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Minimal Systolic Circles

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Remembering Archimedes' words "Do not disturb my circles".

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We control the evolution of convex cyclic polygons by calculating the corresponding evolutionary circumradius (minimal systolic circle) each time a convex polygon is inscribed to a circle until it reaches the termination circle (minimum systolic circle) of the isoperimetric problem. We show that there exists a minimal circumradius for weighted convex quadrilaterals and pentagons such that their sides are given by the variable weights which satisfy the isoperimetric condition of the corresponding inverse weighted Fermat-Torricelli problem and the dynamic plasticity equations in the two dimensional Euclidean space. By splitting the weights along the prescribed rays which meet at the corresponding weighted Fermat-Torricelli point we deduce the generalized plasticity equations for convex polygons and we show that for a large number of variable weights the minimal circumradius approaches the minimum circumradius which corresponds to a regular polygon for equal weights. Furthermore, we obtain that the Gauss' minimal systolic circle of the generalized Gauss problem is smaller than the Fermat's minimal systolic circle of the Fermat-Torricelli problem for convex quadrilaterals.

Keywords: Isoperimetric inequality, polygons, dynamic plasticity, generalized plasticity, inverse Fermat-Torricelli problem, systolic circle, Fermat-Torricelli problem, Gauss problem

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1. Introduction

The isoperimetric problem in the two dimensional Euclidean Space states that:

The area E and the length L of any plane domain with rectifiable boundary satisfy the inequality

$$L^2 \geq 4\pi E.$$

Equality holds only if the plane domain is a circle.

Regarding the main expository contributions to the isoperimetric problem we may consult [5], [4] and [11].

We give the following two restricted isoperimetric problems for convex polygons in $\mathbb{R}^2$ (see in [7] for Theorem 1.2):

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Theorem 1.1. *Among plane polygons with a given length of sides (given perimeter), the largest area is bounded by the convex polygon inscribed to a circle,*

Theorem 1.2. *Among plane polygons with a given number of sides and given perimeter, the largest area is bounded by the regular polygon.*

In this paper, we solve the restricted isoperimetric problem for convex polygons with a given length of sides (Theorem 3.1) by applying a method of angular differentiation of the area of the polygon which has been introduced in [12] for convex quadrilaterals and giving a simple numerical calculation of the radius of the inscribed circle in $\mathbb{R}^2$. Following this result, we show that by maximizing the maximum area of convex polygon already inscribed to a circle, we obtain that the largest area is bounded by a regular polygon (Theorem 3.2). A further synthesis of Theorem 3.1 on a thin partition of the interval $[0, L]$ yields a solution to the isoperimetric problem as a limiting case in the two dimensional Euclidean Space.

Applying the computation of circumradius for convex cyclic polygons in $\mathbb{R}^2$ (systolic circles-definitions 6.12 and 6.13) we prove the existence of a minimal circumradius for weighted triangles, convex quadrilaterals and polygons (Corollary 6.20, Proposition 6.19 and Proposition 6.18, Examples 6.22 and 6.23) having sides with lengths the variable weights which satisfy the condition of the corresponding inverse weighted Fermat-Torricelli problem and the dynamic plasticity equations $\mathbb{R}^2$ (Theorem 6.6). Furthermore, we derive the generalized plasticity equations for convex weighted polygons (Proposition 6.28) by splitting the weights along the prescribed rays which meet at the corresponding weighted Fermat-Torricelli and we show that for a large number of variable weights the minimal circumradius approaches the minimum circumradius which corresponds to a regular polygon for equal weights (Proposition 6.29). Finally, we obtain that the Gauss' minimal systolic circle (Gauss circle-definition 7.4) of the generalized Gauss problem having two Fermat-Torricelli points at the interior of the quadrilateral is smaller than the Fermat's minimal systolic circle (Fermat circle-definition 7.5) of the Fermat-Torricelli problem for convex quadrilaterals (Examples 7.6 and 7.7).

2. The restricted isoperimetric problem for convex quadrilaterals

Let $A_1A_2A_3A_4$ be a convex quadrilateral in $\mathbb{R}^2$. We denote by a_{ij} the length of the line segment A_iA_j and by α_{ijk} the angle which is formulated between A_iA_j and A_jA_k, for $i, j, k = 1, 2, 3, 4$ and $i \neq j \neq k$.

Lemma 2.1 ([12]). *Among plane convex quadrilaterals with a given length of sides (given perimeter), the largest area is bounded by the convex polygon inscribed to a circle.*

Theorem 2.2. *Let $A_1A_2A_3A_4$ be a convex quadrilateral with a given perimeter L. Then the maximum of the maximum area of quadrilateral is obtained by the tetragon.*

Proof of Theorem 2.1. We shall prove that the maximum of the maximum area of convex quadrilaterals attained on a circle is obtained by the tetragon $a_{12} = a_{23} = a_{34} = a_{41} = \frac{L}{4}$.

Let a_{12}, a_{23}, a_{34} and a_{41} are four specific values such that

$$a_{12} + a_{23} + a_{34} + a_{41} = L.$$

From Theorem 2.1, we derive that the class of convex quadrilaterals $A_1A_2A_3A_4$ belong to a family of evolutionary circles.

Hence, the area of convex quadrilaterals inscribed to a circle is given by the Brahmagupta formula:

$$\max E = \sqrt{(S - a_{12})(S - a_{23})(S - a_{34})(S - a_{41})} \tag{1}$$

or

$$2 \ln \max E = \ln \frac{1}{\frac{L}{2} - a_{12}} + \ln \frac{1}{\frac{L}{2} - a_{23}} + \ln \frac{1}{\frac{L}{2} - a_{34}} + \ln \frac{1}{\frac{L}{2} - a_{41}} \tag{2}$$

By differentiating (2) with respect to a_{ii+1}, we get:

$$2 \frac{1}{\max E} \frac{\partial \max E}{\partial a_{ii+1}} = \frac{1}{\frac{L}{2} - a_{ii+1}} - \frac{1}{\sum_{i=1}^{3} a_{jj+1} - \frac{L}{2}} = 0,$$

or

$$a_{ii+1} = \frac{L}{4},$$

for $i = 1, 2, 3, 4$ and $a_{45} = a_{41}$. □

3. The restricted isoperimetric problem for convex polygons

Let $A_1A_2A_3...A_n$ be a convex polygon in $\mathbb{R}^2$.

Theorem 3.1. *Among plane convex polygons with a given length of sides (given perimeter), the largest area is bounded by a convex polygon inscribed to a circle.*

Proof of Theorem 3.1. First, we prove that among convex pentagons with given length of sides a_{ii+1} where $a_{56} = a_{51}$, the largest area is bounded by a convex pentagon inscribed to a circle.

We denote by x_1, the diagonal that connects A_3 with A_1 and by x_5 the diagonal that connects A_3 with A_5 (see Figure 3.1).

By applying the cosine law in $\triangle A_1A_2A_3$ and $A_1A_3A_5$ we get:

$$x_1^2 = a_{12}^2 + a_{23}^2 - 2a_{12}a_{23} \cos \alpha_{123} = a_{51}^2 + x_5^2 - 2a_{51}x_5 \cos \alpha_{153}. \tag{3}$$

or

$$\cos \alpha_{153} = \frac{a_{51}^2 + x_5^2 - a_{12}^2 - a_{23}^2 + 2a_{12}a_{23} \cos \alpha_{123}}{2a_{51}x_5}. \tag{4}$$

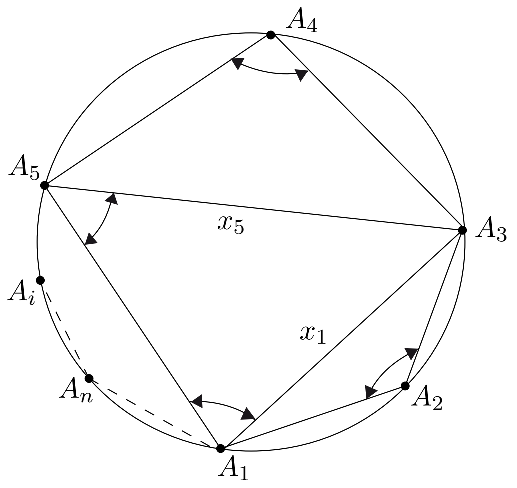

Figure 3.1

By applying the cosine law in $\triangle A_3A_4A_5$, we get:

$$x_5^2 = a_{34}^2 + a_{45}^2 - 2a_{34}a_{45}\cos\alpha_{345}. \tag{5}$$

We express the area of $A_1A_2A_3A_4A_5$ as a function of α_{123} and α_{345} :

$$2E = a_{12}a_{23}\sin\alpha_{123} + a_{15}x_5\sin\alpha_{153} + a_{45}a_{34}\sin\alpha_{345}. \tag{6}$$

By replacing (4) and (5) in (6), we obtain

$$E = E(\alpha_{123}, \alpha_{345}). \tag{7}$$

By differentiating (7) with respect to α_{123} we get:

$$2\frac{\partial E}{\partial \alpha_{123}} = a_{12}a_{23}\cos\alpha_{123} + a_{51}x_5\cos\alpha_{153}\frac{\partial \alpha_{153}}{\partial \alpha_{123}} = 0. \tag{8}$$

By differentiating (4) with respect to α_{123}, we derive:

$$\frac{\partial \alpha_{153}}{\partial \alpha_{123}} = \frac{a_{12}a_{23}\sin\alpha_{123}}{a_{51}x_5\sin\alpha_{153}}. \tag{9}$$

By replacing (9) in (8), we obtain that

$$\sin(\alpha_{123} + \alpha_{153}) = 0,$$

or

$$\alpha_{123} + \alpha_{153} = \pi.$$

Therefore, $A_1A_2A_3A_5$ is inscribed to a circle.

By working cyclically and by connecting A_{i+2} with A_i and A_{i+4}, in order to formulate the diagonals, we derive that: $A_iA_{i+1}A_{i+2}A_{i+3}$ is inscribed to a circle. Therefore, $A_1A_2A_3A_4A_5$ is inscribed to the same circle with $A_iA_{i+1}A_{i+2}A_{i+4}$.

Similarly, by connecting A_{i+3} with $n-3$ vertices of the convex n polygon $A_1A_2A_3...A_n$, we derive a partition of the convex n polygon to convex quadrilaterals. Thus, we derive that $A_iA_{i+1}A_{i+2}A_{i+4}$ is a convex quadrilateral inscribed to a circle for $i = 1, 2, 3, ..., n$. Therefore, $A_1A_2A_3...A_n$ is a convex polygon inscribed to a circle. □

We consider a restriction of Theorem 1.2 for convex polygons.

Theorem 3.2. *Among plane convex polygons with a given number of sides and given perimeter, the largest area is bounded by the regular polygon.*

Proof of Theorem 3.2. Let $A_1A_2A_3..A_n$ be a convex polygon with a given perimeter L :

$$\sum_{i=1}^{n} a_{ii+1} = L, \tag{10}$$

where $a_{nn+1} = a_{n1}$.

By applying Theorem 3.1 for any given values of the length of the sides a_{ii+1}, such that $A_1A_2A_3..A_n$ is a convex polygon, we obtain that $A_1A_2A_3..A_n$ is inscribed to a circle of radius R.

