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# Generalized Steffensen Inequalities and Their Optimal Constants

**Patrick J. Rabier**
*Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA*
*rabier@imap.pitt.edu*

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If $\Phi : [0,\infty) \to \mathbb{R}$ is convex and continuous with $\Phi(0) = 0$ and if $q \in (1,\infty)$, $q' := \frac{q}{q-1}$, we first prove that the inequality $\Phi(\int_0^\infty f(r)dr) \leq C \int_0^\infty f(r)\Phi'(r^{1/q'})dr$ for every $f \geq 0$ in the unit ball of $L^q(0,\infty)$, holds when $C = 1$. In general, both sides may be $\pm\infty$. Related inequalities for $f \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), f \neq 0$ are derived. This inequality is independent of Jensen's inequality and, when $q = \infty$, it is an elaboration on an inequality of Steffensen which was discussed elsewhere by the author.

The next goal of the paper is to identify the range of the admissible constants $C$ and, in particular, to characterize the optimal constant when $\Phi \geq 0$ or $\Phi \leq 0$. It turns out that $C = 1$ is "almost always" optimal, at least in a restricted sense, but not always when $q < \infty$ : Given $q$, the admissible constants lie on an interval containing 1 whose left (right) endpoint is the supremum (infimum) of a function defined on some (left/right dependent) subset of $\mathbb{R}^3$.

If $q = 2$, these extrema can be calculated in a number of examples. Among other things, this reveals that $C = 1$ need not be optimal when $\Phi \geq 0$ and $\Phi'_+(0) = 0$ or when $\Phi \leq 0$ and $\Phi'_+(0) = -\infty$.

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## 1. Introduction and preliminaries

Throughout this paper, the notation $||\cdot||_p$ refers to the $L^p$ norm, $1 \leq p \leq \infty$, either on $(0,\infty)$ or on $\mathbb{R}^N$. The domain will always be clear from the context, which eliminates the need for an explicit mention. When $(0,\infty)$ is replaced by a finite interval $(0,a)$, the notation $||\cdot||_{p,(0,a)}$ will be used.

Let $\Phi : [0,\infty) \to \mathbb{R}$ be a continuous convex function such that $\Phi(0) = 0$. As usual, $\Phi(\infty) := \lim_{r\to\infty} \Phi(r) \in [-\infty,\infty]$. In [8], the author showed that

$$\Phi\left(\int_0^\infty f(r)dr\right) \leq \int_0^\infty f(r)\Phi'(r)dr, \tag{1}$$

for every $f \in L^\infty(0,\infty), f \geq 0$, with $||f||_\infty \leq 1$, (either side may be $\pm\infty$) and, as a by-product, that

$$\omega_N ||f||_\infty \Phi\left(\omega_N^{-1} ||f||_1 ||f||_\infty^{-1}\right) \leq \int_{\mathbb{R}^N} |f(x)|\Phi'(|x|^N)dx, \tag{2}$$

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for every $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), f \neq 0$, where $\omega_N$ is the volume of the unit ball of $\mathbb{R}^N$. The assumption $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ensures that the left-hand side of (2) is finite, but the right-hand side may be $\infty$. Both sides may be finite and negative.

In turn, (2) can be used to obtain non trivial upper and/or lower estimates for integrals of the form $\int_{\mathbb{R}^N} |f(x)|\psi(|x|)dx$ in terms of $||f||_1$ and $||f||_\infty$ when $\psi : (0,\infty) \to \mathbb{R}$ is nondecreasing, but not necessarily nonnegative.

The inequality (1), which has little to no connection with Jensen's inequality in spite of some formal resemblance, is essentially a modern formulation and generalization of an inequality of Steffensen [9]. Although first published in 1918 and revisited numerous times since the late 50s, Steffensen's inequality had only been expressed in an arguably more cryptic and less convenient form. For example, (2) is virtually impossible to derive from Steffensen's original formulation. See [8] for further comments and references.

An advantage of the proof of (1) given in [8] is that a mostly -though not entirely- straightforward variant of it gives the next generalization (a vaguely reminiscent, yet completely different one, is due to Bergh [1]):

$$\Phi\left(\int_0^\infty f(r)dr\right) \leq \int_0^\infty f(r)\Phi'(r^{1/q'})dr, \tag{3}$$

for every $f \in L^q(0,\infty), f \geq 0$, with $||f||_q \leq 1$ and any $q \in (1,\infty)$ where, as usual, $q' := \frac{q}{q-1}$ is the Hölder conjugate of $q$. If $q = 1$, there is no nontrivial variant of (3). On the other hand, there is no conceptual difficulty to generalize (1) further by replacing $L^q(0,\infty)$ by an Orlicz space in (3) – with a concomitant modification of the term $\Phi'(r^{1/q'})$ – but this will not be discussed here beyond Remark 2.2.

As we shall see, (3) implies

$$\omega_N^{1/q'}||f||_q\Phi\left(\omega_N^{-1/q'}||f||_1||f||_q^{-1}\right) \leq \int_{\mathbb{R}^N} |f(x)|\Phi'(|x|^{N/q'})dx, \tag{4}$$

for every $f \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), f \neq 0$. Once again, either side may be $\pm\infty$ in (3) and the right-hand side may be $\infty$ in (4). The applications are similar to those of (1) and (2) and, notably, they provide $L^1 - L^q$ estimates for weighted integrals $\int_{\mathbb{R}^N} |f(x)|\psi(|x|)dx$. Presumably, there are many other potential applications of (3), while (4) has more or less evident ramifications to Sobolev spaces, as briefly discussed in [8].

Neither (3) nor (4) was mentioned in [8], because an issue remained to be settled when $q < \infty$ : While both (1) and (2) are optimal when $\Phi \neq 0$ (equality holds and both sides are finite when $f$ is the characteristic function $\chi_{B_R}$ of *any* ball $B_R$ centered at the origin[1] with radius $R$), it is not obvious whether the same thing is true of (3) and (4) when $q < \infty$.

For example, if $q < \infty$, then $f = \chi_{B_R}$ satisfies $||f||_q \leq 1$ only when $R \leq 1$ and, if so, the inequality (3) is strict unless $\Phi$ is linear on $[0, R^{1/q'}]$. This shows that the

[1] $B_R = [0, R)$ when the domain is $[0,\infty)$.

optimality question does not have the same immediate answer as when $q = \infty$. This question will be the main topic of this paper. Since optimality happens to be the same for (3) and (4) (see Remark 2.3), we henceforth confine attention to the optimality of (3).

It is difficult to give a meaningful account of what is being done without discussing the problem in detail. This will result in a longer Introduction but, on the positive side, several technicalities will not have to be revisited later.

From now on, we shall use the shorter notation

$$\ell(f) := \int_0^\infty f(r)dr, \qquad \ell_{q'}(f) := \int_0^\infty f(r)\Phi'(r^{1/q'})dr, \tag{5}$$

when $f \geq 0$ and $f \in L^q(0,\infty)$. Both $\ell$ and $\ell_{q'}$ are essentially linear forms, but they may assume infinite values. That $\ell(f)$ is unambiguously defined is obvious. The definiteness of $\ell_{q'}(f)$ will be established in Section 2. With this notation, (3) is simply

$$\Phi\left(\ell(f)\right) \leq \ell_{q'}(f), \tag{6}$$

whenever $f \geq 0$ and $||f||_q \leq 1$.

Recall that the convexity of $\Phi$ implies the existence of a right derivative $\Phi'_+$ defined at *every* point of $[0,\infty)$ and nondecreasing, while the derivative $\Phi'$ exists at all but countably many points. In particular, $\Phi' = \Phi'_+$ a.e., which justifies using $\Phi'_+$ instead of $\Phi'$ whenever convenient in measure-theoretic considerations. Other useful properties are that $\Phi'_+$ is finite at every point of $(0,\infty)$ (but $\Phi'_+(0) = -\infty$ is possible) and right-continuous on $[0,\infty)$ (see for example [5], especially [5, Remark 4.2.2, p. 26]). When $\Phi'_+(0) = -\infty$, this means $\lim_{r\to 0+} \Phi'_+(r) = -\infty$.

In the next section, after a brief proof of the inequalities (3) and (4), we show in Theorem 2.4 that (3), that is, (6), is indeed optimal when $\Phi'_+(0) > 0$ or when $-\infty < \Phi'_+(0) < 0$ and the sign of $\Phi$ does not change (hence $\Phi \leq 0$). A partial optimality result is still true if the sign of $\Phi$ changes. However, the problem is much more delicate when $\Phi'_+(0) = 0$ or $\Phi'_+(0) = -\infty$. In addition, Theorem 2.4 does not fully resolve the optimality issue when the sign of $\Phi$ changes on $(0,\infty)$.

To clarify the above statements, observe that (6) implies that there is at least one constant $C > 0$ (specifically, $C = 1$) such that

$$\Phi(\ell(f)) \leq C\ell_{q'}(f), \tag{7}$$

whenever $f \geq 0$ and $||f||_q \leq 1$. It is easy to see that $C \leq 0$ in (7) is impossible unless $\Phi \leq 0$, in which case the inequality has no value, for then $\Phi' \leq 0$ by the convexity of $\Phi$, so that $\ell_{q'}(f) \leq 0$. In fact, $C = 0$ does not even make sense if $|\ell_{q'}(f)| = \infty$ for some $f$ with $||f||_q \leq 1$. This justifies confining attention to positive constants $C$ and, to avoid occasional trivialities, we shall always assume that $\Phi$ is not the 0 function.

Set

$$\gamma := \inf\{r > 0 : \Phi(r) > 0\} \in [0,\infty]. \tag{8}$$

If $\Phi \geq 0$ (hence $\Phi'_+ \geq 0$) on $(0, \infty)$, then $0 \leq \gamma < \infty$ ($\Phi = 0$ was just ruled out) and every constant $C > 0$ for which (7) holds satisfies

$$C \geq C_{q,+} := \sup\left\{\frac{\Phi(\ell(f))}{\ell_{q'}(f)} : f \geq 0, ||f||_q \leq 1, \ell(f) > \gamma, \ell_{q'}(f) < \infty\right\}. \qquad (9)$$

Indeed, if $f \geq 0$, $||f||_q \leq 1$ and $\ell(f) \leq \gamma$, then $0 = \Phi(\ell(f))$ and $\ell_{q'}(f) \geq 0$ since $\Phi' \geq 0$, whence (7) holds with any $C > 0$. The same thing is obviously true if $\ell_{q'}(f) = \infty$. Therefore, restrictions about $C$ arise only when $\ell(f) > \gamma$ and $\ell_{q'}(f) < \infty$, which justifies (9). In the same vein, if $\ell(f) > \gamma$, then $0 < \Phi(\ell(f)) \leq \ell_{q'}(f)$ by (6), so that the definition of $C_{q,+}$ in (9) involves only ratios of strictly positive numbers.

