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Chebyshev Sets and Ball Operators*

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The Chebyshev set of a bounded set K in a normed space is the set of centers of all minimal enclosing balls of K. We use the concept of ball intersection and ball hull operators to derive new properties of Chebyshev sets in normed spaces. These results give a better picture on how Chebyshev sets, ball intersections, ball hulls, and completions of bounded sets are related to each other. It is shown that the Chebyshev set of a bounded set K always contains the Chebyshev set of some completion of K. Moreover, for a special class of sets we obtain a necessary and sufficient condition that the Chebyshev set of the respective set is a singleton. We obtain new results on critical sets of Chebyshev centers, and for that purpose, surprisingly, notions from the combinatorial geometry of convex bodies play an essential role. Also we give a complete geometric description of the ball hull of a finite planar set. This can be taken as starting point for algorithmical constructions of the ball hull of such sets.

Keywords: Ball hull, Ball intersection, Banach space, Chebyshev center, Chebyshev set, complete set, constant width, Jung's constant, Minkowski geometry, normed space, spherical intersection property

2000 Mathematics Subject Classification: 41A50, 41A61, 46B20, 52A21

1. Introduction

Let K be a bounded set in a finite dimensional real Banach (or normed) space. The union of centers of all minimal enclosing balls of K is called the Chebyshev set of K, and its elements are said to be the Chebyshev centers of K. Minimizing the maximal distance to points from K, Chebyshev centers occur in Approximation

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Theory, Banach Space Theory, Location Science, and Computational Geometry; see, e.g., [3], [4], [14], [15], [16], [1], and [2].

On the other hand, the notions of ball intersection and ball hull of given point sets play an essential role in Banach Space Theory, Convex Analysis, and Classical Convexity; see the recent contributions [26], [22], [33], and the references given there. These concepts are also interesting from the purely geometric point of view, because of their close relations to notions like circumballs, minimal enclosing balls, complete sets, sets of constant width, and ball polytopes, and also due to their importance for constructing such point sets; see [17], [1], [23, § 2], [22], [27], [28], and [29].

In this paper we derive new results on Chebyshev sets and Chebyshev centers in real Banach spaces of finite dimension via also new results on ball intersections and ball hulls of bounded sets in such spaces. These results give a better picture on how Chebyshev sets, ball intersections, ball hulls, and completions of bounded sets are related to each other. In particular, we show that for arbitrary sets with unique completion the Chebyshev set and the Chebyshev radius of this set and of its completion coincide. Furthermore, we prove that the Chebyshev set of a bounded set K always contains the Chebyshev set of some completion of K. In Section 4 we investigate the so-called critical set for a given Chebyshev center, i.e., the intersection of the boundary of the corresponding minimal enclosing ball and the original set. It turns out that this set gives further information about the respective Chebyshev set. Surprisingly, in connection with critical sets a basic notion from the combinatorial geometry of convex bodies (namely that of inner illuminating systems) plays an essential role. For the class of centrable sets we derive a necessary and sufficient condition that the Chebyshev set is a singleton. Finally, we give a complete geometric description of the ball hull of a finite set in the plane. This can be used as starting point for algorithmical investigations in this direction.

2. Notation and preliminaries

Let $\mathbb{M}^d = (\mathbb{R}^d, \|\cdot\|)$ be a d-dimensional real Banach space, also called a *normed* (or *Minkowski*) *space*. The *unit ball* B of $\mathbb{M}^d$ is a compact, convex set with non-empty interior (i.e., a *convex body*) centered at the *origin* o, and the boundary of B is the *unit sphere* of $\mathbb{M}^d$. Any homothetical copy $x + \lambda B$ of the unit ball is called the *ball with center* $x \in \mathbb{R}^d$ *and radius* $\lambda > 0$ and denoted by $B(x, \lambda)$, its boundary is the respective *sphere* $S(x, \lambda)$. We use the usual abbreviations int, relint, bd, and conv for *interior, relative interior, boundary*, and *convex hull.* The *line segment* connecting the different points p and q is denoted by $\overline{pq}$, the respective *line* by $\langle p, q\rangle$.