We consider a convex cyclic polygon as a synthesis of isosceles triangles with central angles φ_{i0i+1} which correspond to the side a_{ii+1}, for $i, j = 1, 2, 3, ..., n$ and $a_{nn+1} = a_{n1}$, $\varphi_{n0n+1} = \varphi_{n01}$.

We express the area of the convex cyclic polygon as a function of $(n-1)$ variables φ_{i0i+1}, for $i = 1, 2, 3, ..., n$.

$$E_{convcycl} := \frac{R^2}{2} \sum_{i=1}^{n} \sin \varphi_{i0i+1}, \tag{11}$$

or

$$E_{convcycl} := \frac{R^2}{2} \left(\sum_{i=1}^{n-1} \sin \varphi_{i0i+1} + \sin \left(2\pi - \sum_{i=1}^{n-1} \varphi_{i0i+1} \right) \right), \tag{12}$$

By differentiating (12) with respect to φ_{i0i+1}, we get:

$$\cos \varphi_{i0i+1} = \cos \varphi_{n01},$$

for $i = 1, 2, 3, ..., n-1$. Therefore, we derive that $a_{ii+1} = a_{n1}$. □

Remark 3.3. We note that convex cyclic polygons is a subclass of cyclic polygons in $\mathbb{R}^2$. The generalization of Brahmagupta's formula for the area of cyclic polygons is a difficult task even for cyclic pentagons and hexagons and has been studied in [13]. Considering Robbins's result the formula of the area of convex cyclic polygon cannot be determined only by the formula

$$\max E = \sqrt{(S - a_{12})(S - a_{23})(S - a_{34})(S - a_{41})...(S - a_{n1})}.$$

4. A computational approach to calculate the radius of the circumscribed circle.

Let $A_1A_2...A_n$ be a convex polygon with given length of sides a_{ii+1}, such that:

$$\sum_{i=1}^{n} \frac{a_{ii+1}}{2\pi} > \frac{\max\{a_{ii+1}\}}{2},$$

for $i = 1, 2, 3, .., n$. Then $A_1A_2...A_n$ is inscribed to a circle of radius R.

The radius of the circumscribed circle is calculated by the following formula:

$$\sum_{i=1}^{n} \arcsin \frac{a_{ii+1}}{2R} = \pi,$$

for

$$R^{\circ} = \sum_{i=1}^{n} \frac{a_{ii+1}}{2\pi}.$$

Example 4.1. Given $a_{12} = 1, a_{23} = 2, a_{34} = 3, a_{45} = 4, a_{56} = 5, a_{61} = 6,\ L = 21$, we obtain the initial value of the radius $R^{\circ} = \frac{21}{2\pi}$. By using a numerical method to solve the equation

$$\sum_{i=1}^{n} \arcsin \frac{a_{ii+1}}{2R} = \pi,$$

we obtain $R = 3.64631$ with 5-digit precision.

5. The isoperimetric problem in $\mathbb{R}^2$

We proceed by proving the isoperimetric inequality for closed curves in $\mathbb{R}^2$.

Theorem 5.1. *The area E and the length L of any plane domain with rectifiable boundary satisfy the inequality*

$$L^2 \geq 4\pi E.$$

Equality holds only if the plane domain is a circle.

Proof of Theorem 5.1. V. A. Toponogov proved (see in [14], Problem 1.5.5, page 18) by contradiction that the closed rectifiable curve referring to the isoperimetric problem is convex. By applying the results of Theorem 3.1 and Theorem 3.2, we choose a sufficiently thin partition of the interval $[0, L]$ such that the derived convex polygons approaches the circle of radius $R = \frac{L}{2\pi}$. □

6. Minimal systolic circles

We recall the definition of the dynamic plasticity of convex polygons.

Definition 6.1. We call *dynamic plasticity* of a weighted network in $\mathbb{R}^2$ which is formulated by n weighted prescribed rays meeting at the weighted Fermat-Torricelli point the ability of the network to change their weights preserving the same Fermat-Torricelli point and the boundary of the convex polygon.

Let $A_1A_2\ldots A_n$ be n given non collinear points in $\mathbb{R}^2$. We denote by B_i a positive real number(weight) which corresponds to the vertex A_i, a_{jk}, the length of the line segment A_jA_k and α_{jkl} the angle which is formulated between the line segments A_jA_k and A_kA_l, at A_k for $i = 1, 2, 3\ldots, n$, $j, k, l = 0, 1, 2, 3, \ldots, n$, for $j \neq k \neq l$.

We state the weighted Fermat-Torricelli problem (w.F-T problem) and the inverse weighted Fermat-Torricelli problem (inverse w.F-T problem) in $\mathbb{R}^2$ (see in [1] Chapter II for a detailed exposition for Euclidean Spaces) and we denote by A_0 the weighted Fermat-Torricelli point (w.F-T point) which gives a solution to the w.F-T problem.

Problem 6.2. *Find a point $A_0 \in \mathbb{R}^2$ such that:*

$$f(A_0) = B_1a_{10} + B_2a_{20} + B_3a_{30} + B_4a_{40} + \cdots + B_na_{i0} \tag{13}$$

is minimized.

The existence and uniqueness of the w.F-T point in Euclidean Spaces and a complete characterization of the w.F-T point is established in [1], [10]. We note that the existence and uniqueness of weighted minimal networks is studied in [9] which gives as a specific case the existence and uniqueness of the w.F-T point which is connected by n line segments (branches) and might be considered as a particular case of a distorted weighted minimal network.

Theorem 6.3 ([1, pp. 250: Theorem 18.37], [10]). *The w.F-T point A_0 of $A_1A_2A_3\ldots A_n$ exists and is unique.*

(*i*) *If*

$$\left\| \sum_{j=1}^{n} B_j \vec{u}(A_i, A_j) \right\| > B_i, \quad i \neq j.$$

for $i, j = 1, 2, 3, 4, \ldots, n$, then the w.F-T point does not belong in $\{A_1A_2A_3\ldots A_n\}$ (Weighted Floating Case).

(*ii*) *If there is some i such that*

$$\left\| \sum_{j=1}^{n} B_j \vec{u}(A_i, A_j) \right\| \leq B_i, \quad i \neq j,$$

for $i, j = 1, 2, 3, \ldots, n$, then the w.F-T point is the vertex A_i (Weighted Absorbed Case).

Problem 6.4 (Inverse w.F-T problem). *Given a point A_0 which belongs to the interior of $A_1A_2A_3\dots A_n$, does there exist a unique set of positive weights B_i, such that*

$$B_1 + B_2 + B_3 + B_4 + \cdots + B_n = c = const.,$$

for which A_0 minimizes

$$f(A_0) = B_1a_{10} + B_2a_{20} + B_3a_{30} + B_4a_{40} + \cdots + B_na_{i0}.$$

The inverse normalized w.F-T problem ($c = 1$) was introduced in [8] for triangles in $\mathbb{R}^2$. A positive answer is given for weighted triangles in [8] and a negative answer (dynamic plasticity) is given for weighted quadrilaterals in [16].

Proposition 6.5 ([8]). *For $B_4 = B_5 = \cdots = B_n = 0$ and $c = 1$ there exists a unique set of positive weights B_1, B_2 and B_3 which is uniquely determined by the ratios*

$$\frac{B_i}{B_j} = \frac{\sin\alpha_{j0k}}{\sin\alpha_{i0k}},$$

for $i, j, k = 1, 2, 3$, $i \neq j \neq k$.

We denote by $(B_i)_{123\dots n}$ the positive weight that corresponds to the vertex A_i which lie on the ray A_0A_i, for $i = 1, 2, 3, \dots, n$, $(B_i)_{ijk}$ the positive weight that corresponds to the vertex A_i that lie on the ray A_0A_i with respect to $\triangle A_iA_jA_k$, for $i, j, k = 1, 2, 3, \dots, n$, and $i \neq j \neq k$, $(\frac{B_i}{B_j})_{123\dots n}$ the ratio $\frac{(B_i)_{123\dots n}}{(B_j)_{123\dots n}}$, and $(\frac{B_k}{B_l})_{klm}$ the ratio $\frac{(B_k)_{klm}}{(B_l)_{klm}}$, for $i, j, k, l, m = 1, 2, 3, \dots, n$ and $i \neq j$, $k \neq l \neq m$.

We consider the inverse w.F-T problem for polygons in $\mathbb{R}^2$.

Theorem 6.6 (Dynamic plasticity of weighted polygons). *The following equations point out the* dynamic plasticity *of the weighted distorted network which corresponds to the inverse w.F-T problem such that n prescribed rays meet at the w.F-T point A_0:*

$$\begin{aligned}\left(\frac{B_2}{B_1}\right)_{12\dots n} &= \left(\frac{B_2}{B_1}\right)_{123}\Bigg[1-\left(\frac{B_4}{B_1}\right)_{12\dots n}\left(\frac{B_1}{B_4}\right)_{134}\\ &\qquad -\left(\frac{B_5}{B_1}\right)_{12\dots n}\left(\frac{B_1}{B_5}\right)_{135}-\dots-\left(\frac{B_n}{B_1}\right)_{12\dots n}\left(\frac{B_1}{B_n}\right)_{13n}\Bigg],\\ \left(\frac{B_3}{B_1}\right)_{12\dots n} &= \left(\frac{B_3}{B_1}\right)_{123}\Bigg[1-\left(\frac{B_4}{B_1}\right)_{12\dots n}\left(\frac{B_1}{B_4}\right)_{124}\\ &\qquad -\left(\frac{B_4}{B_1}\right)_{12\dots n}\left(\frac{B_1}{B_5}\right)_{125}-\dots-\left(\frac{B_n}{B_1}\right)_{12\dots n}\left(\frac{B_1}{B_n}\right)_{12n}\Bigg].\end{aligned} \tag{14}$$

Proof of Theorem 6.6. We follow the same process that was used in [16, Proposition 4.4] for a convex n-gon, and we assume that $(n - 3)$ branches grow simultaneously from the point A_0 and belong to the angle $\angle A_1A_0A_4$, such that the vector $\overrightarrow{A_0A_i}$ belongs to the $\angle A_1A_0A_{i-1}$ for $i = 5, 6, \ldots, n$.