By (6) and (9), $0 < C_{q,+} \leq 1$ and $[C_{q,+}, \infty)$ is the set of admissible constants in (7). In addition, $C_{q,+} = 1$ if $\Phi'_+(0) > 0$ (Theorem 2.4), which shows that $C = 1$ is optimal in (7) in this case.

By similar arguments, if $\Phi \leq 0$ (hence $\Phi'_+ \leq 0$ by convexity), then every constant $C > 0$ in (7) satisfies

$$C \leq C_{q,-} := \inf\left\{\frac{\Phi(\ell(f))}{\ell_{q'}(f)} : f \geq 0, ||f||_q \leq 1, \Phi(\ell(f)) > -\infty, \ell_{q'}(f) < 0\right\}. \qquad (10)$$

By (6), $\Phi(\ell(f))$ and $\ell_{q'}(f)$ are strictly negative real numbers in the right-hand side of (10) and $1 \leq C_{q,-} < \infty$. The set of admissible constants in (7) is $(0, C_{q,-}]$. In addition, $C_{q,-} = 1$ if also $-\infty < \Phi'_+(0) < 0$ (Theorem 2.4). Thus, $C = 1$ is once again optimal in (7).

In the two cases $\Phi \geq 0$ or $\Phi \leq 0$ discussed above, only one among $C_{q,+}$ and $C_{q,-}$ is a real number: $C_{q,+} = -\infty$ if $\Phi \leq 0$ because $\ell(f) > \gamma = \infty$ is impossible and $C_{q,-} = \infty$ if $\Phi \geq 0$ (hence $\Phi' \geq 0$) because $\ell_{q'}(f) < 0$ is impossible. However, if the sign of $\Phi$ changes on $(0, \infty)$, then $0 < \gamma < \infty$ in (8), whence $0 < C_{q,+} \leq 1 \leq C_{q,-} < \infty$ (neither extremum is over the empty set). Furthermore, by arguing as above, it is easily checked that (7) holds if and only if $C_{q,+} \leq C \leq C_{q,-}$ ((7) holds with any $C > 0$ if $\ell(f) \leq \gamma$ and $\ell_{q'}(f) \geq 0$, or if $\ell_{q'}(f) = \infty$).

In other words, when the sign of $\Phi$ changes, the interval $[C_{q,+}, C_{q,-}]$ is the set of admissible constants in (7), but none of them is optimal relative to the whole class of functions $f \geq 0$ such that $||f||_q \leq 1$, unless $C_{q,+} = C_{q,-}$ ($= 1$), in which case $C = 1$ is "by default" optimal in (7). Nonetheless, $C_{q,+}$ is always optimal under the additional requirement $\ell(f) > \gamma$ while $C_{q,-}$ is always optimal under the additional requirement $\ell_{q'}(f) < 0$.

Since $C_{q,-} = 1$ when $-\infty < \Phi'_+(0) < 0$ (Theorem 2.4), it follows that if $-\infty < \Phi'_+(0) < 0$ and the sign of $\Phi$ changes, then $C = 1$ optimal in (7) when $f \geq 0, ||f||_q \leq 1$ and $\ell_{q'}(f) < 0$, but this says nothing about its optimality when $\ell(f) > \gamma$. This explains the earlier claim that the optimality question is not fully resolved by Theorem 2.4 in this case.

**Remark 1.1.** From the above, the interval $(0, \infty) \cap [C_{q,+}, C_{q,-}]$ is *always* the set of admissible constants in (7).

Everything would become much simpler if it were always true that $C_{q,+} = 1$ when $\Phi \geq 0$, that $C_{q,-} = 1$ when $\Phi \leq 0$ and that $C_{q,+} = C_{q,-}$ $(= 1)$ when the sign of $\Phi$ changes. While this the case when $q = \infty$ (because $\Phi(\ell(\chi_{(0,a)})) = \ell_1(\chi_{(0,a)})$ for *every* $a > 0$) and in spite of what Theorem 2.4 might suggest, such a simple answer is generally *false* when $q < \infty$. The proof of this fact is the more demanding part of this paper.

Aside from the new but quickly proved inequalities (3) and (4), the main result is that, in both (9) and (10), it suffices to consider nonincreasing functions of the form $f(r) := ((A - B\Phi'_+(r^{1/q'}))\chi_{(0,b)}(r))^{q'/q}$, where $0 < b < \infty$ and $A \in \mathbb{R}$ and $B \geq 0$ are constants such that $A - B\Phi'_+(b^{1/q'}-) \geq 0$ (hence $A - B\Phi'_+(r^{1/q'}) \geq 0$ on $(0, b)$), $||f||_q = 1$ and either $\ell(f) > \gamma$ in (9) or $\ell_{q'}(f) < 0$ in (10). This shows that the calculation of $C_{q,\pm}$ can be reduced to optimizing a function of the real parameters $b, A$ and $B$ in two disjoint sets. Actually, only two parameters are involved due to the constraint $||f||_q = 1$.

It is plain that there are numerous obstacles to obtaining the aforementioned property by showing that the extremum is achieved in (9) or (10). That the domain is not even closed, let alone compact, for any reasonable topology for which the functional is continuous is one of them. Moreover, the existence of an extremum $f$ (which is not always true, anyway) implies *at best* that $f(r)$ must be 0 or $((A - B\Phi'_+(r^{1/q'}))\chi_{(0,b)}(r))^{q'/q}$ for a.e. $r > 0$, a much weaker result.

Our method consists in using an approximation by finite intervals $(0, a)$ to take advantage, among other things, of the embedding $L^q(0, a) \hookrightarrow L^1(0, a)$ and, next, in treating $\ell(f)$ as a parameter $c$. Only then can the existence of an extremum be proved for fixed $c$. Luxemburg's monotone rearrangement inequality (17) is instrumental in establishing the key property that there are always *nonincreasing* extrema. Once this is done, almost everything follows from a careful application of the Lagrange multiplier theorem after setting $f = g^2$ to eliminate the constraint $f \geq 0$. This program is carried out in Section 3 and the conclusions are summarized in Theorems 3.4 and 3.5.

In spite of their conceptual simplicity, it is hardly surprising that Theorems 3.4 and 3.5 rarely lead to a closed-form formula for $C_{q,\pm}$, but they give a simple procedure for their numerical evaluation. While it seems unlikely that any significant further simplification can be introduced in general, a rather dramatic one takes place when $q = 2$, with the help of a change of variables (Section 4). This makes it possible to find the exact values of $C_{2,\pm}$ in several examples (Section 5). These examples show that when it exists, the optimal constant in (7) is not always 1 (Examples 5.1 and 5.3), that the interval $[C_{2,+}, C_{2,-}]$ may be finite and nontrivial (Example 5.2) or just reduce to the single point $\{1\}$ (Example 5.4).

## 2. The generalized Steffensen inequalities

It is always understood that $\Phi : [0, \infty) \to \mathbb{R}$ is convex and continuous with $\Phi(0) = 0$ and that $q \in (1, \infty)$. We begin by giving a quick proof of (3) and (4). Full details when $q = \infty$ are given in [8], so we merely highlight the main steps and the differences. When $p = 1$, Lemma 2.1 below is just the absolute continuity of $\Phi$ on compact intervals ([5, p. 26]).

**Lemma 2.1.** *The function $\Phi'(r^{1/p})$ is in $L^p(0,a)$ for every $a > 0$ and every $1 \leq p < \infty$.*

**Proof.** Since $\Phi' = \Phi'_+$ a.e., it suffices to prove that $\Phi'_+(r^{1/p})$ is in $L^p(0,a)$. This is trivial if $\Phi'_+(0) \in \mathbb{R}$, for then $\Phi'_+(r^{1/p})$ is in $L^\infty(0,a)$ by the monotonicity of $\Phi'_+$ and its finiteness at every point of $(0,\infty)$.

It remains to consider the case when $\Phi'_+(0) = -\infty$. If so, $\Phi'_+ < 0$ on some interval $(0, b^{1/p})$ with $0 < b \leq a$ by the right-continuity of $\Phi'_+$. Since the integrability question is the same over $(0,a)$ and $(0,b)$, we may and shall assume that $b = a$, i.e., $\Phi'_+ < 0$ on $(0, a^{1/p})$. As a result, $\Phi(r^{1/p}) < 0$ for $r \in (0,a)$ since $\Phi(0) = 0$.

It will be more convenient to use $\Psi := -\Phi$, which is concave, continuous on $[0,\infty)$ and satisfies $\Psi(0) = 0$, $\Psi'_+(0) = \infty$ as well as $\Psi' > 0, \Psi > 0$ on $(0, a^{1/p})$. Given $n \in \mathbb{N}$ large enough, the change of variable $r = \rho^p$ yields

$$\int_{\frac{1}{n}}^{a} (\Psi')^p(r^{1/p})dr = \int_{\frac{1}{n}}^{a} (\Psi'_+)^p(r^{1/p})dr = \int_{n^{-1/p}}^{a^{1/p}} p(\Psi'_+)^p(\rho)\rho^{p-1}d\rho.$$

Note that $0 < \Psi'_+(\rho) \leq \frac{\Psi(\rho)}{\rho}$ for $\rho > 0$, the latter by the concavity of $\Psi$. Hence, $(\Psi'_+)^p(\rho) \leq \Psi'_+(\rho)(\frac{\Psi(\rho)}{\rho})^{p-1}$ for $n^{-1/p} \leq \rho \leq a^{1/p}$, so that the above inequality becomes

$$\begin{aligned}\int_{\frac{1}{n}}^{a} (\Psi')^p(r^{1/p})dr &\leq \int_{n^{-1/p}}^{a^{1/p}} p\Psi'(\rho)\Psi^{p-1}(\rho)d\rho \\ &= \Psi^p(a^{1/p}) - \Psi^p(n^{-1/p}) \leq \Psi^p(a^{1/p}),\end{aligned}$$

where the absolute continuity of $\Psi^p$ on $[n^{-1/p}, a^{1/p}]$ was used. That $\Psi'(r^{1/p})$ is in $L^p(0,a)$ follows by letting $n \to \infty$ and monotone convergence. □

Of course, the condition $\Phi(0) = 0$ is actually not needed in Lemma 2.1.