Let K be a bounded set in $\mathbb{M}^d$. We denote by $\operatorname{diam}(K) := \max\{\|x-y\| : x, y \in K\}$ the *diameter of* K. A line segment $\overline{pq}$ with $p, q \in K$ for which this maximum is attained is said to be a *diametrical chord of* K. The λ-*ball intersection* $\operatorname{bi}(K, \lambda)$

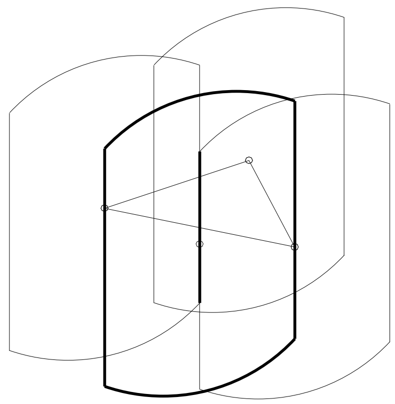

Figure 2.1: Ch(K) does not belong to the convex hull of K

of K is the intersection of all balls of radius λ whose centers are in K:

$$\mathrm{bi}(K,\lambda) = \bigcap_{x\in K} B(x,\lambda).$$

The λ-*ball hull* $\mathrm{bh}(K,\lambda)$ of K is defined as the intersection of all balls of radius λ that contain K:

$$\mathrm{bh}(K,\lambda) = \bigcap_{K\subset B(x,\lambda)} B(x,\lambda).$$

Of course, these notions make only sense if $\mathrm{bi}(K,\lambda) \neq \emptyset$ and $\mathrm{bh}(K,\lambda) \neq \emptyset$, it is clear that $\mathrm{bh}(K,\lambda) \neq \emptyset$ if and only if $\lambda \geq \lambda_K$, where λ_K is the smallest number such that K is contained in a translate of $\lambda_K B$. Such a translate is called a *minimal enclosing ball of* K (or a *circumball of* K), and λ_K is said to be the *minimal enclosing radius* (or *circumradius* or *Chebyshev radius*) of K. Clearly, we have

$$K_1 \subseteq K_2 \implies \lambda_{K_1} \leq \lambda_{K_2}. \tag{1}$$

In the Euclidean subcase the minimal enclosing ball of a bounded set is always unique, but this is no longer true for an arbitrary norm. It is easy to check that

$$\begin{aligned} &\{x \in \mathbb{M}^d : x \text{ is the center of a minimal enclosing disc of } K\} \\ =\ &\mathrm{bi}(K,\lambda_K), \end{aligned} \tag{2}$$

yielding that $\mathrm{bi}(K,\lambda) \neq \emptyset$ if and only if $\lambda \geq \lambda_K$. The set of centers of minimal enclosing balls of K is called the *Chebyshev set of* K, and we denote it by Ch(K). Relation (2) shows that the Chebyshev set of a bounded set is connected and convex. A recent investigation on convexity properties of Chebyshev sets in infinite dimensional normed spaces is [7]. The elements of Ch(K) are said to be the *Chebyshev centers* of K. Note that, in contrast to the Euclidean situation, in general the Chebyshev set of a bounded set does not necessarily belong to the convex hull of this set. An example is shown in Figure 2.1.

According to Theorem 1 in [14], if every bounded set of a normed space $\mathbb{M}^d$ contains a Chebyshev center in its convex hull, then $d = 2$ or, for $d \geq 3$, $\mathbb{M}^d$ is

Euclidean. The best geometric description of the Chebyshev set of a set K known until now is the fact that $\mathrm{Ch}(K)$ is contained in the $(\operatorname{diam} K)$-ball intersection of K (see [6, Theorem 3.3]).

In what follows, when we speak about the λ-ball intersection or λ-ball hull of a set K, we always mean that $\lambda \geq \lambda_K$. It is easy to check that

$$\lambda_K \leq \operatorname{diam} K \leq 2\lambda_K; \tag{3}$$

see also [6]. If $\lambda = \operatorname{diam} K$, then we simply say *ball intersection* and *ball hull* of K, denoting them by $\mathrm{bi}(K)$ and $\mathrm{bh}(K)$, respectively.

The supremum of all λ_K, where K is a bounded set of diameter 2, is *Jung's constant* $J_{\|\cdot\|}$ of the considered normed space, i.e., $J_{\|\cdot\|}$ is the smallest number such that a ball of that diameter can cover any set of diameter ≤ 1.

3. Properties of the operators ball hull and ball intersection

Immediately from the definition of the ball intersection we get

$$\mathrm{bi}\left(\bigcup_j K_j, \lambda\right) = \bigcap_j \mathrm{bi}(K_j, \lambda) \tag{4}$$

with $\lambda \geq \operatorname{diam}(\bigcup_j K_j)$.