Applying the cosine law in $\triangle A_0A_iA_3$, for $i = 1, 2, \ldots, n$ and $i \neq 3$ we express the distance function a_{i0} as a function of the two variables, a_{03} and α_{032} :

$$a_{0i}^2 = a_{03}^2 + a_{3i}^2 - 2a_{03}a_{3i}\cos(\alpha_{23i} - \alpha_{032}).$$

Differentiating the objective function $f(A_0) = \sum_i B_i a_{i0}$ with respect to the variable α_{032} (see Appendix II), we obtain

$$B_1 \sin\alpha_{301} - B_2 \sin\alpha_{302} + B_4 \sin\alpha_{304} + \ldots + B_n \sin\alpha_{30n} = 0. \tag{15}$$

From (15), we have

$$\left(\frac{B_2}{B_1}\right)_{12\ldots n} = \frac{\sin\alpha_{301}}{\sin\alpha_{302}}\left[1 + \left(\frac{B_4}{B_1}\right)_{12\ldots n}\frac{\sin\alpha_{304}}{\sin\alpha_{301}} + \ldots + \left(\frac{B_n}{B_1}\right)_{12\ldots n}\frac{\sin\alpha_{30n}}{\sin\alpha_{301}}\right].$$

Taking into account Proposition 6.5 with respect to $\triangle A_1A_2A_3$ and $\triangle A_1A_3A_i*$, where A_i* is the symmetrical point of A_i with respect to A_0, for $i = 4, 5, \ldots, n$, we get

$$\left(\frac{B_2}{B_1}\right)_{123} = \frac{\sin\alpha_{301}}{\sin\alpha_{302}}, \qquad -\left(\frac{B_1}{B_n}\right)_{13n} = \frac{\sin\alpha_{30n}}{\sin\alpha_{301}}.$$

Similarly, by expressing the objective function $f(A_0) = \sum_i B_i a_{0i}$with respect to a_{02} and α_{023} and by differentiating the objective function $f(A_0$ with respect to the variable α_{023}, we obtain

$$-B_1 \sin\alpha_{201} + B_3 \sin\alpha_{203} + B_4 \sin\alpha_{204} + \ldots + B_n \sin\alpha_{20n} = 0. \tag{16}$$

Taking into account Proposition 6.5 with respect to $\triangle A_1A_2A_i$ for $i = 3, 4, \ldots, n$, we obtain

$$\left(\frac{B_3}{B_1}\right)_{123} = \frac{\sin\alpha_{201}}{\sin\alpha_{203}}, \qquad \left(\frac{B_1}{B_i}\right)_{12i} = \frac{\sin\alpha_{20i}}{\sin\alpha_{201}}. \qquad \square$$

Remark 6.7. Choosing a proper orientation of angles that was used in [21], we could also derive the plasticity equations (14) by applying the first variation formula of line segments with respect to arc length and the parametrization that was used in [17].

Concerning the inverse F-T problem for quadrilaterals we derive the following corollary:

Corollary 6.8 ([15, Proposition 4.4, pp. 417]). *The following equations point out the* plasticity *of the weighted distorted network which corresponds to the inverse w.F-T problem such that four prescribed rays meet at the w.F-T point* A_0 :

$$\left(\frac{B_2}{B_1}\right)_{1234} = \left(\frac{B_2}{B_1}\right)_{123}\left[1 - \left(\frac{B_4}{B_1}\right)_{1234}\left(\frac{B_1}{B_4}\right)_{134}\right] \tag{17}$$

$$\left(\frac{B_3}{B_1}\right)_{1234} = \left(\frac{B_3}{B_1}\right)_{123}\left[1 - \left(\frac{B_4}{B_1}\right)_{1234}\left(\frac{B_1}{B_4}\right)_{124}\right] \tag{18}$$

The weight $(B_i)_{1234}$ corresponds to the vertex A_i that lie in the line A_0A_i, $i = 1,2,3,4$ and the weight $(B_j)_{jkl}$ corresponds to the vertex A_j that lie in the A_0A_j regarding the triangle $\triangle A_jA_kA_l$, $j,k,l = 1,2,3,4$.

We set

$$\sum\nolimits_{12\ldots n} B := (B_1)_{12\ldots n}\left(1 + \sum\nolimits_{j=2} \frac{B_2}{B_1} + \frac{B_3}{B_1} + \frac{B_4}{B_1} + \cdots + \frac{B_n}{B_1}\right)_{12\ldots n}.$$

Corollary 6.9. *If $\sum_{12\ldots n} B = \sum_{123} B = \sum_{124} B = \sum_{134} B = \ldots = \sum_{1(n-1)n} B$, then*

$$(B_i)_{12\ldots n} = b_{i,4}(B_4)_{12\ldots n} + b_{i,5}(B_5)_{12\ldots n} + \cdots + b_{i,n}(B_n)_{12\ldots n} + b_{i,n+1},$$

for $i = 1,2,3$:

$$(b_{1,4}, b_{1,5}, \ldots, b_{1,n}, b_{1,n+1}) = \left(\frac{(\frac{B_1}{B_4})_{134}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_4})_{124}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, \frac{(\frac{B_1}{B_5})_{135}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_5})_{125}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, \ldots \frac{(\frac{B_1}{B_n})_{13n}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_n})_{12n}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, (B_1)_{123}\right)$$

$$(b_{2,4}, b_{2,5}, \ldots, b_{2,n}, b_{2,n+1}) = \left(b_{1,4}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{134}\left(\frac{B_2}{B_1}\right)_{123}, b_{1,5}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_5}\right)_{135}\left(\frac{B_2}{B_1}\right)_{123}, \ldots, b_{1,n}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_n}\right)_{13n}\left(\frac{B_2}{B_1}\right)_{123}, (B_2)_{123}\right)$$

$$(b_{3,4}, b_{3,5}, \ldots, b_{3,n}, b_{3,n+1}) = \left(b_{1,4}\left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{124}\left(\frac{B_3}{B_1}\right)_{123}, \ldots b_{1,n}\left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B_n}\right)_{12n}\left(\frac{B_3}{B_1}\right)_{123}, (B_3)_{123}\right).$$

Proof of Corollary 6.9. From (14) and by taking into account that

$$\sum\nolimits_{12\ldots n} B := (B_1)_{12\ldots n}\left(1 + \sum\nolimits_{j=2} \frac{B_2}{B_1} + \frac{B_3}{B_1} + \frac{B_4}{B_1} + \ldots + \frac{B_n}{B_1}\right)_{12\ldots n} = \sum\nolimits_{ijk} B$$

for $i, j, k \in \{1, 2, \ldots, n\}$ and by solving with respect to $(B_i)_{12\ldots n}$ we derive the plasticity equations for $i = 1, 2, 3$ which depend on $n - 3$ independent variables $(B_i)_{12\ldots n}$ for $i = 4, 5, \ldots, n$. $\square$

We mention a dynamic plasticity principle of convex quadrilateral in $\mathbb{R}^2$ which has been proved on a surface of bounded(continuous) curvature in $\mathbb{R}^3$.

Theorem 6.10 ([21]). *Given four prescribed rays arcs which meet at the w.F-T point A_0 and their endpoints form a convex quadrilateral in $\mathbb{R}^2$ and the w.F-T point belongs to the interior of this convex quadrilateral, an increase of the weight that corresponds to a prescribed ray causes a decrease to the two weights that correspond to the two neighboring rays and an increase to the weight that corresponds to the opposite ray.*

We extend the dynamic plasticity principle for convex weighted polygons (given in [16] and [21] for convex quadrilaterals) which satisfy the dynamic plasticity equations of Theorem 6.6.

Consider the inverse w.F-T problem for three prescribed rays which meet at A_0. We assume at time $t = 0$ that $(n - 3)$ branches (fourth, fifth and $(n - 3)th$ branch) grow simultaneously from the w.F-T point A_0 which is located at the interior of the triangle $\triangle A_1A_2A_3$ such that the $(n-3)$ branches belong to the angle $\angle A_1A_0A_4$, and the vector $\overrightarrow{A_0A_i}$ belongs to the angle $\angle A_1A_0A_{i-1}$ for $i = 5, 6, \ldots, n$.

Theorem 6.11. *Given n prescribed rays arcs which meet at the w.F-T point A_0 and their endpoints form a convex polygon in $\mathbb{R}^2$ and the w.F-T point belongs to the interior of this convex quadrilateral, an increase of the weight that corresponds to the fourth,fifth, $\ldots (n-3)th$ prescribed ray causes a decrease to the two weights that correspond to the first and third ray and an increase to the weight that corresponds to the second ray.*

Proof of Theorem 6.11. Taking into account the weighted floating case of Theorem 6.3 the w.F-T point is a weighted equilibrium point:

$$\sum_{i=4}^{n} B_i\vec{u}(A_0, A_i) + \sum_{i=1}^{3} B_i\vec{u}(A_0, A_i) = \vec{0}.$$

The composition of $\sum_{i=4}^{n} B_i\vec{u}(A_0, A_i)$ yield a vector $\vec{B}_{comp} = B_{comp}\vec{u}(A_0, A_{comp})$ such that:

$$B_{comp}\vec{u}(A_0, A_{comp}) + \sum_{i=1}^{3} B_i\vec{u}(A_0, A_i) = \vec{0}.$$

Therefore, we obtain the desired result from the dynamic plasticity principle of convex quadrilaterals in $\mathbb{R}^2$. $\square$

Consider the inverse w.F-T problem for n prescribed rays which meet at A_0 in $\mathbb{R}^2$. The dynamic plasticity equations of the corresponding weighted network are given by Corollary 6.9.

Definition 6.12. We call a *minimal systolic circle* the circle with the minimum circumradius which is derived from all the possible values of the circumradius which correspond to a class of convex n-gons inscribed to a circle with sides the weights B_i which satisfy the dynamic plasticity equations of Corollary 6.9, such that:

$$B_1 + B_2 + B_3 + \cdots + B_n = c.$$

Definition 6.13. The minimum of the classes of minimal systolic circles for a given number n which corresponds to the inverse w.F-T problem of a weighted convex k-gon for $k = 3, 4, \ldots, n$ is called a *minimum systolic circle.*

Remark 6.14. We consider as a family of systolic circles the homocentric circles having their corresponding circumradius decreased such that for the same constant c, the isoperimetric condition holds:

$$B_1 + B_2 + B_3 + \cdots + B_n = c.$$

Proposition 6.15. *If $B_i = B_0$ for $i = 1, 2, 3, \ldots, n$ then the circumradius of the minimum systolic circle is $R_{\min} = \frac{B_0}{2\sin\frac{\pi}{n}}$.*

Proof. Let $C(O, R)$ be the circle with center O and circumradius R.

Taking into account the isoperimetric condition for a given constant c, we get:

$$B_i = \frac{c}{n},$$

for $i = 1, 2, 3..., n$. From the triangle $\triangle A_i O A_{i+1}$, we obtain $R(n) = \frac{c}{2n\sin\frac{\pi}{n}}$. The function $\frac{c}{2x\sin\frac{\pi}{x}}$ is a monotone decreasing function with respect to x. Therefore, the minimum value is attained at $x = n$. □

Corollary 6.16. *For a large number n the minimum systolic circle is the termination systolic circle of the isoperimetric problem in $\mathbb{R}^2$, having circumradius $R_{term} = \frac{c}{2\pi}$.*

Proof of Corollary 6.16. It is a direct consequence from the well known limit

$$\lim_{y\to 0} \frac{\sin y}{y} = 1,$$

where $y = \frac{\pi}{x}$ and $x \to \infty$. □

Example 6.17. The minimum circumradius $R(n) = \frac{2.366}{2n\sin\frac{\pi}{n}}$, approximates the circumradius $R_{term} = \frac{2.366}{2\pi}$ of the termination systolic circle for $c = 2.366$ and $n = 850$ with five digit precision $R_{\min} = 0.376561$ (see Figure 6.1)