**Proof of the inequality (3).** First and foremost, since $f \geq 0$, both sides of (3) are well defined, possibly $\pm\infty$. This is clear for the left-hand side. The definiteness of the right-hand side (that is, of $\ell_{q'}(f)$ when the notation (5) is used) is due to the sign of $\Phi'$ being constant a.e. on some interval $(\eta, \infty)$ with $\eta \geq 0$, so that $\int_{\eta^{q'}}^{\infty} f(r)\Phi'(r^{1/q'})dr$ is defined, possibly $\pm\infty$, whereas $\int_0^{\eta^{q'}} f(r)\Phi'(r^{1/q'})dr \in \mathbb{R}$ by Lemma 2.1 since $f \in L^q(0, \eta^{q'})$.

The main step in the proof of (3) is as follows: Given $a > 0$, let $f \in L^q(0,a)$ be such that $f > 0$ on $(0,a)$ and $||f||_{q,(0,a)} \leq 1$. Set $F(r) := \int_0^r f(s)ds$, so that $\Phi\left(\int_0^a f(r)dr\right) = \Phi(F(a)) = \int_0^{F(a)} \Phi'_+(r)dr = \int_0^a \Phi'_+(F(s))f(s)ds$ after the change of variable $r := F(s)$. By Hölder's inequality, $F(s) \leq ||f||_{q,(0,a)} s^{1/q'} \leq s^{1/q'}$, whence $\Phi\left(\int_0^a f(r)dr\right) \leq \int_0^a f(s)\Phi'_+(s^{1/q'})ds = \int_0^a f(s)\Phi'(s^{1/q'})ds$ by the monotonicity of $\Phi'_+$.

The case when $f \geq 0$ on $(0,a)$ can be deduced from $f > 0$ by an approximation argument (replace $f$ by $f_\varepsilon := (1-\varepsilon)f + \varepsilon a^{-1/q}$, so that $f_\varepsilon \geq 0$ and $||f_\varepsilon||_{q,(0,a)} \leq 1$ and note that $\Phi'(r^{1/q'})$ is in $L^1(0,a)$ by Lemma 2.1). This proves (3) when $f = 0$ on some interval $(a,\infty)$. The full proof of (3) follows by a limiting process based on the monotone convergence theorem (see [8, Theorem 2.2]).

The above even shows that $\Phi\left(\int_0^\infty f(r)dr\right) \leq \int_0^\infty f(s)\Phi'(||f||_q s^{1/q'})ds$ (it is only a bit trickier to use monotone convergence), which is more intricate than (3) and not sharper when $||f||_q = 1$. That only $||f||_q = 1$ is needed to derive (4) was a compelling reason not to discuss this form of the inequality.

**Remark 2.2.** Since the classical Hölder inequality is valid in Orlicz spaces equipped with the Orlicz norm and there is an explicit formula for the norm of the characteristic function of $(0,s)$ ([6, pp. 72 and 74]), the above arguments lead to further generalizations of (3), but we shall not elaborate.

When $q = \infty$, it is shown in [8, Theorem 2.4] that if $\Phi$ is absolutely continuous on compact intervals and $\Phi(0) = 0$, (3) implies the convexity of $\Phi$. In fact, it is even true that (3) for step functions suffices (recall that step functions are not dense in $L^\infty(0,\infty)$). If $q < \infty$, this is no longer true in general. For instance, if $\Phi \geq 0$ and $C_{q,+} < 1$ (the optimal constant in (7) in this case; see Example 5.1 in Section 5), it is not hard to see that the addition of a suitable "bump" function produces a nonconvex function for which (3) continues to hold. Thus, the class of functions for which (3) holds provides a $q$-dependent generalization of convexity. It seems unlikely that this class can be characterized by a property expressed solely in terms of the values of $\Phi$ (like convexity).

**Proof of the inequality (4).** Upon setting $f := \frac{g}{||g||_q}$ in (3), we get

$$||g||_q \Phi\left(||g||_q^{-1}\int_0^\infty g(r)dr\right) \leq \int_0^\infty g(r)\Phi'(r^{1/q'})dr, \tag{11}$$

for every $g \in L^q(0,\infty), g \geq 0, g \neq 0$.

If now $f \in L^q(\mathbb{R}^N), f \neq 0$, let $f_S(r) := \frac{1}{N\omega_N}\int_{\mathbb{S}^{N-1}} |f(r\sigma)|d\sigma$ denote the spherical mean of $|f|$, where $\mathbb{S}^{N-1}$ is the unit sphere of $\mathbb{R}^N$, with measure $N\omega_N$ and let $g(r) := r^{(N-1)/q} f_S(r)$, so that $g \geq 0$, $g \neq 0$ and $||g||_q \leq (N\omega_N)^{-1/q}||f||_q$. Furthermore, $\int_0^\infty g(r)dr = (N\omega_N)^{-1}\int_{\mathbb{R}^N} |f(x)||x|^{(1-N)/q'}dx$ and $\int_0^\infty g(r)\Phi'(r^{1/q'})dr = (N\omega_N)^{-1}\int_{\mathbb{R}^N} |f(x)||x|^{(1-N)/q'}\Phi'(|x|^{1/q'})dx$. Thus, by (11),

$$\begin{aligned} &N\omega_N||g||_q\Phi\left((N\omega_N||g||_q)^{-1}\int_{\mathbb{R}^N}|f(x)||x|^{(1-N)/q'}dx\right) \\ &\leq \int_{\mathbb{R}^N}|f(x)||x|^{(1-N)/q'}\Phi'(|x|^{1/q'})dx. \end{aligned} \tag{12}$$

As noted earlier, $||g||_q \leq (N\omega_N)^{-1/q}||f||_q$, so that $N\omega_N||g||_q \leq (N\omega_N)^{1/q'}||f||_q$ and $\lambda\Phi(R/\lambda)$ is a nonincreasing function of $\lambda > 0$ for every $0 \leq R \leq \infty$ since $\Phi$ is convex and $\Phi(0) = 0$. Together with (12), this yields

$$(N\omega_N)^{1/q'}||f||_q\Phi\left((N\omega_N)^{-1/q'}||f||_q^{-1}\int_{\mathbb{R}^N}|f(x)||x|^{(1-N)/q'}dx\right)$$
$$\leq \int_{\mathbb{R}^N}|f(x)||x|^{(1-N)/q'}\Phi'(|x|^{1/q'})dx$$

and (4) follows by replacing $f(x)$ by $N^{-1/q}|x|^{(1-N)/q}f(x|x|^{(1-N)/N})$ in this inequality. Indeed, this leaves $||f||_q$ unchanged while $\int_{\mathbb{R}^N}|f(x)||x|^{(1-N)/q'}dx$ and $\int_{\mathbb{R}^N}|f(x)||x|^{(1-N)/q'}\Phi'(|x|^{1/q'})dx$ are changed into $N^{1/q'}\int_{\mathbb{R}^N}|f(x)|dx$ $(= N^{1/q'}||f||_1$ if $f \in L^1(\mathbb{R}^N))$ and $N^{1/q'}\int_{\mathbb{R}^N}|f(x)|\Phi'(|x|^{N/q'})dx$, respectively (if $x = y|y|^{N-1}$, then $dx = N|y|^{N(N-1)}dy$).

**Remark 2.3.** More generally, if $C > 0$ and $\Phi(\int_0^\infty f(r)dr) \leq C\int_0^\infty f(r)\Phi'(r^{1/q'})dr$ for every $f \geq 0$ in $L^q(0,\infty)$ with $||f||_q \leq 1$, the same procedure as above yields $\omega_N^{1/q'}||f||_q\Phi(\omega_N^{-1/q'}||f||_1||f||_q^{-1}) \leq C\int_{\mathbb{R}^N}|f(x)|\Phi'(|x|^{N/q'})dx$ for every $f \in L^q(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), f \neq 0$. That the converse is true is easily seen by confining attention to nonnegative radially symmetric functions of the form $f(x) = g(|x|^N)$ with $g \in C_0^\infty([0,\infty))$. It is then straightforward to infer that both inequalities have the same admissible and/or optimal constants (after restriction to a suitable class if the sign of $\Phi$ changes on $(0,\infty)$; see the Introduction).

As a follow-up on Remark 2.3, we now address the optimality of the inequalities (3) and (4).

**Theorem 2.4.** *Let $C_{q,\pm}$ be given by* (9) *and* (10). *Then, $C_{q,+} = 1$ if $\Phi'_+(0) > 0$ and $C_{q,-} = 1$ if $-\infty < \Phi'_+(0) < 0$.*
*In particular, if either $\Phi'_+(0) > 0$, or $-\infty < \Phi'_+(0) < 0$ and $\Phi \leq 0$, the inequalities* (3) *(for $f \geq 0$ and $||f||_q \leq 1$) and* (4) *(for $f \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), f \neq 0$) are optimal. In addition, if $-\infty < \Phi'_+(0) < 0$ and the sign of $\Phi$ changes on $(0,\infty)$, then* (3) *is optimal when $f \geq 0, ||f||_q \leq 1$ and $\int_0^\infty f(r)\Phi'(r^{1/q'})dr < 0$ and* (4) *is optimal when $f \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N}|f(x)|\Phi'(|x|^{N/q'})dx < 0$.*

**Proof.** Recall that $\ell$ and $\ell_{q'}$ were defined in (5). If $f_n := n^{1/q}\chi_{(0,\frac{1}{n})}$, then, $f_n \geq 0$, $||f_n||_q = 1$ and $\ell(f_n) = n^{-1/q'}$. Since $\Phi'_+$ is right continuous with $\Phi'_+(0) \in \mathbb{R}$, it follows from $\Phi(r) = \int_0^r \Phi'_+(s)ds$ that $\Phi(r) = \Phi'_+(0)r + o(r)$ for $r > 0$ small enough. In particular, $\Phi(\ell(f_n)) = \Phi'_+(0)n^{-1/q'} + o(n^{-1/q'})$.