Our first proposition shows that the ball intersection operator is non-increasing with respect to sets, and non-decreasing with respect to radii. Conversely, the ball hull operator is non-decreasing with respect to sets, and non-increasing with respect to radii. The Euclidean subcase of the last statement is used by Edelsbrunner (see [12, p. 309]) for approximating the shape of a given point set.

Proposition 3.1.

$$K_1 \subseteq K_2 \Longrightarrow \mathrm{bi}(K_1, \lambda) \supseteq \mathrm{bi}(K_2, \lambda) \textit{ and } \mathrm{bh}(K_1, \lambda) \subseteq \mathrm{bh}(K_2, \lambda), \tag{5}$$

$$\lambda_1 \leq \lambda_2 \Longrightarrow \mathrm{bi}(K, \lambda_1) \subseteq \mathrm{bi}(K, \lambda_2) \textit{ and } \mathrm{bh}(K, \lambda_1) \supseteq \mathrm{bh}(K, \lambda_2). \tag{6}$$

Proof. The inclusions in (5) and the first inclusion in (6) follow directly from the definitions of ball intersections and ball hulls. Thus, only the second inclusion in (6) has to be proved. Let $y \in \mathrm{bh}(K, \lambda_2)$, and we suppose that there exists $B(x, \lambda_1)$ such that $K \subset B(x, \lambda_1)$ and $y \notin B(x, \lambda_1)$. Let x' be a point in $\langle x, y\rangle$ and such that $\lambda_2 - \lambda_1 = \|x - x'\| < \|y - x'\|$. We prove the following for the ball $B(x', \lambda_2)$:

(i) $B(x, \lambda_1) \subset B(x', \lambda_2)$ and
(ii) $y \notin B(x', \lambda_2)$.

Indeed, if $z \in B(x, \lambda_1)$, then $\|z - x'\| \leq \|z - x\| + \|x - x'\| \leq \lambda_1 + \lambda_2 - \lambda_1 = \lambda_2$ and $\|y - x'\| = \|y - x\| + \|x - x'\| = \|y - x\| + \lambda_2 - \lambda_1 > \lambda_2$. Hence there exists a ball $B(x', \lambda_2)$ which contains K such that $y \notin B(x', \lambda_2)$, a contradiction. □

From [32, Proposition 2.1.1] we have the following

Proposition 3.2. *Let K be a set in a normed space. Then*

(*i*) $\mathrm{bh}(K) = \mathrm{bi}(\mathrm{bi}(K), \mathrm{diam}(K))$,
(*ii*) $\mathrm{bi}(K) = \mathrm{bi}(\mathrm{bh}(K), \mathrm{diam}(K))$.

Note that if $\lambda \geq \mathrm{diam}\, K$, then we have the inclusions

$$K \subseteq \mathrm{bh}(K, \lambda) \subseteq \mathrm{bi}(K, \lambda). \tag{7}$$

The first inclusion follows immediately from the definition of the ball hull. For the second inclusion we take $x \in \mathrm{bh}(K, \lambda)$. Thus x belongs to all balls $B(y, \lambda)$ which contain K. If y is an arbitrary point from K, then $B(y, \lambda) \supseteq B(y, \mathrm{diam}(K)) \supseteq K$. Therefore every ball with radius λ and center from K contains K.

If $\lambda = \mathrm{diam}\, K$, then

$$K \subseteq \mathrm{bh}(K) \subseteq K^c \subseteq \mathrm{bi}(K), \tag{8}$$

where K^c is a *completion* of K, i.e., a complete set that contains K and has the same diameter. (Recall that a set in $\mathbb{M}^d$ is said to be *complete* if it cannot be enlarged by additional points without increasing its diameter.) The inclusion $\mathrm{bh}(K, \lambda) \subseteq K^c$ was given in the proof of Theorem 5 in [17]. The completeness of a set K is equivalent of the fact that $K = \mathrm{bi}(K)$; see [13, p. 167, (E)]. It also known that completeness and constant width are equivalent in the Euclidean subcase and in every two-dimensional normed space. (For the definition and many properties of *sets of constant width* in normed spaces we refer to Section 2 of the survey [23].) In a normed space of dimension at least three, constant width implies completeness, but not vice versa; cf. [13] and [23, pp. 98–99].