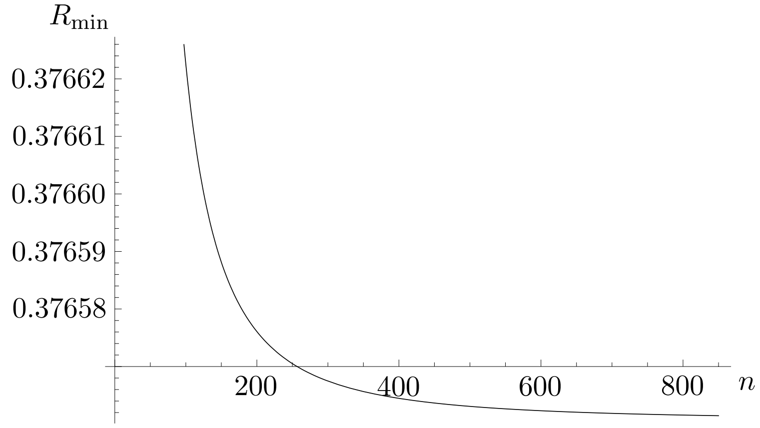

Figure 6.1

Proposition 6.18. *There exists a minimal systolic circle of a weighted convex polygon with N vertices and n a sufficiently large number of equal weights which satisfy the equations such that $n < N$:*

$$(B_i)_{12\ldots n} = b_{i,4}(B_4)_{12\ldots n} + b_{i,5}(B_5)_{12\ldots n} + \cdots + b_{i,n}(B_n)_{12\ldots n} + b_{i,n+1},$$

for $i = 1, 2, 3$:

$$(b_{1,4}, b_{1,5}, \ldots, b_{1,n}, b_{1,n+1}) = \Bigg(\frac{(\frac{B_1}{B_4})_{134}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_4})_{124}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, \frac{(\frac{B_1}{B_5})_{135}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_5})_{125}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, \ldots \frac{(\frac{B_1}{B_n})_{13n}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_n})_{12n}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, (B_1)_{123} \Bigg)$$

$$(b_{2,4}, b_{2,5}, \ldots, b_{2,n}, b_{2,n+1}) = \Bigg(b_{1,4}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{134}\left(\frac{B_2}{B_1}\right)_{123}, b_{1,5}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_5}\right)_{135}\left(\frac{B_2}{B_1}\right)_{123}, \ldots, b_{1,n}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_n}\right)_{13n}\left(\frac{B_2}{B_1}\right)_{123}, (B_2)_{123} \Bigg)$$

$$(b_{3,4}, b_{3,5}, \ldots, b_{3,n}, b_{3,n+1}) = \Bigg(b_{1,4}\left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{124}\left(\frac{B_3}{B_1}\right)_{123}, \ldots b_{1,n}\left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B_n}\right)_{12n}\left(\frac{B_3}{B_1}\right)_{123}, (B_3)_{123} \Bigg).$$

Proof of Proposition 6.18. We consider the formula for the calculation of the circumradius R of a convex polygon with N vertices with a sufficient large number of given sides n :

$$B_4 = B_5 = \cdots = B_n = B_0$$

$$\sum_{i=1}^{N} \arcsin\left(\frac{a_{ii+1}}{2R}\right) = \pi.$$

Taking into account that $B_4 = B_5 = \cdots = B_0$, we get:

$$\pi = \sum_{i=1}^{N} \arcsin\left(\frac{a_{ii+1}}{2R}\right) = \sum_{i=1}^{n} \arcsin\left(\frac{B_0}{2R}\right))$$
$$+ \sum_{i=1}^{3} \arcsin\left(\frac{B_i}{2R}\right) \approxeq \sum_{i=1}^{n} \arcsin\left(\frac{B_0}{2R}\right)). \qquad \square$$

Proposition 6.19. *There exists a minimal systolic circle of a weighted convex quadrilateral which satisfy the dynamic plasticity equations:*

$$(B_i)_{1234} = b_{i,4}(B_4)_{1234} + b_{i,5}, \quad i = 1, 2, 3 :$$

$$(b_{1,4}, b_{1,5}) = \left(\frac{(\frac{B_1}{B_4})_{134}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_4})_{124}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, (B_1)_{123}\right)$$

$$(b_{2,4}, b_{2,5}) = \left(b_{1,4}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{134}\left(\frac{B_2}{B_1}\right)_{123}, (B_2)_{123}\right)$$

$$(b_{3,4}, b_{3,5}) = \left(b_{1,4}\left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{124}\left(\frac{B_3}{B_1}\right)_{123}, (B_3)_{123}\right).$$

Proof of Proposition 6.19. The circumradius of a convex quadrilateral inscribed to a circle is given by the well known formula:

$$R_{\min} = (1/4)\sqrt{\frac{((B_1B_3 + B_2B_4)(B_1B_4 + B_2B_3)(B_1B_4 + B_3B_4))}{(s - B_1)(s - B_2)(s - B_3)(s - B_4)}}$$

where $s = \frac{B_1+B_2+B_3+B_4}{2}$.

Therefore, we obtain that R is a rational function of B_4 and from Weirstrass Theorem it possesses a maximum and a minimum value in the interval $[0, (B_4)_{\max}]$. $\square$

Corollary 6.20. *There exists a maximum of the minimal systolic circles which is derived at time zero starting from the inverse w.F-T problem for three prescribed rays meeting at A_0 such that:*

$$B_1 + B_2 + B_3 = c.$$

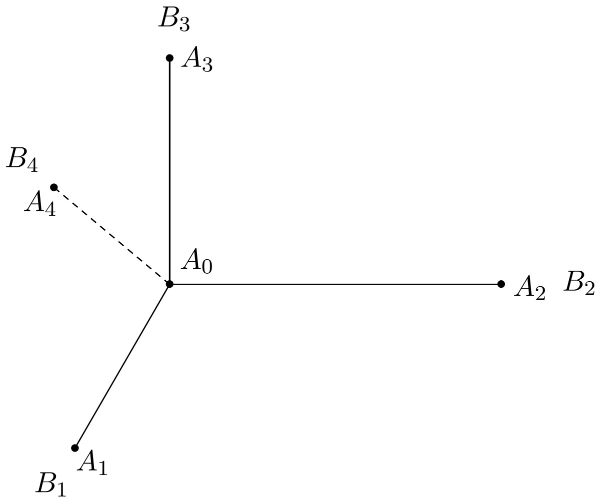

Figure 6.2

Proof of Corollary 6.20. The circumradius of a triangle is uniquely determined by the well known formula:

$$R = \frac{B_1 B_2 B_3}{\sqrt{(B_1 + B_2 + B_3)(B_2 + B_3 - B_1)(B_3 + B_1 - B_2)(B_1 + B_2 - B_3)}}$$

Therefore, the circumradius is uniquely determined by B_1, B_2 and B_3. □

Remark 6.21. The derivation of the circumaradius with respect to the weights B_1 B_2 and B_3 is also given in [15, pp. 59, formula (19)].

Example 6.22. We use the data of [16, Example 4.7, pp. 418], in order to calculate the minimal systolic circle of weighted convex quadrilateral in $\mathbb{R}^2$. We start from the inverse w.F-T problem which corresponds to three prescribed rays which meet at the w.F-T point A_0, with angles $\alpha_{102} = 120^\circ$, $\alpha_{203} = 90^\circ$ and $c = 2.366$.

By applying Corollary 6.5, we calculate the weights $(B_i)_{123}$, for $i = 1, 2, 3$.

$$\triangle 123(A_1A_2A_3) : (B_1)_{123} = 1,\ (B_2)_{123} = 0.50,\ (B_3)_{123} = 0.866$$

$$\sum_{123} B = 2.366$$

By applying Corollary 6.20, the circumradius of the triangle $\triangle B_1B_2B_3$ with sides $B_iB_j = (B_k)_{123}$ for $i \neq j \neq k$ and $i, j, k = 1, 2, 3$, is $R = 0.5$.

Then, we consider the inverse w.F-T problem for convex quadrilaterals in $\mathbb{R}^2$.

A fourth branch grows from the w.F-T point A_0 such that $\alpha_{304} = 50^\circ$ and $\sum_{1234} B = \sum_{123} B = \sum_{124} B = \sum_{134} B = 2.366$ (see Figure 6.2). By applying Corollaries 6.5 and 6.9 the following results are derived:

$$\triangle 123(A_1A_2A_3) : (B_1)_{123} = 1, (B_2)_{123} = 0.50, (B_3)_{123} = 0.866$$

$$\sum_{123} B = 2.366$$

$$\triangle 124(A_1A_2A_4) : (B_1)_{124} = 0.606, (B_2)_{124} = 0.942, (B_4)_{124} = 0.818$$

$$\sum_{124} B = 2.366$$

$$\triangle 134(A_1A_3A_4) : (B_1)_{134} = 1.449, (B_3)_{134} = 1.863, (B_4)_{134} = -0.946$$

$$\sum_{134} B = 2.366$$

$$(B_1)_{1234} - (B_1)_{123} = -0.475(B_4)_{1234} \tag{19}$$

$$(B_2)_{1234} - (B_2)_{123} = 0.528(B_4)_{1234} \tag{20}$$

$$(B_3)_{1234} - (B_3)_{123} = -1.053(B_4)_{1234} \tag{21}$$

and $\sum_{1234} B = 2.366$. The range of $(B_4)_{1234}, (B_1)_{1234}$, $(B_2)_{1234}$, $(B_3)_{1234}$ is:

$$0 \leqslant (B_4)_{1234} \leqslant 0.821$$

$$1 \geqslant (B_1)_{1234} \geqslant 0.61$$

$$0.5 \leqslant (B_2)_{1234} \leqslant 0.935$$

$$0.866 \geqslant (B_3)_{1234} \geqslant 0.$$

We note that the weight $(B_{4^\star})_{134^\star}$ which corresponds to the symmetrical point $A_{4^\star}$ of A_4 with respect to A_0 is positive and the weight $(B_4)_{134}$ is negative but the weights $(B_i)_{1234}$ are positive real numbers for $i = 1, 2, 3, 4$. By applying Corollary 6.5 in $\triangle A_1A_3A_4$, we obtain $(B_4)_{134} = -0.946$ with respect to the angles $\alpha_{304} = 50^\circ$, $\alpha_{104} = 100^\circ$ and $360^\circ - \alpha_{103} = 210^\circ$.

From Proposition 6.19, we obtain a function of the minimal circumradius of systolic circles which corresponds to the weighted convex quadrilateral $B_1'B_2'B_3'B_4'$ having sides $B_i'B_{i+1}' = (B_i)_{1234}$ for $i = 1, 2, 3, 4$, and for $i = 4$ we set $B_4'B_5' = B_4'B_1' = (B_4)_{1234}$.

Hence, we get:

$$R_{\min}((B_4)_{1234}) = (1/4)\frac{H(B_1, B_2, B_3, B_4)H(B_2, B_1, B_3, B_4)H(B_1, B_3, B_2, B_4)}{P((B_4)_{1234})},$$

where

$$H(B_i, B_j, B_k, B_4) := B_i((B_4)_{1234})B_k((B_4)_{1234}) + B_j((B_4)_{1234})(B_4)_{1234},$$

$$P((B_4)_{1234}) := (s - B_1((B_4)_{1234}))(s - B_2((B_4)_{1234}))(s - B_3((B_4)_{1234}))(s - (B_4)_{1234}),$$

where $s = \frac{c}{2}$ and c is the constant of the isoperimetric weighted condition of the inverse w.F-T problem for weighted quadrilaterals.