Next, $\ell_{q'}(f_n) = n^{1/q}\int_0^{\frac{1}{n}}\Phi'(r^{1/q'})dr = n^{1/q}\int_0^{\frac{1}{n}}\Phi'_+(r^{1/q'})dr$. Once again by the right-continuity of $\Phi'_+$, this yields $\ell_{q'}(f_n) = \Phi'_+(0)n^{-1/q'} + o(n^{-1/q'})$. Therefore, since $\Phi'_+(0) \neq 0$, we infer that

$$\frac{\Phi(\ell(f_n))}{\ell_{q'}(f_n)} = \frac{\Phi'_+(0)n^{-1/q'} + o(n^{-1/q'})}{\Phi'_+(0)n^{-1/q'} + o(n^{-1/q'})} = 1 + o(1),$$

that is, $\lim_{n\to\infty}\frac{\Phi(\ell(f_n))}{\ell_{q'}(f_n)} = 1$.

If $\Phi'_+(0) > 0$, then $\gamma = 0$ in (8), so that $C_{q,+} \geq \frac{\Phi(\ell(f_n))}{\ell_{q'}(f_n)}$ by (9) since $\ell(f_n) > 0$. Therefore, $C_{q,+} = 1$ since $C_{q,+} \leq 1$ is always true (see Section 1). If now $-\infty < \Phi'_+(0) < 0$, then $C_{q,-} \leq \frac{\Phi(\ell(f_n))}{\ell_{q'}(f_n)}$ by (10) (to see that $\ell_{q'}(f_n) < 0$, use the right-continuity of $\Phi'_+$ at 0). Therefore, $C_{q,-} = 1$ since $C_{q,-} \geq 1$ is always true.

The "in particular" part follows from the discussion in Section 1 and from Remark 2.3. □

## 3. Finite dimensional characterization of $C_{q,\pm}$

In this section, we describe a procedure to simplify the calculation of the constants $C_{q,\pm}$ in (9) and (10), respectively. Of course, this has value only when they are not visibly $\pm\infty$ or known to be 1 by Theorem 2.4. The strategy to overcome the difficulties associated with this issue was already outlined in the Introduction and will now be implemented. As before, $q \in (1, \infty)$ and $\Phi : [0, \infty) \to \mathbb{R}$ is convex and continuous, $\Phi(0) = 0$ and $\Phi$ is not the 0 function. We first discuss the evaluation of $C_{q,+}$.

### 3.1. Reduction to a finite interval

Since $C_{2,+} = -\infty$ if $\gamma = \infty$ in (8), we shall assume that $\Phi(r) > 0$ for some $r > 0$ (i.e., $\Phi \geq 0$ or the sign of $\Phi$ changes) so that $0 \leq \gamma < \infty$ and the set $\{f \geq 0, ||f||_q \leq 1, \ell(f) > \gamma, \ell_{q'}(f) < \infty\}$ is not empty because it contains the function $f = n^{-1/q}\chi_{(0,n)}$ for $n$ large enough (the finiteness of $\ell_{q'}(f)$ follows from Lemma 2.1).

The first remark is that the integrals $\ell(f)$ and $\ell_{q'}(f)$ in the definition (9) of $C_{q,+}$ are always finite and positive, which follows from (6) and the conditions $\ell(f) > \gamma$ and $\ell_{q'}(f) < \infty$. As a result, this definition is unchanged if it is also required that $f$ vanishes over some infinite interval $(a, \infty)$.

In turn, with $a > 0$ being fixed, set $f_a := f\chi_{(0,a)}$ for every measurable function $f$ on $(0, \infty)$ and

$$C_{q,+}(a) := \sup \left\{ \frac{\Phi(\ell(f_a))}{\ell_{q'}(f_a)} : f_a \geq 0, ||f_a||_q \leq 1, \ell(f_a) > \gamma \right\}.$$

By Lemma 2.1, the function $f_a(r)\Phi'(r^{1/q'})$ is integrable, so that the definition of $C_{q,+}(a)$ is unchanged if the immaterial restriction $\ell_{q'}(f_a) < \infty$ is incorporated. If so, it is readily seen from (9) that

$$C_{q,+} = \sup_{a>0} C_{q,+}(a) = \lim_{a\to\infty} C_{q,+}(a), \tag{13}$$

the latter since $C_{q,+}(a)$ is clearly a nondecreasing function of $a$.

We now focus on the evaluation of $C_{q,+}(a)$. For simplicity of notation, we drop the subscript "$a$" in $f_a$, but $f$ will henceforth denote a function of the space $L^q(0, a)$, with norm $|| \cdot ||_{q,(0,a)}$. Then,

$$\ell(f) = \int_0^a f(r)dr, \qquad \ell_{q'}(f) = \int_0^a f(r)\Phi'(r^{1/q'})dr = \int_0^a f(r)\Phi'_+(r^{1/q'})dr \tag{14}$$

and

$$C_{q,+}(a) = \sup\left\{\frac{\Phi(\ell(f))}{\ell_{q'}(f)} : f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) > \gamma\right\}$$
$$= \sup_{c>\gamma}\sup\left\{\frac{\Phi(c)}{\ell_{q'}(f)} : f \geq 0, ||f||_{q,(0,a)} = 1, \ell(f) = c\right\}. \tag{15}$$

By Hölder's inequality, the set $\{f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) = c\}$ is nonempty if and only if $a > \gamma^{q'}$ and $c \in (\gamma, a^{1/q'}]$ (in particular, $C_{q,+}(a) = -\infty$ if $a \leq \gamma^{q'}$). Thus, in the right-hand side of (15), the supremum over $c > \gamma$ is actually the supremum over $c \in (\gamma, a^{1/q'}]$ and, for such values of $c$,

$$\sup\left\{\frac{\Phi(c)}{\ell_{q'}(f)} : f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) = c\right\}$$
$$= \frac{\Phi(c)}{\inf\{\ell_{q'}(f) : f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) = c\}} > 0.$$

(In general, if $G$ is a nonnegative functional, the relation $\sup\frac{1}{G} = \frac{1}{\inf G}$ is true only if the extrema are taken over a nonempty set; same thing with $\inf\frac{1}{G} = \frac{1}{\sup G}$.) As a result,

$$C_{q,+}(a) = \sup_{c\in(\gamma,a^{1/q'}]} \frac{\Phi(c)}{\inf\{\ell_{q'}(f) : f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) = c\}}. \tag{16}$$

The next step is to show that not only the infimum in (16) is a minimum, but also that it is achieved at a *nonincreasing* right-continuous function on $(0, a)$. To do this, we shall use the special case (17) below of an inequality of Luxemburg[2] (see [2, p. 942] or [7, p. 102]).

If $g$ is a measurable function on $(0, a)$, call $\delta_g$ the nonincreasing rearrangement of $g$ and $\iota_g$ the nondecreasing rearrangement of $g$, i.e., $\iota_g(r) := \delta_g((a - r)-)$. Unlike in many other treatments in which the monotone rearrangements of a function $g$ are those of $|g|$ and hence nonnegative, the definition used in [2] and [7] has the property that $\delta_g = g$ a.e. if $g$ is nonincreasing (hence $\iota_g = g$ a.e. if $g$ is nondecreasing) regardless of the sign of $g$. If $g \geq 0$, all the definitions coincide. In particular, $||g||_{p,(0,a)} = ||\delta_{|g|}||_{p,(0,a)}$ for $1 \leq p < \infty$; see for example [10, Theorem 1.8.5].

Luxemburg's inequality states that, if $f$ and $g$ are measurable and $\delta_{|f|}\delta_{|g|} \in L^1(0, a)$ (e.g., $f \in L^q(0, a)$ and $g \in L^{q'}(0, a)$), then

$$\int_0^a \delta_f(r)\iota_g(r)dr \leq \int_0^a f(r)g(r)dr \leq \int_0^a \delta_f(r)\delta_g(r)dr. \tag{17}$$

**Lemma 3.1.** *If $a > \gamma^{q'}$ and $c \in (\gamma, a^{1/q'}]$, there is a nonincreasing right-continuous function $f_c \in L^q(0, a)$, $f_c \neq 0$, such that $f_c \geq 0$, $||f_c||_{q,(0,a)} \leq 1$, $\ell(f_c) = c$ and $\ell_{q'}(f_c) = \inf\{\ell_{q'}(f) : f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) = c\}$.*

[2] In discrete form, this inequality is already in [4, Theorem 368].

**Proof.** The hypotheses about $a$ and $c$ ensure that $\{f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) = c\} \neq \emptyset$, so that there is a sequence $(f_n)$ from that set that minimizes $\ell_{q'}$. By the reflexivity of $L^q(0,a)$, this sequence contains a weakly convergent subsequence, still denoted $(f_n)$ for simplicity, say $f_n \rightharpoonup f_c$ in $L^q(0,a)$.

Since $L^q(0,a) \hookrightarrow L^1(0,a)$, it follows that $c = \int_0^a f_n(r)dr = \int_0^a f_c(r)dr = \ell(f_c)$. Also, $||f_c||_q \leq 1$ (but $f_c \neq 0$ since $c > \gamma \geq 0$) and $f_c \geq 0$ since the convex cone of nonnegative functions is weakly closed in $L^q(0,a)$.

To see that $f_c$ may also be chosen nonincreasing, note that $\delta_{f_c} \geq 0$ since $f_c \geq 0$, whence $||\delta_{f_c}||_{q,(0,a)} = ||f_c||_{q,(0,a)}$. Also, once again since $f_c, \delta_{f_c} \geq 0$, it follows that $c = \ell(f_c) = ||f_c||_{1,(0,a)} = ||\delta_{f_c}||_{1,(0,a)} = \ell(\delta_{f_c})$.