The next statement shows relations between the notions of ball hull, ball intersection and completion of a given set.

Proposition 3.3. *Let K be a bounded set in a normed space. Then*

$$\mathrm{bh}(K) = \bigcap(\textit{all completions of}\, K) \tag{9}$$

and

$$\mathrm{bi}(K) = \bigcup(\textit{all completions of}\, K). \tag{10}$$

Proof. First we prove (10). Due to (8) we only need to check whether for every point x of $\mathrm{bi}(K)$ there exists a completion of K such that x belongs to this completion. Since $x \in \mathrm{bi}(K)$, for any point y of K the inequality $\|x - y\| \leq \mathrm{diam}(K)$ holds. This means that $\mathrm{diam}(K \bigcup \{x\}) = \mathrm{diam}(K)$, and there exits a completion K^c of the set $K \bigcup \{x\}$ which is also a completion of K.

Statement (9) follows from (10) and (4). □

Remark 3.4. The Euclidean planar subcase of Proposition 3.3 was proved by Bavaud; see [8, Theorem IV]. Another proof of this statement was given by

Moreno in [25]; see Proposition 2 there. Note that in that paper, as well as in the papers of Moreno and Schneider cited here, the notation $\eta(K)$ and $\theta(K)$, respectively, for ball intersection and ball hull of K is used. In fact, $\theta(K)$ is defined as $\cap_{x\in\eta(K)}B(x, \operatorname{diam} K)$. Due to Proposition 3.2 (i), the latter is the ball hull $\operatorname{bh}(K)$. In [27] and [28] Moreno and Schneider call $\operatorname{bi}(K)$ and $\operatorname{bh}(K)$, respectively, the *wide spherical hull* and the *tight spherical hull* of K. Also we note that Baronti and Papini used in [6] the notation K' for ball intersection and K^c for ball hull of K.

The first corollary of Proposition 3.3 follows immediately from (10).

Corollary 3.5. *Every ball with radius $\lambda = \operatorname{diam} K$ and center from K which contains K also contains any completion of K.*

As it was announced in Section 2, we have that $\operatorname{Ch}(K) \subset \operatorname{bi}(K)$; see [6, Theorem 3.3]. This fact and (10) from Proposition 3.3 yield

Corollary 3.6. *Let $B(x, \lambda_K)$ be a minimal enclosing ball of a set K. Then there exists a completion K^c of K which contains x.*

For the next considerations we need Theorem 3.5 from [6] which says: for every ball of radius $\lambda \geq \frac{1}{2} \operatorname{diam} K$ which contains K there exists a completion K^c of K contained in this ball. This theorem immediately implies

Lemma 3.7. *Let $B(x, \lambda_K)$ be a minimal enclosing ball of K. Then $B(x, \lambda_K)$ contains some completion K^c of K.*

Proposition 3.8. *Every minimal enclosing ball of a set K contains its ball hull.*

Proof. This follows from Lemma 3.7 and (8). □

Lemma 3.9. *For any bounded set $K \subset \mathbb{M}^d$ there exists some completion K^c such that the relations $\lambda_K = \lambda_{K^c}$ and $\operatorname{Ch}(K^c) \subseteq \operatorname{Ch}(K)$ hold.*

Proof. By Lemma 3.7 and inequality (1) there exists a completion K^c with the first relation. In view of $\operatorname{Ch}(K) = \operatorname{bi}(K, \lambda_K)$, $\operatorname{Ch}(K^c) = \operatorname{bi}(K^c, \lambda_K)$, and (5) we get $\operatorname{Ch}(K^c) \subseteq \operatorname{Ch}(K)$. □

Lemma 3.10. *For any bounded set $K \subset \mathbb{M}^d$ and $x \in \operatorname{Ch}(K)$ there exists some completion K^c such that $x \in \operatorname{Ch}(K^c)$.*

Proof. Let $x \in \operatorname{Ch}(K)$. By Lemma 3.7 there exists a completion K^c contained in $B(x, \lambda_K)$ with $\lambda_{K^c} = \lambda_K$. Therefore, $\|x - y\| \leq \lambda_{K^c}$ for every $y \in K^c$ and $x \in \cap_{y\in K^c}B(y, \lambda_{K^c})$. We conclude that $x \in \operatorname{bi}(K^c, \lambda_{K^c}) = \operatorname{Ch}(K^c)$. □

The next theorem gives a relation between the Chebyshev set of a set and the Chebyshev sets of its completions.