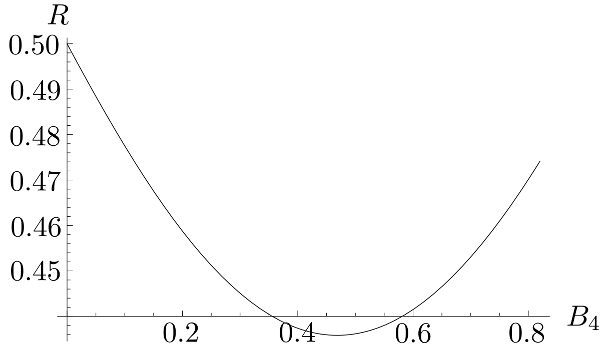

Figure 6.3

The minimum value of the class of minimal systolic circle is attained at $R_{\min} = 0.43582$ for $B_4 = 0.468591$ (see Figure 6.3).

The maximum radius of the class of minimal systolic circles start from $R_{\mathrm{maxmin}} = 0.5$ corresponds to $\triangle B_1B_2B_3$ which deals with the inverse w.F-T problem at $t = 0$, of $\triangle A_1A_2A_3$.

When a fourth branch A_0A_4 starts to grow, the values of $(B_4)_{1234}$ and $(B_2)_{1234}$ are increased and the values of $(B_1)_{1234}$ and $(B_3)_{1234}$ are decreased such that the corresponding w.F-T point A_0 of $A_1A_2A_3A_4$ remains always invariant (plasticity principle of convex quadirlaterals). Each time we derive a weighted quadrilateral $B_1'B_2'B_3'B_4'$ which is inscribed to a circle. Considering these circles with respect to a common center O from the function of the circumradius with respect to $(B_4)_{1234}$ we deduce a family of minimal systolic circles for $(B_4)_{1234} \in$[0,0.468591] and a family of maximal diastolic circles for $(B_4)_{1234} \in [0.468591,0.821]$ (see Figure 6.3).

For instance, we calculate a family of four systolic circles for $B_4 \in \{0.1, 0.2, 0.3, 0.468591\}$.

By replacing $(B_4)_{1234} = 0.1$ in (19), (20) and (21) we obtain the weighted convex quadrilateral with sides $(B_1)_{1234}(0.1) = 0.9525$, $(B_2)_{1234}(0.1) = 0.5529$, $(B_3)_{1234}(0.1) = 0.7606$ inscribed to the circle $c_1(0, 0.477517)$.

By replacing $(B_4)_{1234} = 0.2$ in (19), (20) and (21) we obtain the weighted convex quadrilateral with sides $(B_1)_{1234}(0.2) = 0.905$, $(B_2)_{1234}(0.2) = 0.6058$, $(B_3)_{1234}(0.2) = 0.6552$ inscribed to the circle $c_2(0, 0.458763)$.

By replacing $(B_4)_{1234} = 0.3$ in (19), (20) and (21) we obtain the weighted convex quadrilateral with sides $(B_1)_{1234}(0.3) = 0.8575$, $(B_2)_{1234}(0.3) = 0.6587$, $(B_3)_{1234}(0.3) = 0.5498$ inscribed to the circle $c_3(0, 0.445081)$.

By replacing $(B_4)_{1234} = 0.468591$ in (19), (20) and (21) we obtain the weighted convex quadrilateral with sides $(B_1)_{1234}(0.468591) = 0.777419$, $(B_2)_{1234}(0.468591) = 0.747885$, $(B_3)_{1234}(0.468591) = 0.372105$ inscribed to the circle $c_4(0, 0.43582)$.

The family of systolic circles $c(0, 0.5)$, $c_1(0, 0.477517)$, $c_2(0, 0.458763)$, $c_3(0, 0.445081)$, and $c_4(0, 0.43582)$, and the corresponding weighted convex quadrilaterals inscribed to these circles are given in Figure 6.4.

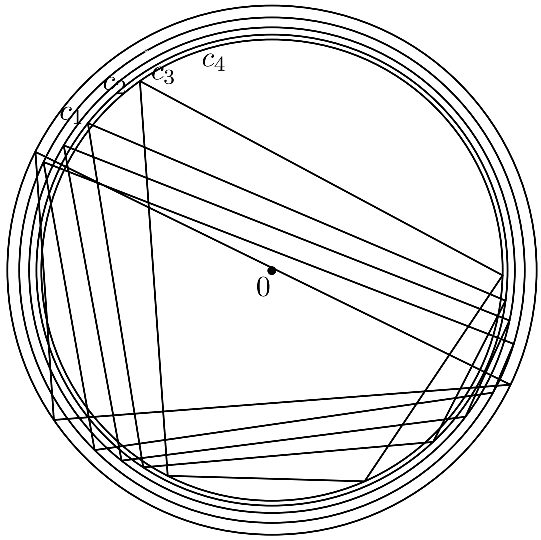

Figure 6.4

Example 6.23. We use the data of [16, Example 4.7, pp. 418] and Example 6.22, in order to calculate the minimal systolic circle of a weighted convex pentagon in $\mathbb{R}^2$.

We start from the inverse w.F-T problem which corresponds to three prescribed rays which meet at the w.F-T point A_0, with angles $\alpha_{102} = 120°$, $\alpha_{203} = 90°$ and $c = 2.366$

$$\triangle 123(A_1A_2A_3) : (B_1)_{123} = 1, (B_2)_{123} = 0.50, (B_3)_{123} = 0.866$$

$$\sum_{123} B = 2.366.$$

Then, we consider the inverse w.F-T problem for convex pentagons in $\mathbb{R}^2$.

A fourth and a fifth branch grow simultaneously from the w.F-T point A_0 such that $\alpha_{304} = 50°$, $\alpha_{405} = 30°$ and $\sum_{1234} B = \sum_{123} B = \sum_{124} B = \sum_{125} B = \sum_{134} B = 2.366$. By applying Corollaries 6.5 and 6.9 the following results are derived:

$$\triangle 123(A_1A_2A_3) : (B_1)_{123} = 1, (B_2)_{123} = 0.50, (B_3)_{123} = 0.866$$

$$\sum_{123} B = 2.366$$

$$\triangle 124(A_1A_2A_4) : (B_1)_{124} = 0.606, (B_2)_{124} = 0.942, (B_4)_{124} = 0.818$$

$$\sum_{124} B = 2.366$$

$$\triangle 134(A_1A_3A_4) : (B_1)_{134} = 1.449, (B_3)_{134} = 1.863, (B_4)_{134} = -0.946$$

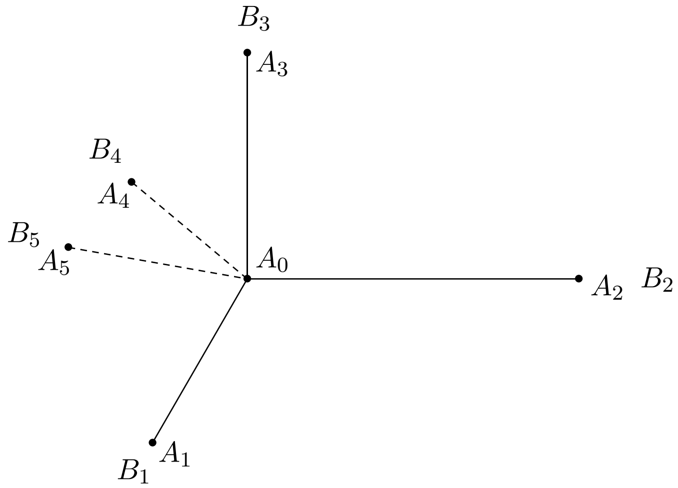

Figure 6.5

$$\sum_{134} B = 2.366$$

$$\triangle 125(A_1A_2A_5) : (B_1)_{125} = 0.207, (B_2)_{125} = 1.123, (B_5)_{125} = 1.035$$

$$\sum_{125} B = 2.366$$

$$\triangle 135(A_1A_3A_5) : (B_1)_{135} = 1.635, (B_3)_{135} = 1.561, (B_5)_{135} = -0.83.$$

$$\sum_{135} B = 2.366$$

$$(B_1)_{12345} - (B_1)_{123} = -0.475(B_4)_{12345} - 0.766(B_5)_{12345} \tag{22}$$

$$(B_2)_{12345} - (B_2)_{123} = 0.528(B_4)_{12345} + 0.602(B_5)_{12345} \tag{23}$$

$$(B_3)_{12345} - (B_3)_{123} = -1.053(B_4)_{12345} - 0.836(B_5)_{12345} \tag{24}$$

and $\sum_{12345} B = 2.366$. The range of $(B_4)_{12345}$, $(B_5)_{12345}$, $(B_1)_{12345}$, $(B_2)_{12345}$, $(B_3)_{12345}$ is:

$$0 \leqslant (B_5)_{12345} \leqslant 0.722$$

$$0 \leqslant (B_4)_{12345} \leqslant 0.821$$

$$1 \geqslant (B_1)_{12345} \geqslant 0.61$$

$$0.5 \leqslant (B_2)_{12345} \leqslant 0.935$$

$$0.866 \geqslant (B_3)_{12345} \geqslant 0$$

We note that the weight $(B_{5^\star})_{135^\star}$ which corresponds to the symmetrical point $A_{5^\star}$ of A_5 with respect to A_0 is positive and the weight $(B_5)_{135}$ is negative but the weights $(B_i)_{12345}$ are positive real numbers for $i = 1, 2, 3, 4, 5$. By applying Corollary 6.5 in $\triangle A_1A_3A_5$, we obtain $(B_5)_{135} = -0.83$ with respect to the angles $\alpha_{305} = 80°$, $\alpha_{105} = 70°$ and $360° - \alpha_{105} - \alpha_{305} = 210°$.

We consider the following formula that we used in Section 4, in order to calculate the radius R of a convex cyclic polygon in $\mathbb{R}^2$:

$$\sum_{i=1}^{3} \arcsin\left(\frac{(B_i(B_4,B_5))_{12345}}{2R}\right) + \arcsin\left(\frac{(B_4)_{12345}}{2R}\right) + \arcsin\left(\frac{(B_5)_{12345}}{2R}\right) = \pi. \tag{25}$$

By using computational tools (Mathematica) we derive from (25) the implicit function of the circumradius R of systolic (diastolic) circles with respect to $(B_4)_{12345}$ and $(B_5)_{12345}$ which corresponds to the weighted convex pentagon $B_1'B_2'B_3'B_4'B_5'$ having sides $B_i'B_{i+1}' = (B_i)_{12345}$ for $i = 1,2,3,4,5$ and for $i = 5$ we set $B_5'B_6' = B_5'B_1' = (B_5)_{12345}$ (see Figure 6.6).

By assuming that the fourth and fifth branch grow simultaneously, we may assume that $(B_4)_{12345} = (B_5)_{12345}$ which is consistent with the plasticity principle of polygons given in Theorem 6.11.

Thus, by replacing $(B_4)_{12345} = (B_5)_{12345}$ in (25) we get:

$$\sum_{i=1}^{3} \arcsin\left(\frac{(B_i(B_4,B_4))_{12345}}{2R}\right) + 2\arcsin\left(\frac{(B_4)_{12345}}{2R}\right) = \pi. \tag{26}$$

From (26) we obtain that the minimum value of the class of minimal systolic circles is attained at $R_{\min} = 0.429$ for $(B_4)_{12345} = (B_5)_{12345} = 0.235$ with three digit precision (see Figure 6.7).

The maximum radius of the class of minimal systolic circles start from $R_{\text{maxmin}} = 0.5$ corresponds to $\triangle B_1B_2B_3$ which deals with the inverse w.F-T problem at $t = 0$, of $\triangle A_1A_2A_3$.