The first inequality in (17) with $f = f_c$ and $g(r) = \Phi'_+(r^{1/q'})$ yields

$$\int_0^a \delta_{f_c}(r)\Phi'_+(r^{1/q'})dr \leq \int_0^a f_c(r)\Phi'_+(r^{1/q'})dr,$$

since $f \in L^q(0,a), g \in L^{q'}(0,a)$ (Lemma 2.1) and $\iota_g = g$ a.e. by the monotonicity of $\Phi'_+$. Since $\Phi' = \Phi'_+$ a.e., the left-hand side is $\ell_{q'}(\delta_{f_c})$ and the right-hand side is $\ell_{q'}(f_c)$, so that $\ell_{q'}(\delta_{f_c}) \leq \ell_{q'}(f_c)$. Since $\delta_{f_c}$ satisfies the required constraints, $\ell_{q'}(\delta_{f_c}) = \ell_{q'}(f_c)$ by the optimality of $f_c$. Thus, it suffices to replace $f_c$ by $\delta_{f_c}$ to obtain the desired monotonicity of $f_c$ with no prejudice to the other properties. In principle, the right-continuity requires only a modification on a countable set, but that $\delta_{f_c}$ is right-continuous is well known anyway ([10, p. 26]). $\square$

### 3.2. Construction of a maximizing sequence and characterization of $C_{q,+}$

With $a > 0$ being fixed, much more information can be obtained about the minimizers $f_c$ of Lemma 3.1. First, it should be noticed that $f \in L^q(0,a)$ and $f \geq 0$ if and only if $f = g^2$ with $g \in L^{2q}(0,a)$. If so, $||f||_{q,(0,a)} \leq 1$ if and only if $||g||_{2q,(0,a)} \leq 1$. From (16), it follows that if $a > \gamma^{q'}$, then

$$C_{q,+}(a) = \sup_{c \in (\gamma, a^{1/q'}]} \frac{\Phi(c)}{\inf\{\ell_{q'}(g^2) : ||g||_{2q,(0,a)} \leq 1, \ell(g^2) = c\}} \tag{18}$$

and the infimum in the right-hand side of (18) is achieved when $g = g_c := \sqrt{f_c}$ with $f_c$ given by Lemma 3.1. In particular, $g_c$ is nonincreasing on $(0,a)$ and right-continuous. For future reference, we record that

$$||g_c||_{2q,(0,a)} \leq 1, \quad \ell(g_c^2) = c \quad \text{and} \quad \ell_{q'}(g_c^2) = \inf\{\ell_{q'}(g^2) : ||g||_{2q,(0,a)} \leq 1, \ell(g^2) = c\}. \tag{19}$$

Let $\Gamma : L^{2q}(0,a) \to \mathbb{R}^2$ denote the mapping

$$\Gamma(g) := \begin{pmatrix} \ell(g^2) \\ ||g||_{2q,(0,a)}^{2q} \end{pmatrix}. \tag{20}$$

Then, $\Gamma$ is $C^1$ since $g \in L^{2q}(0,a) \to \ell(g^2) \in \mathbb{R}$ is quadratic and continuous, hence $C^\infty$ and since also $|| \cdot ||_{p,(0,a)}^p$ is well known to be $C^1$ on $L^p(0,a)$ if $p \in (1,\infty)$.

**Lemma 3.2.** *If $g_0 \in L^{2q}(0,a)$ is nonnegative, nonincreasing and right-continuous on $(0,a)$, then either $g_0$ is a nonnegative multiple of the characteristic function $\chi_{(0,b)}$ of some interval $(0,b)$ with $0 \leq b \leq a$, or the derivative $D\Gamma(g_0)$ is onto $\mathbb{R}^2$.*

**Proof.** With no loss of generality, assume $g_0 \neq 0$ since the result is obvious if $g_0 = 0$. In what follows, $g^{2q-1}$ should be understood as $(g^2)^{q-1}g$, which is well defined regardless of the sign of $g$ at any point. If $g \in L^{2q}(0,a)$, the derivative $D\Gamma(g)$ is the mapping

$$h \in L^{2q}(0,a) \mapsto \begin{pmatrix} 2\ell(gh) \\ 2q\int_0^a g^{2q-1}(r)h(r)dr \end{pmatrix} \in \mathbb{R}^2.$$

Thus, if $D\Gamma(g_0)$ is not onto $\mathbb{R}^2$, the two linear forms $h \mapsto \ell(g_0 h) = \int_0^a g_0 h$ and $h \mapsto \int_0^a g_0^{2q-1}(r)h(r)dr$ on $L^{2q}(0,a)$ are collinear. These linear forms correspond to the two functions $g_0$ and $g_0^{2q-1}$, respectively, of the space $L^{(2q)'}(0,a)$ dual to $L^{2q}(0,a)$ (note $2q > 2 > (2q)'$, so $g_0 \in L^{(2q)'}(0,a)$) and they are collinear if and only if $\lambda g_0 + \mu g_0^{2q-1} = 0$ a.e. for some real constants $\lambda$ and $\mu$, not both 0.

Given $r \in (0,a)$, this means that either $g_0(r) = 0$ or that $\mu \neq 0$ and $g_0^{2q-2}(r) = -\lambda/\mu$. Thus, $-\lambda/\mu > 0$ since $g_0 \neq 0$ and, since $g_0 \geq 0$, it follows that for $r \in (0,a), g_0(r)$ can only be 0 or $\sqrt{-\lambda/\mu}$. But since $g_0 \neq 0$ is also nonincreasing and right-continuous, it must equal $\sqrt{-\lambda/\mu}$ on some interval $(0,b)$ with $0 < b \leq a$ and 0 on its complement, so that $g_0 = \sqrt{-\lambda/\mu}\chi_{(0,b)}$. □

**Lemma 3.3.** *The (nonincreasing, right-continuous) minimizer $f_c$ of Lemma 3.1 may be chosen of the form $f_c(r) = (A - B\Phi'(r^{1/q'}))^{q'/q}\chi_{(0,b)}(r)$, where $0 < b \leq a$ and $A$ and $B \geq 0$ are real constants, not both 0, such that $A - B\Phi'_+(r^{1/q'}) \geq 0$ for $0 < r < b$.*

**Proof.** In this proof, $f_c$ denotes one of the minimizers found in Lemma 3.1, chosen once and for all. First, we discuss the (very) special case when $\Phi$ is linear on $[0, b^{1/q'}]$ where $0 < b \leq a$ *and* $f_c = 0$ on $[b,a)$ if $b < a$ (no condition about $f_c$ if $b = a$). Thus, $\Phi(r) = \lambda r$ for $0 \leq r \leq b^{1/q'}$ and some $\lambda \in \mathbb{R}$ and $\ell_{q'}(f_c) = \int_0^a f_c(r)\Phi'(r^{1/q'})dr = \int_0^b f_c(r)\Phi'(r^{1/q'})dr = \lambda\int_0^b f_c(r)dr = \lambda\ell(f_c) = \lambda c$.

Also, since $c = \ell(f_c) = \int_0^a f_c(r)dr = \int_0^b f_c(r)dr$ and $||f_c||_{q,(0,b)} = ||f_c||_{q,(0,a)} \leq 1$, it follows from Hölder's inequality that $c \leq b^{1/q'}$. Now, define $\widetilde{f}_c := cb^{-1}\chi_{(0,b)}$, so that $\widetilde{f}_c \geq 0$ (recall $c > \gamma \geq 0$), $||\widetilde{f}_c||_{q,(0,a)} = cb^{-1/q'} \leq 1, \ell(\widetilde{f}_c) = c$ and $\ell_{q'}(\widetilde{f}_c) = cb^{-1}\int_0^b \lambda dr = \lambda c = \ell_{q'}(f_c)$. This shows that both $f_c$ and $\widetilde{f}_c$ minimize $\ell_{q'}(f)$ under the same constraints about $f$. Furthermore, $\widetilde{f}_c$ has the desired form with $B = 0$ and $A = (cb^{-1})^{q-1} > 0$ and the lemma is proved by replacing $f_c$ by $\widetilde{f}_c$.

Accordingly, we assume from now on that $\Phi$ is not linear on $[0, a^{1/q'}]$ and that, if it is linear on some interval $[0, b^{1/q'}]$ with $0 < b < a$, then $f_c$ is not 0 on $[b,a)$.

There is nothing to prove if $f_c = A\chi_{(0,b)}$ with $A > 0$ and $0 < b \leq a$. Since $f_c \neq 0$ neither $A = 0$ nor $b = 0$ can occur. Thus, from now on, we assume with no loss of generality that $f_c \neq A\chi_{(0,b)}$ for any $A \geq 0$ and any $0 \leq b \leq a$. As a result, $g_c = \sqrt{f_c}$ is not a nonnegative multiple of any characteristic function $\chi_{(0,b)}, 0 \leq b \leq a$, so that $D\Gamma(g_c)$ is onto $\mathbb{R}^2$ by Lemma 3.2.

If $d := ||g_c||_{2q,(0,a)}^{2q}$, then $0 < d \leq 1$ (since $g_c \neq 0$) and, by (20),

$$g_c \in S := \left\{ g \in L^{2q}(0,a) : \Gamma(g) = \begin{pmatrix} c \\ d \end{pmatrix} \right\}$$
$$\subset \{g \in L^{2q}(0,a) : ||g||_{2q,(0,a)} \leq 1, \ell(g^2) = c\}.$$

Therefore, it follows from (19) that, *a fortiori*, $g_c \in S$ minimizes $\ell_{q'}(g^2)$ for $g \in S$. By the standard Lagrange multiplier theorem ([3, p. 333]), the surjectivity of $D\Gamma(g_c)$ ensures that the derivative of $\ell_{q'}(g^2)$ at $g_c$ is a linear combination of the derivatives of the two scalar components of $D\Gamma(g_c)$, that is,

$$g_c(r)\Phi'_+(r^{1/q'}) = \lambda g_c(r) + \mu g_c^{2q-1}(r),$$

for a.e. $r \in (0,a)$, where $\lambda$ and $\mu$ are real constants. For any such $r$, the equality can hold only if $g_c(r) = 0$ or if $\Phi'_+(r^{1/q'}) = \lambda + \mu g_c^{2q-2}(r)$.

Suppose first that $g_c(r) \neq 0$ for every $r \in (0,a)$. Then, $\mu \neq 0$ since $\Phi$ is not linear on $[0, a^{1/q'}]$. Next, if $g_c(r) = 0$ for some $r \in (0,a)$, then $g_c = 0$ on some maximal interval $[b,a)$ with $0 < b < a$ since $g_c \neq 0$ is nonincreasing and right-continuous. Thus, if $\mu = 0$, it follows that $\Phi'_+(r^{1/q'}) = \lambda$ for $0 < r < b$. But then, $\Phi$ is linear on $[0, b^{1/q'}]$ while $f_c = g_c^2 = 0$ on $[b,a)$, which is ruled out by the standing assumptions.