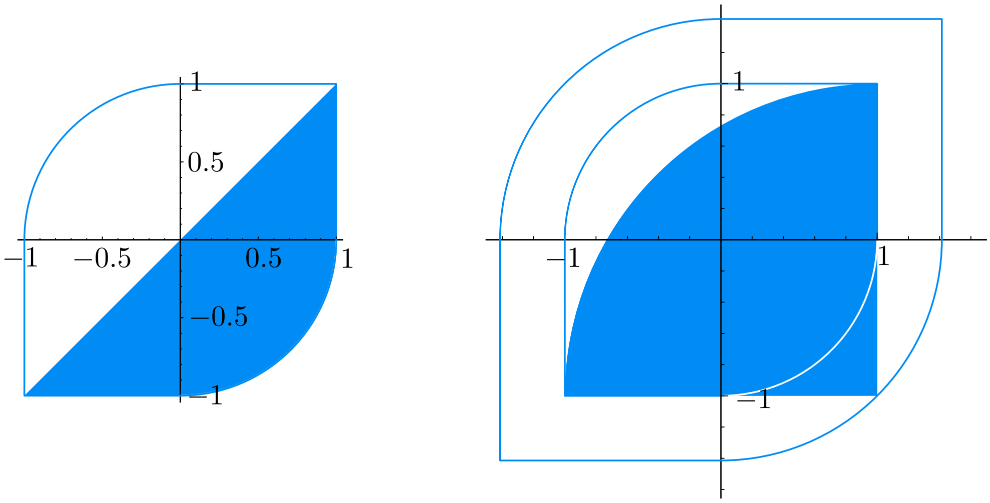

Figure 3.1: A set K and a completion K^c

Theorem 3.11. *For any bounded set $K \subset \mathbb{M}^d$ the inclusions*

$$\bigcap \mathrm{Ch}(K^c) \subseteq \mathrm{Ch}(K) \subseteq \bigcup \mathrm{Ch}(K^c)$$

hold.

Proof. The first inclusion follows by Lemma 3.9 and the second one by Lemma 3.10. □

Note that the equality $\mathrm{Ch}(K) = \cup\,\mathrm{Ch}(K^c)$ is not true in general. If we consider in $\mathbb{R}^2$ the normed space with unit ball as in Figure 3.1 (left), then the Chebyshev set of the set $K = \{(x, y) \in B(0, 1) : x - y < 0\}$ is the point $(0, 0)$, but the Chebyshev set of the completion K^c shown in Figure 3.1 (right) is not this point. It should also be noticed that the location of the Chebyshev set of a bounded set as given in Theorem 3.11 sharpens the inclusion $\mathrm{Ch}(K) \subset \mathrm{bi}(K)$ given in [6, Theorem 3.3]. Indeed, by $\mathrm{Ch}(K) \subset \mathrm{bi}(K)$ and (5), we have

$$\begin{aligned} \bigcup \mathrm{Ch}(K^c) &\subseteq \bigcup \mathrm{bi}(K^c, \mathrm{diam}(K^c)) \\ &= \bigcup \mathrm{bi}(K^c, \mathrm{diam}(K)) \subseteq \bigcup \mathrm{bi}(K) = \mathrm{bi}(K). \end{aligned}$$

The last corollary of this section follows immediately from Theorem 3.11.

Corollary 3.12. *If K is a bounded set with unique completion K^c, then the relations $\lambda_K = \lambda_{K^c}$ and $\mathrm{Ch}(K) = \mathrm{Ch}(K^c)$ hold.*

It should be noticed that the class of sets with unique completion is relatively large and well studied; see [23] and the references given there. We hope that Corollary 3.12 will be of importance for further investigations of Chebyshev sets and Chebyshev radii of sets from this class. Due to this corollary, one does not