When a fourth and a fifth branch A_0A_4 and A_0A_5 starts to grow simultaneously (not necessarily), the values of $(B_4)_{12345}$, $(B_5)_{12345}$ and $(B_2)_{12345}$ are increased and the values of $(B_1)_{12345}$ and $(B_3)_{12345}$ are decreased such that the corresponding w.F-T point A_0 of $A_1A_2A_3A_4A_5$ remains always invariant (plasticity principle of convex pentagons). Each time we derive a weighted pentagon $B_1'B_2'B_3'B_4'B_5'$ which is inscribed to a circle. Considering these circles with respect to a common center O from the function of the circumradius with respect to $(B_4)_{12345} = (B_5)_{12345}$ we deduce a family of minimal systolic circles for $(B_4)_{12345} = (B_5)_{12345} \in$[0,0.235] and a family of maximal diastolic circles for $(B_4)_{12345} = (B_5)_{12345} \in$ [0.235,0.44] (see Figure 6.7).

For instance, we calculate a family of three systolic circles for $(B_4) = (B_5)_{12345} \in \{0.1, 0.165, 0.235, \}$.

By replacing $(B_4)_{12345} = (B_5)_{12345} = 0.1$ in (22), (23) and (24) we obtain the weighted convex pentagon with sides $(B_1)_{12345}(0.1) = 0.876$, $(B_2)_{12345}(0.1) = 0.613$, $(B_3)_{12345}(0.1) = 0.677$ inscribed to the circle $c_1(0, 0.451)$.

By replacing $(B_4)_{12345} = (B_5)_{12345} = 0.165$ in (22), (23) and (24) we obtain the weighted convex pentagon with sides $(B_1)_{12345}(0.165) = 0.795$, $(B_2)_{12345}(0.165) = 0.686$, $(B_3)_{12345}(0.165) = 0.554$ inscribed to the circle $c_2(0, 0.434)$.

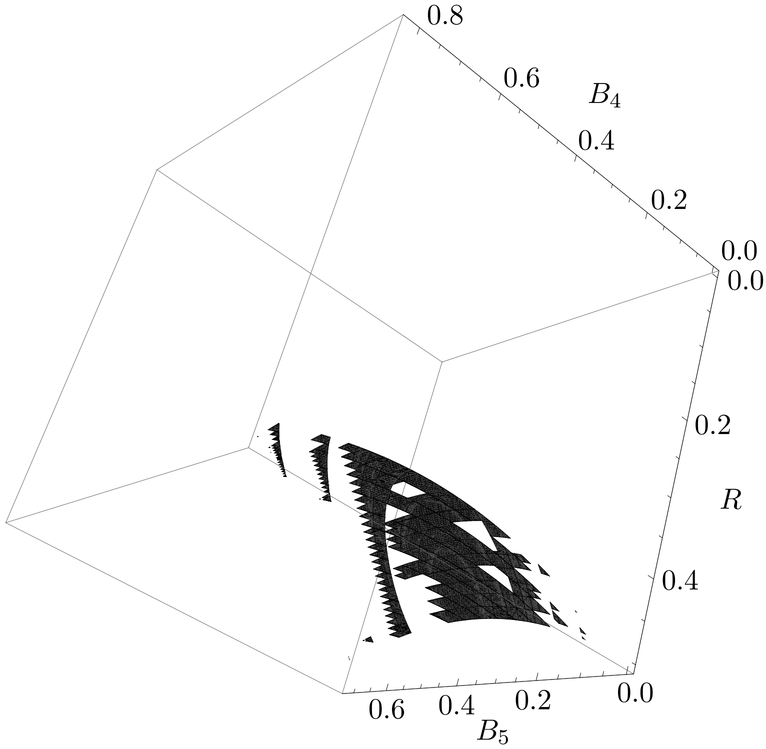

Figure 6.6

By replacing $(B_4)_{12345} = (B_5)_{12345} = 0.235$ in (22), (23) and (24) we obtain the weighted convex pentagon with sides $(B_1)_{12345}(0.235) = 0.708$, $(B_2)_{12345}(0.235) = 0.766$, $(B_3)_{12345}(0.235) = 0.422$ inscribed to the circle $c_3(0, 0.429)$.

The family of systolic circles $c(0, 0.5)$, $c_1(0, 0.451)$, $c_2(0, 0.434)$, and $c_3(0, 0.429)$, and the corresponding weighted convex pentagons inscribed to these circles are given in Figure 6.8.

Remark 6.24. By comparing the results of Examples 6.22 and 6.23 the minimum of the systolic circles which correspond to the family of weighted convex cyclic pentagons for $(B_4)_{12345} = (B_5)_{12345}$ is less than the minimum of the systolic circles which correspond to the family of weighted convex cyclic quadrilaterals.

We continue by mentioning the generalized plasticity of polygons in $\mathbb{R}^2$, which could be easily derived from the generalized plasticity of closed polyhedra in $\mathbb{R}^3$ due to the strict convexity of the Euclidean distance. The generalized plasticity considers the split of weights along n prescribed rays which meet at the w.F-T point such that the generalized w.F-T point of the new weighted network remains invariant.

Definition 6.25. We call *generalized plasticity* of a weighted network in $\mathbb{R}^2$ which is formulated by n weighted prescribed rays meeting at the w.F-To point the ability of the network to change and split their weights preserving the point and the convex hull of the polygon.

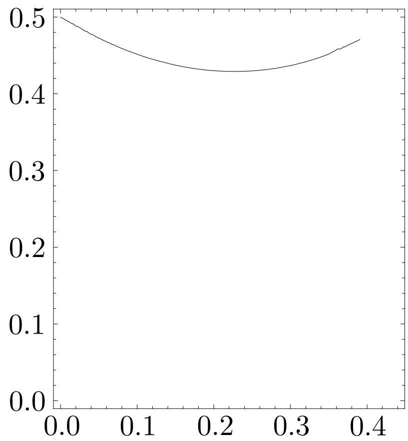

Figure 6.7

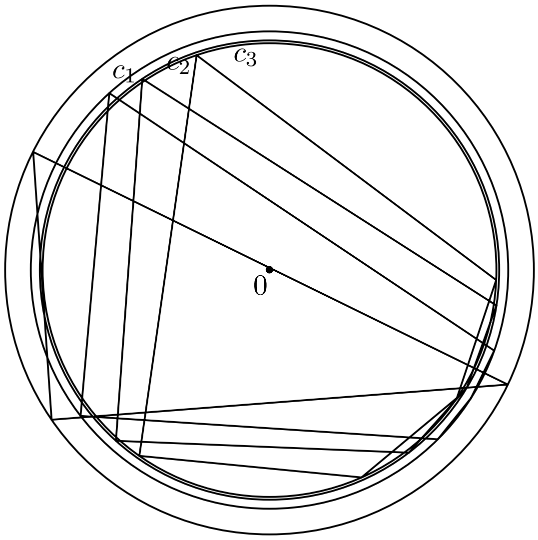

Figure 6.8

The generalized plasticity is introduced in [20, Proposition 4.1, pp. 848] for pentahedra and could be extended for closed polyhedra in $\mathbb{R}^3$, by following the same proof given in [20, Proposition 4.1, pp. 848].

Proposition 6.26 (Generalized plasticity principle in $\mathbb{R}^3$). *Let $A_1A_2A_3...A_n$ be a closed polyhedron in $\mathbb{R}^3$ and with non-negative weights B_i that correspond to each vertex A_i, respectively, which satisfy the weighted inequalities of the floating case and A_0 is the corresponding w.F-T point. Assume that every non-negative weight B_i is split into n_i non-negative weights B_{ik} :*

$$\sum_{k=1}^{n_i} B_{ik} = B_i,$$

for $i = 1, 2, 3, \ldots, n$. The weight $B_{i,k}$ corresponds to every vertex $A_{i,k}$ which belongs to the line segment A_0A_i, for every $k \neq n_i$ and the weight B_{i,n_i} corresponds

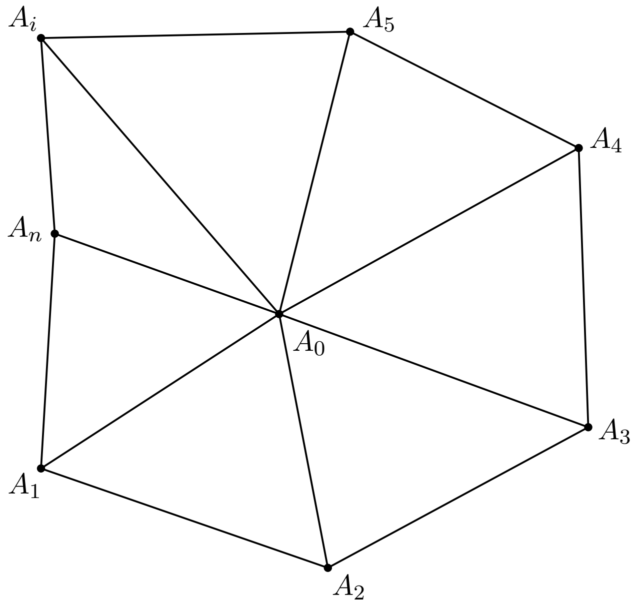

Figure 6.9

to the vertex $A_i = A_{i,n_i}$. Then the generalized F-T point of $\{A_{i,k}\}$ coincides with the w.F-T point of $\{A_1 A_2 \ldots A_n\}$.

A direct consequence of Proposition 6.26 is the derivation of the generalized plasticity principle of closed polygons in $\mathbb{R}^2$ which is given by the following Corollary (see Figures 6.9 and 6.10):

Corollary 6.27 (Generalized plasticity principle in $\mathbb{R}^2$). *Let $A_1 A_2 A_3 \ldots A_n$ be a polygon in $\mathbb{R}^2$ and with non-negative weights B_i that correspond to each vertex A_i, respectively, which satisfy the weighted inequalities of the floating case and A_0 is the corresponding w.F-T point. Assume that every non-negative weight B_i is split into n_i non-negative weights B_{ik} :*

$$\sum_{k=1}^{n_i} B_{ik} = B_i,$$

for $i = 1, 2, 3, \ldots .n$. The weight $B_{i,k}$ corresponds to every vertex $A_{i,k}$ which belongs to the line segment $A_0 A_i$, for every $k \neq n_i$ and the weight B_{i,n_i} corresponds to the vertex $A_i = A_{i,n_i}$. Then the generalized F-T point of $\{A_{i,k}\}$ coincides with the w.F-T point of $\{A_1 A_2 ... A_n\}$.

By combining Corollary 6.27, Theorem 6.6 and Corollary 6.9 we obtain the generalized plasticity equations of weighted polygons in $\mathbb{R}^2$.