As a result, $\mu \neq 0$ in all cases and, hence, $g_c^{2q-2}(r) = -\lambda/\mu + (1/\mu)\Phi'_+(r^{1/q'}) > 0$ whenever $g_c(r) \neq 0$. By the monotonicity and right-continuity of $g_c$, it must be that $g_c = 0$ on some maximal interval $[b,a)$ with $0 < b \leq a$ and $g_c \neq 0$ on $(0,b)$. Therefore, $g_c^{2q-2}(r) = A - B\Phi'_+(r^{1/q'}) > 0$ for $0 < r < b$, where $A := -\lambda/\mu$ and $B := -1/\mu$. Note that $B > 0$ since $g_c^{2q-2}$ is nondecreasing and $\Phi'_+$ is nonincreasing. Since $q/q' = q-1$, this proves that $f_c(r) = g_c^2(r) = (A - B\Phi'_+(r^{1/q'}))^{q'/q}\chi_{(0,b)}(r)$. □

**Theorem 3.4.** *The constant $C_{q,+}$ in* (9) *is characterized by*

$$C_{q,+} = \sup_{b > \gamma^{q'}} \sup_{f \in \Sigma_{q,+}(b)} \frac{\Phi(\ell(f))}{\ell_{q'}(f)}, \tag{21}$$

*where $\Sigma_{q,+}(b)$ denotes the set of functions $f \in L^q(0,b)$ (extended by 0 in $[b,\infty)$) of the form $f(r) = (A - B\Phi'_+(r^{1/q'}))^{q'/q}$, where $A$ and $B$ are real constants which are further restricted by the conditions $B \geq 0$, $A - B\Phi'_+(b^{1/q'}-) \geq 0, ||f||_{q,(0,b)} = 1$ and $\ell(f) > \gamma$ (i.e., $\int_0^b f(r)dr > \gamma$; this is redundant if $\gamma = 0$ in* (8)*).*

**Proof.** Since (21) is obvious when $\gamma = \infty$ (both sides are $-\infty$), we henceforth assume that $\gamma < \infty$. Let $(a_n) \subset (0,\infty)$ be such that $\lim_{n\to\infty} a_n = \infty$, so that $\lim_{n\to\infty} C_{q,+}(a_n) = C_{q,+}$ by (13). With no loss of generality, assume $a_n^{1/q'} > \gamma$ and choose $c_n \in (\gamma, a_n^{1/q'}]$ such that $\frac{\Phi(c_n)}{\inf\{\ell_{q'}(f): f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) = c_n\}} \geq C_{q,+}(a_n) - \frac{1}{n}$, which is possible by (18). If $f_{c_n}$ is given by Lemma 3.3 with $c = c_n$, this reads

$$(C_{q,+}(a_n) \geq) \frac{\Phi(\ell(f_{c_n}))}{\ell_{q'}(f_{c_n})} \geq C_{q,+}(a_n) - \frac{1}{n},$$

whence $\lim_{n\to\infty} \frac{\Phi(\ell(f_{c_n}))}{\ell_{q'}(f_{c_n})} = C_{q,+}$.

Since $f_{c_n}(r) = (A_n - B_n\Phi'_+(r^{1/q'}))^{q'/q}\chi_{(0,b_n)}$ for some $b_n > 0$, $A_n \in \mathbb{R}$ and $B_n \geq 0$ such that $A_n - B_n\Phi'_+(r^{1/q'}) > 0$ for $0 < r < b_n$ (hence $A_n - B_n\Phi'_+(b_n^{1/q'}-) \geq 0$), $f_{c_n}$ has all the properties required for membership to $\Sigma_{q,+}(b_n)$, except that $0 < ||f_{c_n}||_q < 1$ is possible. However, it is readily checked that $h_n := \frac{f_{c_n}}{||f_{c_n}||_q} \in \Sigma_{q,+}(b_n)$ and that

$$\frac{\Phi(\ell(h_n))}{\ell_{q'}(h_n)} = \frac{||f_{c_n}||_q \Phi\left(\frac{1}{||f_{c_n}||_q}\ell(f_{c_n})\right)}{\ell_{q'}(f_{c_n})} \geq \frac{\Phi(\ell(f_{c_n}))}{\ell_{q'}(f_{c_n})},$$

the latter since $\lambda\Phi(R/\lambda)$ is a nonincreasing function of $\lambda > 0$ for any $R \geq 0$ by the convexity of $\Phi$ (and $||f_{c_n}||_q \leq 1, \ell_{q'}(f_{c_n}) > 0$; indeed, $\ell_{q'}(f_{c_n}) \geq \Phi(\ell(f_{c_n})) > 0$ by (6) since $\ell(f_{c_n}) = c_n > \gamma$).

From the above, $\sup_{b>0}\sup_{f\in\Sigma_{q,+}(b)} \frac{\Phi(\ell(f))}{\ell_{q'}(f)} \geq \sup_n \frac{\Phi(\ell(h_n))}{\ell_{q'}(h_n)} \geq C_{q,+}$. Conversely, $\sup_{b>0}\sup_{f\in\Sigma_{q,+}(b)} \frac{\Phi(\ell(f))}{\ell_{q'}(f)} \leq C_{q,+}$ by (9) since $\Sigma_{q,+}(b) \subset \{f \geq 0, ||f||_q \leq 1, \ell(f) > \gamma, \ell_{q'}(f) < \infty\}$ for every $b > 0$ (if $f \in \Sigma_{q,+}(b)$, then $\ell_{q'}(f) < \infty$ by Lemma 2.1). This proves (21) with $\sup_{b>\gamma^{q'}}$ replaced by $\sup_{b>0}$, but there is no difference since $\int_0^b f(r)dr > \gamma$ and $||f||_{q,(0,b)} = 1$ are not compatible if $b \leq \gamma^{q'}$. □

### 3.3. Characterization of $C_{q,-}$

With appropriate modifications, a similar procedure can be followed to find a simpler characterization of $C_{q,-}$ in (10). The only case of interest is when $\gamma > 0$ in (8) (if $\Phi \geq 0$, then $C_{q,-} = \infty$), an assumption which is retained in the remainder of this section.

By (6), both $\Phi(\ell(f))$ and $\ell_{q'}(f)$ are finite and strictly negative in the right-hand side of (10), but this does not always require $\ell(f) < \infty$ because $\ell(f) = \infty$ is possible if $\Phi(\infty)$ is finite (hence negative). However, this is not an obstacle to reduce once again the problem to finite intervals: It is straightforward to check that

$$C_{q,-} = \inf_{a>0} C_{q,-}(a) = \lim_{a\to\infty} C_{q,-}(a),$$

with

$$C_{q,-}(a) := \inf_{f\in E} \frac{\Phi(\ell(f))}{\ell_{q'}(f)} = \inf_{c\in(0,\gamma)} \inf_{f\in E_c} \frac{\Phi(c)}{\ell_{q'}(f)},$$

where $E_c := \{f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell_{q'}(f) < 0, \ell(f) = c\}$ and $E := \cup_{c>0}E_c = \cup_{c\in(0,\gamma)}E_c$ (if $c \geq \gamma$, (6) shows that the conditions $\ell_{q'}(f) < 0$ and $\ell(f) = c$ are not compatible, so $E_c = \emptyset$).

From now on, $a > 0$ is fixed. Since it is inconvenient to discuss the minimization of the ratio of two negative quantities, we shall temporarily use

$$\Psi := -\Phi \quad \text{and} \quad k_{q'}(f) := \int_0^a f(r)\Psi'(r^{1/q'})dr = -\ell_{q'}(f).$$

With this notation,

$$C_{q,-}(a) = \inf_{c\in(0,\gamma)} \inf_{f\in E_c} \frac{\Psi(c)}{k_{q'}(f)}$$

and $E_c = \{f \geq 0, ||f||_{q,(0,a)} \leq 1, k_{q'}(f) > 0, \ell(f) = c\}$.

The condition $c \in (0,\gamma)$ does not ensure that $E_c \neq \emptyset$ (for example, $E_c = \emptyset$ if $a^{1/q'} < \gamma$ and $a^{1/q'} < c < \gamma$), but it is clear that $E_c \neq \emptyset$ if $c > 0$ is small enough and that, if $E_{c_0} \neq \emptyset$ for some $c_0 > 0$, then $E_c \neq \emptyset$ for $0 < c \leq c_0$ (if $f \in E_{c_0}$, then $\frac{c}{c_0} f \in E_c$).

This shows that $E_c \neq \emptyset$ for $c$ in a maximal open subinterval $(0, c_\gamma)$ of $(0,\gamma)$ with $c_\gamma > 0$, although $E_{c_\gamma}$ may or may not be empty. This is immaterial for our purposes, for if $E_{c_\gamma} \neq \emptyset$, then every $f \in E_{c_\gamma}$ can be approximated in $L^q(0,a)$ by a sequence $(f_n)$ with $f_n \in E_{c_\gamma - \frac{1}{n}}$ (for example, $f_n := (1 - \frac{1}{nc_\gamma})f$). It follows that

$$C_{q,-}(a) = \inf_{c\in(0,c_\gamma)} \inf_{f\in E_c} \frac{\Psi(c)}{k_{q'}(f)},$$

regardless of whether $E_{c_\gamma}$ is empty or not. Since $E_c \neq \emptyset$ for every $c \in (0, c_\gamma)$, this is also

$$C_{q,-}(a) = \inf_{c\in(0,c_\gamma)} \frac{\Psi(c)}{\sup_{f\in E_c} k_{q'}(f)}.$$

When $E_c \neq \emptyset$ (in particular, $c \in (0, c_\gamma)$), it is obvious from the definition of $E_c$ that $\sup_{f\in E_c} k_{q'}(f) = \sup\{k_{q'}(f) : f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) = c\}$. As a result,

$$C_{q,-}(a) = \inf_{c\in(0,c_\gamma)} \frac{\Psi(c)}{\sup\{k_{q'}(f) : f \geq 0, ||f||_{q,(0,a)} \leq 1, \ell(f) = c\}}.$$

It is straightforward to check that the supremum is a maximum. In addition, it is achieved at some (nonzero) $f_c$ which is nondecreasing and right-continuous. To see this, argue as in the proof of Lemma 3.1, but now using the second inequality in (17) with $f = f_c$ and $g(r) = \Psi'_+(r^{1/q'})$ and note that $g = \delta_g$ a.e. by the monotonicity of $\Psi'_+$.

