Banach space, Bull. Math. Soc. Sci. Math. Répub. Soc. Roum., Nouv. Sér. 21 (1977) 259–262.

[4] R. Deville, G. Godefroy, V. Zizler: A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions, J. Funct. Anal. 111 (1993) 197–212.

[5] R. Deville, G. Godefroy, V. Zizler: Smoothness and Renormings in Banach Spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics 64, Longman Scientific & Technical, Harlow (1993).

[6] J. Diestel, J. J. Uhl: Vector Measures, American Mathematical Society, Providence (1977).

[7] K. Kuratowski: Topology II, Academic Press, New York (1968).

[8] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Comprehensive Studies in Mathematics 317, Springer, Berlin (1998).

[9] E. C. Titchmarsh: The Theory of Functions, 2nd Ed., Oxford University Press, London (1952).