We assume that $\sum_{12\ldots n} B = \sum_{123} B = \sum_{124} B = \sum_{134} B = \ldots = \sum_{1(n-1)n} B$ and

$$\sum_{k=1}^{n_i} B_{ik} = B_i,$$

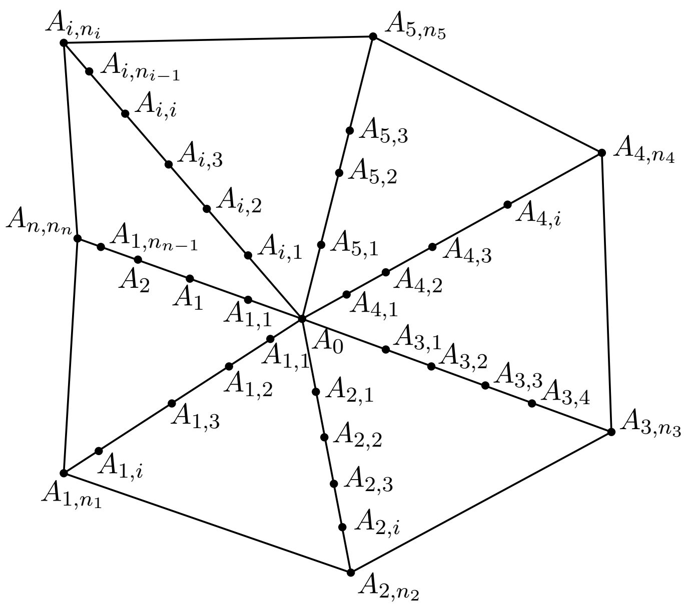

Figure 6.10

where $B_{i,k}$ corresponds to every vertex $A_{i,k}$ which belongs to the line segment A_0A_i, for every $k \neq n_i$ and the weight B_{i,n_i} corresponds to the vertex $A_i = A_{i,n_i}$.

Proposition 6.28 (Generalized plasticity equations). *The following equations point out the generalized dynamic plasticity of the generalized weighted network* $\{A_0, A_{i,k}\}$:

$$(\sum_{k=1}^{n_i} B_{ik})_{12\ldots n} = b_{i,4}(\sum_{k=1}^{n_4} B_{4k})_{12\ldots n} + b_{i,5}(\sum_{k=1}^{n_5} B_{ik})_{12\ldots n} + \cdots + b_{i,n}(\sum_{k=1}^{n_n} B_{nk})_{12\ldots n} + b_{i,n+1},$$

for $i = 1, 2, 3$*:*

$$(b_{1,4}, b_{1,5}, \ldots, b_{1,n}, b_{1,n+1}) = \left(\frac{(\frac{B_1}{B_4})_{134}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_4})_{124}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, \frac{(\frac{B_1}{B_5})_{135}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_5})_{125}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, \ldots \frac{(\frac{B_1}{B_n})_{13n}(\frac{B_2}{B_1})_{123} + (\frac{B_1}{B_n})_{12n}(\frac{B_3}{B_1})_{123} - 1}{1 + (\frac{B_2}{B_1})_{123} + (\frac{B_3}{B_1})_{123}}, (B_1)_{123} \right)$$

$$(b_{2,4}, b_{2,5}, \ldots, b_{2,n}, b_{2,n+1}) = \left(b_{1,4}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{134}\left(\frac{B_2}{B_1}\right)_{123}, b_{1,5}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_5}\right)_{135}\left(\frac{B_2}{B_1}\right)_{123}, \ldots, b_{1,n}\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_n}\right)_{13n}\left(\frac{B_2}{B_1}\right)_{123}, (B_2)_{123} \right)$$

$$(b_{3,4}, b_{3,5}, \ldots, b_{3,n}, b_{3,n+1}) = \left(b_{1,4} \left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{124} \left(\frac{B_3}{B_1}\right)_{123}, \right.$$
$$\left. \ldots b_{1,n} \left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B_n}\right)_{12n} \left(\frac{B_3}{B_1}\right)_{123}, (B_3)_{123} \right).$$

By splitting a sufficiently large amount of knots with equal weights B_{ik} along n prescribed rays which meet at the corresponding w.F-T point, we derive a stronger result than the one given in Proposition 6.18, because we do not assume that a large amount of branches shall grow simultaneously from A_0 with respect to the inverse w.F-T problem which correspond to three prescribed rays which meet at A_0 at time zero.

Proposition 6.29. *There exists a minimal systolic circle of a weighted polygon $\{B_{ik}\}$ for $i = 1, 2, 3 \ldots m$ and $k = 1, 2, 3, \ldots, n_i$ with $\sum_{i=1}^{m} n_i$ vertices which is derived by the inverse w.F-T problem of a convex m-gon $A_1A_2 \ldots A_m$ by splitting the weights B_{ik} which correspond to the knots A_{ik} along the m prescribed rays which meet at A_0, such that $n_i - 1$ is a sufficiently large number of knots with equal weights which satisfy the generalized plasticity equations:*

$$(\sum_{k=1}^{n_i} B_{ik})_{12\ldots m}$$
$$= b_{i,4}(\sum_{k=1}^{n_4} B_{4k})_{12\ldots m} + b_{i,5}(\sum_{k=1}^{n_5} B_{ik})_{12\ldots m} + \cdots + b_{i,m}(\sum_{k=1}^{n_m} B_{mk})_{12\ldots n} + b_{i,m+1},$$

and

$$B_{jk} = B_0,$$

for $i = 1, 2, 3,\ j = 1, 2, 3, \ldots, m$ and $k = 1, 2, 3, \ldots, (n_j - 1)$.

Proof of Proposition 6.29. By taking into account that the weights split equally along the m prescribed rays

$$B_{ik} = B_0,$$

we get:

$$(n_i - 1)B_0 + B_{in_i} = B_i \tag{27}$$

where B_{in_i} is a sufficiently small number for $i = 1, 2, 3, \ldots, m$ and $k = 1, 2, 3 \ldots, (n_i - 1)$.

We consider the formula for the calculation of the circumradius R of a weighted convex polygon with $\sum_{i=1}^{m} n_i$ vertices

$$\sum_{i=1}^{\sum_{j=1}^{m} n_j} \arcsin\left(\frac{a_{ii+1}}{2R}\right) = \pi$$

or

$$\pi = \sum_{i=1}^{\sum_{j=1}^{m} n_j} \arcsin\left(\frac{a_{ii+1}}{2R}\right)$$
$$= \sum_{i=1}^{m}(n_i - 1)\arcsin\left(\frac{B_0}{2R}\right)) + \sum_{i=1}^{m}\arcsin\left(\frac{B_{in_i}}{2R}\right) \approxeq \sum_{i=1}^{m}(n_i - 1)\arcsin\left(\frac{B_0}{2R}\right)).$$

Therefore, we derive a good approximation of the circumradius R which corresponds to a regular weighted polygon with weights $B'_j = B_0$ for $j = 1, 2, 3, \ldots, \sum_{j=1}^{m}(n_i - 1)$, where $B_0 \approxeq \frac{c}{\sum_{j=1}^{m}(n_i-1)}$ and c is a constant real number which corresponds to the isoperimetric condition for the weights with respect to the inverse w.F-T problem of the connected network of $\{A_{ik}\}$ points with A_0 in $\mathbb{R}^2$. □

Example 6.30. We consider the evolution of convex quadrilaterals with the same data given in Example 6.22.

Referring to the minimal circumradius of systolic circles which corresponds to the weighted convex quadrilateral $B'_1B'_2B'_3B'_4$ having sides $B'_iB'_{i+1} = (B_i)_{1234}$ for $i = 1, 2, 3, 4$, and for $i = 4$ by setting $B'_4B'_5 = B'_4B'_1 = (B_4)_{1234}$, we obtain the minimum systolic circle with minimum circumradius $R_{\min} = 0.43582$ for $B_4 = 0.468591$ (see Figure 6.3).

From the dynamic plasticity equations (19),(20) and (21) for $(B_4)_{1234} = 0.469$ we derive the weighted convex quadrilateral with sides $(B_1)_{1234} = 0.777, (B_2)_{1234} = 0.748$, $(B_3)_{1234} = 0.372$ inscribed to the circle $c_4(0, 0.436)$, with three digit precision.

We split:

(a) the weight B_1 along the ray A_0A_1 by creating 77 points $A_{1,i} \in A_0A_i$ with corresponding weights $B_{1,i} = B_0$ for $i = 1, 2, 3, \ldots, 77$ and $A_{1,78} = A_1$

(b) the weight B_2 along the ray A_0A_2 by creating 74 points $A_{2,i} \in A_0A_2$ with corresponding weights $B_{2,i} = B_0$ for $i = 1, 2, 3, \ldots, 74$ and $A_{2,75} = A_2$

(c) the weight B_3 along the ray A_0A_3 by creating 37 points $A_{3,i} \in A_0A_i$ with corresponding weights $B_{3,i} = B_0$ for $i = 1, 2, 3, \ldots, 37$ and $A_{3,38} = A_3$

(d) the weight B_4 along the ray A_0A_4 by creating 46 points $A_{4,i} \in A_0A_i$ with corresponding weights $B_{4,i} = B_0$ for $i = 1, 2, 3, \ldots, 46$ and $A_{4,47} = A_4$ (see Figures 6.11 and 6.12).

$$77B_0 + B_{177} = (B_1)_{1234} \tag{28}$$

$$74B_0 + B_{274} = (B_2)_{1234} \tag{29}$$

$$37B_0 + B_{337} = (B_3)_{1234} \tag{30}$$

$$46B_0 + B_{446} = (B_4)_{1234} \tag{31}$$

By replacing $B_0 = 0.01005$ in (28), (29), (30) and (31) we get $B_{177} = 0.00315$, $B_{274} = 0.0043$, $B_{337} = 0.00015$, and $B_{446} = 0.0067$, respectively.

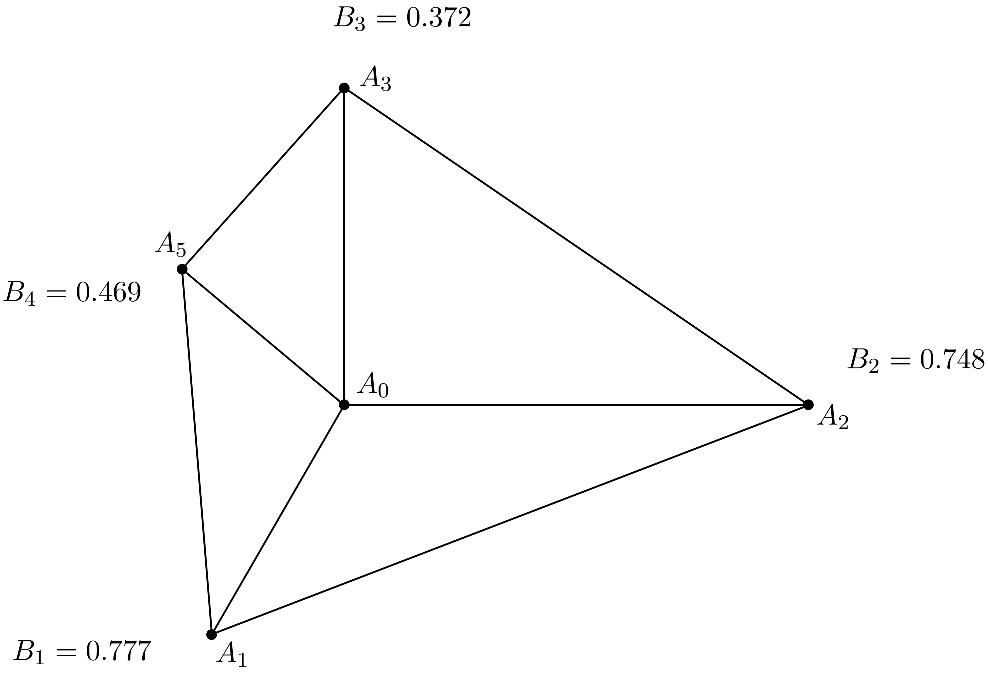

Figure 6.11

Taking into account the given sides of the weighted 238-gon $\{B_{ik}\}$ for $i = 1, 2, 3, 4$ and $k = 1, 2, 3, \ldots n_i$ where $n_1 = 78$, $n_2 = 75$, $n_3 = 38$ and $n_4 = 47$, we get:

$$234\frac{\arcsin(0.01005)}{2R} + \frac{\arcsin(0.00315)}{2R} + \frac{\arcsin(0.0043)}{2R} + \frac{\arcsin(0.00015)}{2R} + +\frac{\arcsin(0.0067)}{2R} = \pi, \tag{32}$$

By using a numerical method (for instance Newton method, bisection method) for $R_0 = \frac{\sum_{i=1}^{4}\sum_{k=1}^{n_i} B_{ik}}{2\pi}$ we obtain $R_{\min} = 0.376572$ and approximates the termination systolic circle of the isoperimetric weighted condition with terminal radius $R_{\min} = \frac{2.366}{2\pi}$ with three digit precision.