From the above, it suffices to replace minimization by maximization in the proof of Lemma 3.3 to find out that, with no loss of generality, $f_c$ may be assumed to have the form $f_c(r) = (A - B\Phi'(r^{1/q'}))^{q'/q}\chi_{(0,b)}(r)$, where $0 < b \leq a$ and $A$ and $B \geq 0$ are real constants, not both 0, such that $A - B\Phi'_+(r^{1/q'}) \geq 0$ for $0 < r < b$. Then, an obvious modification of the proof of Theorem 3.4 yields (returning to $\Phi$ and $\ell_{q'}$ instead of $\Psi$ and $k_{q'}$):

**Theorem 3.5.** *The constant $C_{q,-}$ in* (10) *is characterized by*

$$C_{q,-} = \inf_{b>0} \inf_{f\in\Sigma_{q,-}(b)} \frac{\Phi(\ell(f))}{\ell_{q'}(f)}, \tag{22}$$

*where $\Sigma_{q,-}(b)$ denotes the set of functions $f \in L^q(0,b)$ (extended by $0$ in $(b,\infty)$) of the form $f(r) = (A - B\Phi'_+(r^{1/q'}))^{q'/q}$, where $A$ and $B$ are real constants which are further restricted by the conditions $B \geq 0, A - B\Phi'_+(b^{1/q'}-) \geq 0, ||f||_{q,(0,b)} = 1$ and $\ell_{q'}(f) < 0$ (i.e., $\int_0^b f(r)\Phi'_+(r^{1/q'})dr < 0$; this is redundant if $\gamma = \infty$ in* (8)*).*

While Theorems 3.4 and 3.5 considerably restrict the class of functions needed to evaluate the constants $C_{q,\pm}$, this task often remains analytically nontrivial. It is shown in the next section that the problem can be greatly simplified when $q = 2$, but a number of other questions remain open at this time. For example, issues about the $q$-dependence of $C_{q,\pm}$, such as monotonicity or continuity. It is clear from (9) and (10) that $C_{q,+}$ ($C_{q,-}$) is lsc (usc), but continuity is more elusive and may well depend on extra properties of $\Phi$. Another question, closely related to Theorem 2.4, is whether $C_{q,+} < 1$ ($C_{q,-} > 1$) whenever $\Phi'_+(0) = 0$ ($\Phi'_+(0) = -\infty$).

## 4. The case $q = 2$

When $q = 2$, the calculation of $C_{2,\pm}$ can be reduced to an optimization problem much simpler than described in Theorems 3.4 and 3.5. Details follow.

From now on, $\Phi$ is given and $q = 2$ $(= q')$. Consistent with Theorems 3.4 and 3.5, let $f(r) := A - B\Phi'_+(r^{1/2})$ for $r \in (0, b)$ with $b > 0$, where $A \in \mathbb{R}$ and $B \geq 0$ are constant. We also assume

$$A - B\Phi'_+(b^{1/2}-) \geq 0 \tag{23}$$

and $||f||_2 = 1$. Set

$$I(b) := \int_0^b \Phi'(r^{1/2})dr \quad \text{and} \quad J(b) = \int_0^b \Phi'(r^{1/2})^2 dr - b^{-1}I(b)^2 \tag{24}$$

and note that, by the Cauchy-Schwarz inequality, $J(b) > 0$ unless $\Phi$ is linear on $[0, b^{1/2}]$. From now on, we assume that $J(b) > 0$. For the case when $\Phi$ is linear on $[0, b^{1/2}]$ for some $b > 0$, see the comments after Remark 4.1.

A simple calculation reveals that the condition $||f||_2 = 1$ amounts to

$$X^2 + Y^2 = 1, \tag{25}$$

where (recall $B \geq 0$)

$$X := Ab^{1/2} - Bb^{-1/2}I(b) \quad \text{and} \quad Y := BJ(b)^{1/2} \geq 0 \tag{26}$$

Since $J(b) > 0$, these relations can be inverted to yield

$$A = b^{-1/2}X + b^{-1}I(b)J(b)^{-1/2}Y \quad \text{and} \quad B = J(b)^{-1/2}Y. \tag{27}$$

In terms of $X$ and $Y$, the inequality (23) becomes $X \geq b^{1/2}J(b)^{-1/2}K(b)Y$, where

$$K(b) := \Phi'_+(b^{1/2}-) - b^{-1}I(b) > 0. \tag{28}$$

Above, the positivity of $K(b)$ follows from the monotonicity and non-constancy of $\Phi'_+$ on $(0, b^{1/2})$ (since $\Phi$ is not linear on $[0, b]$). Thus, $X \geq 0$ since $Y \geq 0$ and so, from (25), $Y = (1 - X^2)^{1/2}$. It follows that $X \geq b^{1/2}J(b)^{-1/2}K(b)Y$ if and only if (note that the right-hand side is strictly less than 1)

$$X \geq L(b) := b^{1/2}K(b)(J(b) + bK(b)^2)^{-1/2} > 0. \tag{29}$$

Another elementary calculation shows that

$$\ell(f) = b^{1/2}X \quad \text{and} \quad \ell_2(f) = b^{-1/2}I(b)X - J(b)^{1/2}(1-X^2)^{1/2}. \tag{30}$$

Conversely, if $b > 0$ is fixed and $X \in [b^{1/2}K(b)(J(b) + bK(b)^2)^{-1/2}, 1]$ is arbitrary, then the function $f(r) = A - B\Phi'_+(r^{1/2})$ with $A$ and $B$ given by (27) with $Y = (1 - X^2)^{1/2}$ satisfies (23) (hence $f \geq 0$ on $(0, b)$) and $||f||_2 = 1$. Furthermore, by (30) and (24), $\ell(f) > \gamma$ (as required in Theorem 3.4) if and only if $X > b^{-1/2}\gamma$ and $\ell_2(f) < 0$ (as required in Theorem 3.5) if and only if $X < M(b) \leq 1$, where

$$M(b) := \begin{cases} 1 & \text{if } I(b) < 0, \\ J(b)^{1/2}\left(\int_0^b \Phi'(r^{1/2})^2 dr\right)^{-1/2} & \text{if } I(b) \geq 0. \end{cases} \tag{31}$$

As a result, if $\gamma < \infty$ in (8), then

$$C_{2,+} = \sup_{b>\gamma^2, X\in[L(b),1], X>b^{-1/2}\gamma} \frac{\Phi\left(b^{1/2}X\right)}{b^{-1/2}I(b)X - J(b)^{1/2}(1-X^2)^{1/2}} \tag{32}$$

(and $C_{2,+} = -\infty$ if $\gamma = \infty$) and, irrespective of $\gamma$ (but see Remark 4.1 below when $\gamma = 0$)

$$C_{2,-} = \inf_{b>0, X\in[L(b),M(b))} \frac{\Phi\left(b^{1/2}X\right)}{b^{-1/2}I(b)X - J(b)^{1/2}(1-X^2)^{1/2}}. \tag{33}$$

The ratios in the right-hand sides of (32) and (33) are always positive for the specified values of the parameters. The advantage of (32) and (33) lies in the fact that the $(b, X)$-dependence of the function to be optimized *and* of the constraints is much simpler than the analogous $(b, A, B)$-dependence in Theorems 3.4 and 3.5.

**Remark 4.1.** It is not immediately clear that (33) yields $C_{2,-} = \infty$ if $\gamma = 0$, but this follows from $M(b) \leq L(b)$ for every $b > 0$ if $\Phi \geq 0$ (hence $\Phi' \geq 0$ a.e. by convexity). Indeed, a short calculation and $I(b) \geq 0$ show that $M(b) \leq L(b)$ is equivalent to $J(b) \leq I(b)K(b)$. In turn, by (24) and (28), the latter inequality is just $\int_0^b \Phi'(r^{1/2})^2 dr \leq \Phi'_+(b^{1/2}-)\int_0^b \Phi'(r^{1/2})dr$, which holds since $\Phi' \geq 0$ is nondecreasing.

If $\Phi$ is linear on some interval $[0, b^{1/2}]$ with $b > 0$, then $J(b) = K(b) = 0$ and $L(b)$ is not defined, but this has no negative impact on the end results.

First, it is readily seen that $\Phi$ cannot be linear on any interval $[0, b^{1/2}]$ with $b > \gamma^2$ (i.e., linear on an interval strictly longer than $[0, \gamma]$) if $0 < \gamma < \infty$. Thus, since every $b > 0$ involved in (32) satisfies $b > \gamma^2$, the case when $\Phi$ is linear on $[0, b^{1/2}]$ never happens and (32) remains valid when $0 < \gamma < \infty$ even if $\Phi$ is linear on some interval $[0, b]$ (with, of necessity, $b \leq \gamma^2$).

If $\gamma = \infty$, it is obvious that $C_{2,+} = -\infty$ and, if $\gamma = 0$, the fact that $\Phi$ is linear on some nontrivial interval $[0, b]$ implies $\Phi'_+(0) > 0$, so that $C_{2,+} = 1$ by Theorem 2.4.

The case of (33) is similar: If $\gamma = 0$, then $C_{2,-} = \infty$ whereas, if $0 < \gamma \leq \infty$ and $\Phi$ is linear on some nontrivial interval $[0, b]$, then $-\infty < \Phi'_+(0) < 0$, so that $C_{2,-} = 1$ by Theorem 2.4.

In summary, irrespective of $\Phi$, the value of $C_{2,+}$ or that of $C_{2,-}$ is always given by one of the following:

(i) It is trivially infinite.
(ii) It is 1 by Theorem 2.4.
(iii) It is given by the unambiguously defined relevant formula (32) or (33).

## 5. Examples

We shall use the formulas (32) and (33) to find $C_{2,\pm}$ in a few simple enough but instructive examples. These constants can be exactly calculated in a number of other cases. For brevity, we skip the straightforward calculations of $I(b), J(b), K(b)$ and $L(b)$ (notation of the previous section). In all four examples, it turns out that $L(b)$ is independent of $b$. This is somewhat surprising, but just a coincidence. In general, $L(b)$ does depend upon $b$.