If we consider the approximation of the 238-gon by the regular 234-gon, we get $R_{\min} = (1/2)\frac{2.366}{234\sin(\frac{\pi}{234})} = 0.374296$.

7. Gauss' minimal systolic circle

In 1836, C. F. Gauss posed the following problem to the astronomer Schumacher (see [1], page 326, [6], Chapter 2):

Problem 7.1. *How to find a railway network of minimal total length which connects the four cities Bremen, Harburg, Hannover, and Braunschweig.*

We note that the solution to Gauss problem is that the cities Harburg, Hannover and Bremen are connected with the F-T point $A_{0,123}$ of the triangle $A_1A_2A_3$ which

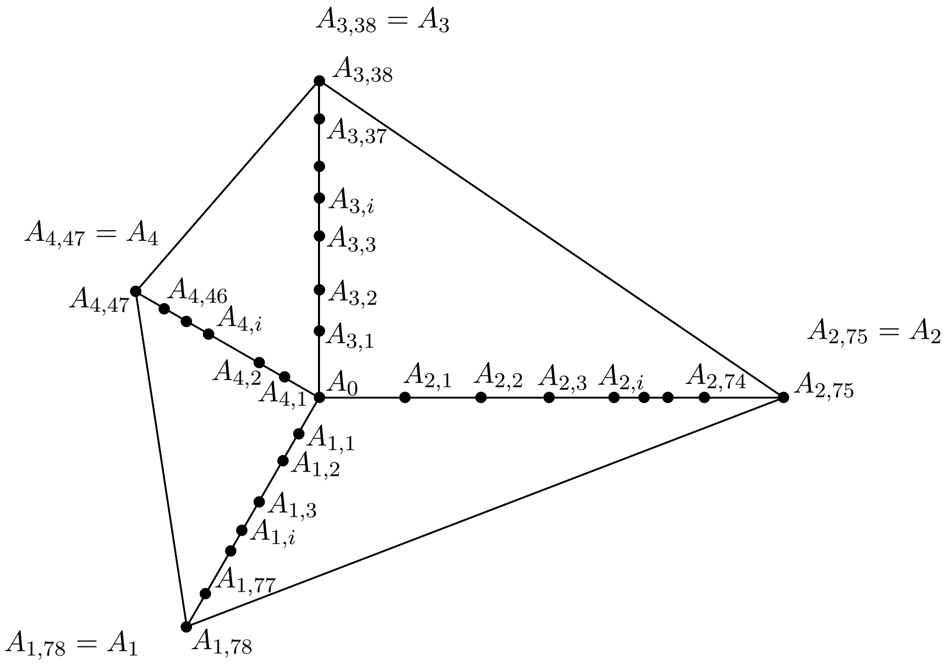

Figure 6.12

is formulated by the three cities and the cities Hannover and Braunschweig are connected with a segment. We may consider the solution to Gauss problem as an "ideal case" because the angle $\angle A_{0,123}A_3A_4 = 120°$. A solution to the generalized Gauss problem is K. Bopp's solution for any four given points in $\mathbb{R}^2$ (see [2], [1], page 327) which is the unweighted (full) Steiner minimal tree of a convex quadrilateral $A_1A_2A_3A_4$. A further generalization of the generalized Gauss problem is the Courant-Robbins problem ([3, pp. 360]) for n given points in $\mathbb{R}^2$.

A solution of the generalized weighted Gauss problem is given in [19] on the K-plane (Spherical plane, Hyperbolic plane and Euclidean plane).

Theorem 7.2 ([19]). *A weighted (full) Steiner minimal tree of $A_1A_2A_3A_4$ consists of two w.F-T points A_0, A_0' which are located at the interior convex domain with corresponding weights $B_0{=}B_{0'}{=}B_5$ and minimizes the objective function:*

$$B_1a_{10} + B_2a_{20} + B_3a_{30} + B_4a_{40} + B_5d = minimum, \tag{33}$$

such that:

$$|B_i - B_j| < B_k < B_i + B_j \tag{34}$$

and

$$|B_l - B_m| < B_n < B_l + B_m \tag{35}$$

for $i,j,k \in \{1,4,5\}$, $l,m,n \in \{2,3,5\}$ and $i \neq j \neq k$, $l \neq m \neq n$.

Corollary 7.3. *A (full) Steiner minimal tree of $A_1A_2A_3A_4$ consists of two F-T points A_0, A_0' which are located at its interior domain and minimizes the objective*

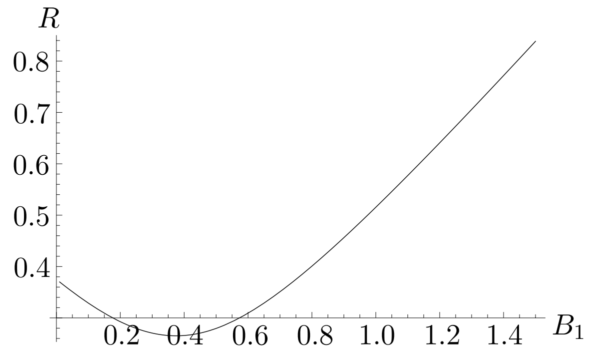

Figure 7.1

function:

$$a_{10} + a_{20} + a_{30} + a_{40} + d = minimum. \tag{36}$$

such that:

$$B_i = B_5,$$

for $i = 1, 2, 3, 4$.

Definition 7.4. We call Gauss circle, the circle which is circumscribed on a given weighted convex hexagon $B'_1B'_2B'_3B'_4B'_5B'_6$ in $\mathbb{R}^2$ which corresponds to the generalized weighted Gauss problem on the K-plane with sides $B'_iB'_{i+1} = B_i$ for $i = 1, 2, 3, 4$ and $B'_5B'_6 = B'_6B'_1 = B_5$.

Definition 7.5. We call Fermat circle, the circle which is circumscribed on a given weighted convex quadrilateral $B'_1B'_2B'_3B'_4$ in $\mathbb{R}^2$ which corresponds to the w.F-T problem on the K-plane with sides $B'_iB'_{i+1} = B_i$ for $i = 1, 2, 3$, and $B'_4B'_5 = B'_4B'_1 = B_4$.

Example 7.6. Consider the Fermat-Torricelli problem of a given convex quadrilateral $A_1A_2A_3A_4$ with equal weights $B_i = 0.375$ for $i = 1, 2, 3, 4$. The F-T point A_0 of $A_1A_2A_3A_4$ is the intersection of the diagonals A_1A_3 and A_2A_4. The Fermat circle of the weighted tetragon $B'_1B'_2B'_3B'_4$ is a circle with circumradius $R_F = \frac{B_i}{\sqrt{2}} \approxeq 0.2652$.

If we consider the Fermat-Torricelli problem of the same given convex quadrilateral $A_1A_2A_3A_4$ with weights $B_1 = B_3$, $B_2 = B_4$ such that $B_1+B_2+B_3+B_4 = 1.5$, the F-T point A_0 of $A_1A_2A_3A_4$ is the intersection of the diagonals A_1A_3 and A_2A_4 and the circumradius of the family of weighted quadrilaterals inscribed to a circle is given by a function with respect to B_1 (see Figure 7.1)

$$R(B_1) = \frac{(0.75 - B_1)^2 + B_1^2}{2}. \tag{37}$$

Therefore, we obtain from (37) a family of minimal systolic circles for $B_1 \in [0, 0.375]$ and a family of diastolic circles for $B_1 \in [0.375, 1.5]$. The minimum value of the circumradius is attained at $R = R_F$ for $B_1 = 0.375$.

Example 7.7. Consider the generalized Gauss problem of a given convex quadrilateral $A_1A_2A_3A_4$ with equal weights $B_i = 0.25$ for $i = 1, 2, 3, 4$ and a generalized solution for equal weights $B_0 = B_0' = 0.25$ which corresponds to two F-T points A_0 and A_0' which are located at the interior $A_1A_2A_3A_4$. The circumradius of Gauss circle with respect to the weighted regular hexagon $R_G = \frac{1.5}{12\sin\frac{\pi}{6}} = 0.25$.

Remark 7.8. By comparing the Fermat circle taken from Example 7.6 with the Gauss circle taken from Example 7.7 we derive that the Gauss circle is smaller than the Fermat circle with respect to the same isoperimetric condition of the weights ($c = 1.5$) because $R_F > R_G$.

Taking into consideration that Nature searches for the minimum communication among evolutionary networks, we conclude with the following question: Does Nature selects the minimum of the class of minimal systolic circles for every evolutionary structure of weighted polygons by creating knots on their convex hull?

A. Appendix

We mention two methods of differentiating a line segment with respect to: (I) a variable line segment and (II) a variable angle.

By applying the cosine law in $\triangle A_0A_iA_3$ we get:

$$a_{0i}^2 = a_{03}^2 + a_{3i}^2 - 2a_{03}a_{3i}\cos(\alpha_{23i} - \alpha_{032}). \tag{38}$$

or

$$a_{0i} = a_{0i}(a_{03}, \alpha_{032}).$$

I. A method of diferentiating the length of a line segment with respect to the length of a variable line segment is given in [18], [17, Corollary 2, pp. 51], for a triangle in $\mathbb{R}^2$. Specifically, by differentiating (38) with respect to a_{03}, and by replacing in the derived equation $\cos(\alpha_{23i} - \alpha_{032})$ taken from (38), we obtain:

$$\frac{\partial a_{0i}}{\partial a_{03}} = \cos(\alpha_{30i}). \tag{39}$$

II. A method of diferentiating a line segment with respect to a variable angle is given in [15, Proposition 2.6(b), pp. 60]. Specifically, by differentiating (38) with respect to α_{032}, we get:

$$\frac{\partial a_{0i}}{\partial \alpha_{032}} = -a_{03}\frac{a_{3i}}{a_{0i}}\sin(\alpha_{23i} - \alpha_{032}). \tag{40}$$

From the sine law in $\triangle A_iA_0A_3$, we get:

$$\frac{a_{3i}}{\sin(\alpha_{30i})} = \frac{a_{0i}}{\sin(\alpha_{23i} - \alpha_{032})}. \tag{41}$$

By replacing (41) in (40), we obtain:

$$\frac{\partial a_{0i}}{\partial \alpha_{032}} = -a_{03} \sin(\alpha_{30i}). \tag{42}$$

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