**Example 5.1.** Let $\Phi(r) := r^{\beta+1}$ with $\beta > 0$. Then, $C_{2,-} = -\infty$ since $\Phi \geq 0$ and the optimal constant $C$ in (7) is $C_{2,+}$. Theorem 2.4 is not applicable since $\Phi'_+(0) = 0$, but since $I(b) = \frac{2(\beta+1)}{\beta+2} b^{\frac{\beta}{2}+1}, J(b) = \frac{(\beta+1)\beta^2}{(\beta+2)^2} b^{\beta+1}, K(b) = \frac{\beta(\beta+1)}{\beta+2} b^{\frac{\beta}{2}}$ and $L(b) = (\frac{\beta+1}{\beta+2})^{1/2}$, (32) yields (note the $b$-independence)

$$C_{2,+} = \frac{\beta+2}{(\beta+1)^{1/2}} \sup_{X \in \left[\left(\frac{\beta+1}{\beta+2}\right)^{1/2}, 1\right]} \frac{X^{\beta+1}}{2(\beta+1)^{1/2} X - \beta(1-X^2)^{1/2}}.$$

The function in the right-hand side is decreasing on $[(\frac{\beta+1}{\beta+2})^{1/2}, 1]$ (its derivative vanishes only at the left endpoint and tends to $-\infty$ when $X \to 1-$). Thus,

$$C_{2,+} = \left(\frac{\beta+1}{\beta+2}\right)^{\beta/2} < 1.$$

From Remark 2.3, it follows that

$$||f||_1 \leq C_{\beta,N} ||f||_2^{\frac{\beta}{\beta+1}} \left( \int_{\mathbb{R}^N} |f(x)||x|^{\frac{N\beta}{2}} dx \right)^{\frac{1}{\beta+1}}, \tag{34}$$

for every $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, where $C_{\beta,N} := \omega_N^{\frac{\beta}{2\beta+2}} (\beta+1)^{\frac{1}{\beta+1}} (\frac{\beta+1}{\beta+2})^{\frac{\beta}{2\beta+2}}$ is optimal. This shows that while $||f||_1$ cannot be controlled by $||f||_2$, it can be controlled by a (nonlinear) combination of $||f||_2$ and $\int_{\mathbb{R}^N} |f(x)||x|^{N\beta/2} dx$, although the latter may be infinite. With $\alpha > 0$ and $\beta := 2\alpha/N$, (34) may also be written as

$$\int_{\mathbb{R}^N} |f(x)||x|^{\alpha} dx \geq C'_{\alpha,N} ||f||_1^{1+2\alpha/N} ||f||_2^{-\frac{2\alpha}{N}}, \tag{35}$$

for every $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), f \neq 0$, where $C'_{\alpha,N}$ is an optimal constant readily obtained from $C_{\beta,N}$.

**Example 5.2.** Let $\Phi(r) := r^3 - r$. First, $\Phi'_+(0) = -1$, so that $C_{2,-} = 1$ by Theorem 2.4, but it remains to find $C_{2,+}$. Since $\gamma = 1$, $I(b) = \frac{3b^2}{2} - b$, $J(b) = \frac{3b^3}{4}$, $K(b) = \frac{3b}{2}$ and $L(b) = \frac{\sqrt{3}}{2}$, (32) takes the form

$$C_{2,+} = \sup_{b>1, X\in\left[\frac{\sqrt{3}}{2},1\right], X>b^{-1/2}} \frac{2(X^3 b - X)}{(3b-2)\,X - b\sqrt{3}(1-X^2)^{1/2}}.$$

If $X \in [\frac{\sqrt{3}}{2}, 1]$ is fixed, the ratio in the right-hand side is an increasing function of $b$. Indeed, $-4X^4 + 6X^2 - 2\sqrt{3}X(1-X^2)^{1/2} = 2X(3X - 2X^3 - \sqrt{3}(1-X^2)^{1/2}) > 0$ on $[\frac{\sqrt{3}}{2}, 1]$, which can for instance be seen from $(1-X^2)^{1/2} \leq (\frac{2}{\sqrt{3}} - 1)X$ on that interval. By letting $b$ tend to $\infty$ (so that $X > b^{-1/2}$ holds), it follows that $C_{2,+} = \sup_{X\in[\frac{\sqrt{3}}{2},1]} \frac{2X^3}{3X-\sqrt{3}(1-X^2)^{1/2}}$. It is easily checked that this function is decreasing on $[\frac{\sqrt{3}}{2}, 1]$, so that its maximum is achieved when $X = \frac{\sqrt{3}}{2}$. Thus,

$$C_{2,+} = \frac{3}{4}.$$

From Remark 2.3,

$$||f||_1 \left(\omega_N^{-1}||f||_1^2||f||_2^{-2} - 1\right) \leq C \int |f(x)|(3|x|^N - 1)dx, \tag{36}$$

for every $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $f \neq 0$ and every $C \in [\frac{3}{4}, 1]$. There is no optimal $C$ in this inequality, but $C = \frac{3}{4}$ is optimal when $||f||_1 > \sqrt{\omega_N}||f||_2$ whereas $C = 1$ is optimal when $\int_0^\infty |f(x)|(3|x|^N - 1)dx < 0$.

**Example 5.3.** Let $\Phi(r) := -r^\beta$ with $0 < \beta < 1$, so that $\Phi'_+(0) = -\infty$. Since $\Phi \leq 0$, it follows that $C_{2,+} = -\infty$ and the optimal constant $C$ in (7) is $C_{2,-}$. Theorem 2.4 is not applicable but $I(b) = -\frac{2\beta}{\beta+1}b^{(\beta+1)/2}$, $J(b) = \frac{\beta(1-\beta)^2}{(\beta+1)^2}b^\beta$, $K(b) = \frac{\beta(1-\beta)}{\beta+1}b^{(\beta-1)/2}$ and $L(b) = (\frac{\beta}{\beta+1})^{1/2}$, $M(b) = 1$ for every $b > 0$. By (33),

$$C_{2,-} = \inf_{X\in\left[\left(\frac{\beta}{\beta+1}\right)^{1/2},1\right)} \frac{(\beta+1)X^\beta}{2\beta X + \beta^{1/2}(1-\beta)(1-X^2)^{1/2}}.$$

The function in the right-hand side is increasing on $[(\frac{\beta}{\beta+1})^{1/2}, 1)$ (its derivative vanishes only at the left endpoint and tends to $\infty$ as $X \to 1-$). As a result,

$$C_{2,-} = \left(\frac{\beta}{\beta+1}\right)^{(\beta-1)/2} > 1.$$

The corresponding inequality for $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is

$$\int_{\mathbb{R}^N} \frac{|f(x)|}{|x|^{N(1-\beta)/2}dx} \leq C_{\beta,N}||f||_1^\beta||f||_2^{1-\beta}, \tag{37}$$

with optimal constant $C_{\beta,N} := \omega_N^{(1-\beta)/2}\frac{1}{\beta}(\frac{\beta}{\beta+1})^{(1-\beta)/2}$. That an inequality of this type holds can easily be seen by splitting the integral in the left-hand side over a ball $B_R$ and its complement, estimating both terms and optimizing $R$. However, this only provides (37) with $C_{\beta,N}$ replaced by $\omega_N^{(1-\beta)/2}\frac{1}{\beta}(\frac{\beta}{1-\beta})^{(1-\beta)/2}$, which is not even as sharp as (4) (with $q = 2$) for this example.

**Example 5.4.** Let $\Phi(r) := r \operatorname{Log} r - r$, so that $\Phi'_+(0) = -\infty$ and $\gamma = e$. Theorem 2.4 does not provide any information about $C_{2,+}$ or $C_{2,-}$. In this example, $I(b) = \frac{b}{2}(\operatorname{Log} b - 1)$, $J(b) = \frac{b}{4}$, $K(b) = \frac{1}{2}$ and $L(b) = \frac{\sqrt{2}}{2}$. Therefore, (32) yields

$$C_{2,+} = \sup_{b>e^2, X\in\left[\frac{\sqrt{2}}{2},1\right], X>b^{-1/2}e} \frac{X \operatorname{Log} b + 2X(\operatorname{Log} X - 1)}{X \operatorname{Log} b - X - (1-X^2)^{1/2}}.$$

If $X \in [\frac{\sqrt{2}}{2}, 1]$ is fixed, the ratio in the right-hand side is an increasing function of $\operatorname{Log} b$ and hence of $b$. Furthermore, the condition $X > b^{-1/2}e$ holds as soon as $b > 2e^2$. The monotonicity claim follows from the fact that $X^2 - X(1-X^2)^{1/2} - 2X^2 \operatorname{Log} X = X(X-(1-X^2)^{1/2} - 2X \operatorname{Log} X) > 0$ because $X-(1-X^2)^{1/2} \geq 0$ and $\operatorname{Log} X \leq 0$ when $X \in [\frac{\sqrt{2}}{2}, 1]$ and at least one inequality is strict. Thus, by letting $b \to \infty$, it follows that $C_{2.,+} \geq 1$, whence $C_{2,+} = 1$ since the reverse inequality is always true.

Next, by (33), $C_{2,-} = \inf_{b>0,[\frac{\sqrt{2}}{2},M(b))} \frac{X \operatorname{Log} b + 2X(\operatorname{Log} X - 1)}{X \operatorname{Log} b - X - (1-X^2)^{1/2}}$, where $M(b)$ is given by (31). In general, $M(b)$ depends upon $b$, but $M(b) = 1$ if $b < e$ since $I(b) < 0$ in this case. Thus,

$$C_{2,-} \leq \inf_{b\in(0,e), X\in\left[\frac{\sqrt{2}}{2},1\right)} \frac{X \operatorname{Log} b + 2X(\operatorname{Log} X - 1)}{X \operatorname{Log} b - X - (1-X^2)^{1/2}}.$$

Once again, if $X \in [\frac{\sqrt{2}}{2}, 1)$, the ratio in the right-hand side is an increasing function of $\operatorname{Log} b$, so that its infimum for $b \in (0, e)$ is obtained by letting $b \to 0$. This shows that $C_{2,-} \leq 1$ and hence that $C_{2,-} = 1$ since the reverse inequality is always true. In summary,

$$C_{2,+} = C_{2,-} = 1,$$

which means that $C = 1$ is the only possible constant in (7) when $q = 2$.

Therefore,

$$||f||_1 \left( \operatorname{Log} \frac{||f||_1}{\sqrt{\omega_N}||f||_2} - 1 \right) \leq \frac{N}{2} \int_{\mathbb{R}^N} |f(x)| \operatorname{Log} |x| dx, \tag{38}$$

for every $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $f \neq 0$ and $\frac{N}{2}$ cannot be replaced by any other constant. For example, (38) implies that $\int_{\mathbb{R}^N} |f(x)| \operatorname{Log} |x| dx > 0$ as soon as $||f||_1 > e\sqrt{\omega_N}||f||_2$, which is not obvious in the first place.

**Remark 5.5.** Of course, (4) yields inequalities similar to (34), (36), (37) and (38) (and countless others) when $f \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for any $q \in (1, \infty)$, but not necessarily with the best constant.

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