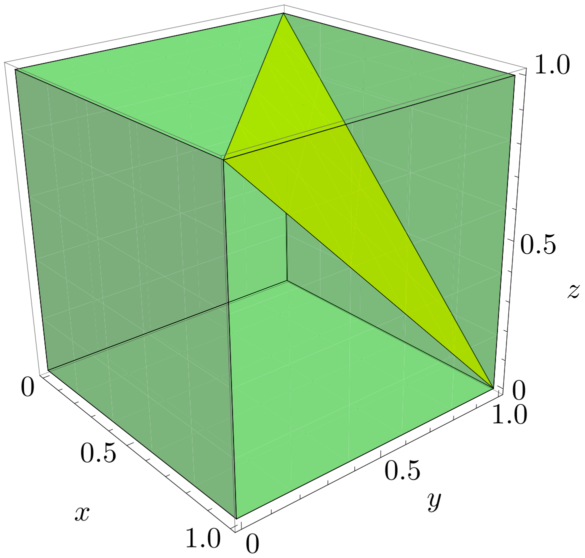

Figure 3.2: Ch(K) does not intersect the convex hull of K

have to investigate these notions for an arbitrary set from this class. It is sufficient to study them only for complete sets.

Remark 3.13. We would like to comment the chain (8) of inclusions as follows. For a bounded convex set K, in general $\mathrm{Ch}(K) \nsubseteq K$; see again Theorem 1 in [14]. According to [6, Theorem 3.3] we have that $\mathrm{Ch}(K) \subseteq \mathrm{bi}(K)$ and, more precisely, Corollary 3.6 implies that there exists a completion K^c of K with $\mathrm{Ch}(K) \subset K^c$. Moreover, in general normed spaces there exist sets whose Chebyshev sets do not necessarily intersect their convex hull. In Figure 3.2 a set with this property is shown. Consider the maximum norm in $\mathbb{R}^3$. Then the set $K = \{(1,1,0),(1,0,1),(0,1,1)\}$ has only one Chebyshev center which does not belong to the convex hull of these points. It is easy to construct a similar example with non-empty interior.

4. Critical sets

Let $B(x, \lambda_K)$ be a minimal enclosing ball of a set K. The set of all boundary points of K which belong to $\mathrm{bd}\, B(x, \lambda_K)$ is called the *critical set* of K with respect to x and denoted by $\mathrm{cr}(K, x)$. The elements of $\mathrm{cr}(K, x)$ are said to be *critical points*. The notion of critical set was introduced by Garkavi in [15], and it turns out that such sets yield important information on Chebyshev sets. E.g., via critical points we can completely clarify, for the class of centrable sets, whether or not the Chebyshev set is a singleton; see Proposition 4.7. Sets K with the property $\mathrm{diam}\, K = 2\lambda_K$ are called *centrable sets*. In other words, a set is centrable if the second inequality in (3) is an equality for it. Centrable sets are treated, e.g., in [30], where real Banach spaces with the property that every finite set is centrable are characterized.

Theorem 4.1. *Let there be given a non-centrable compact convex set K in a*

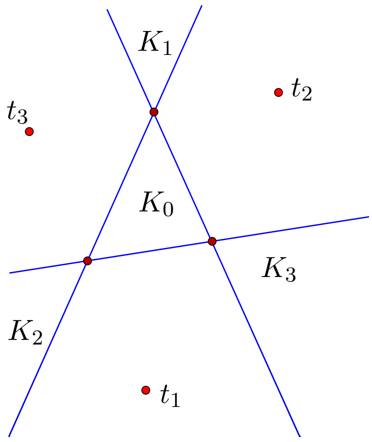

Figure 4.1

normed plane $\mathbb{M}^2$, and $B(x, \lambda_K)$ be a minimal enclosing ball of K. Then there exist $m \in \{2,3\}$ points $t_1, \dots, t_m \in \operatorname{cr}(K,x)$ being the vertices of an $(m-1)$-simplex such that

$$x \in \operatorname{relint}(\operatorname{conv}(t_1, \dots, t_m)).$$

Proof. By Lemma 2.2 in [11], there exist m critical points $t_1, \dots, t_m$ ($m \in \{2,3\}$) such that the minimal enclosing radius of the set $S = \operatorname{conv}(t_1, \dots, t_m)$ is λ_K.

If $m = 2$ and $S = \operatorname{conv}(t_1, t_2)$, then $x \in B(t_1, \lambda_K) \cap B(t_2, \lambda_k)$. The intersection of these two balls has no interior points, because in that case there exists a ball containing t_1 and t_2 with radius smaller than λ_K. Therefore (see [24, Section 3.3]), the intersection is a segment and K is centrable due to $\|t_1 - t_2\| = 2\lambda_K$.

Let us assume that $m = 3$ and $S = \operatorname{conv}(t_1, t_2, t_3)$. The center x of any minimal enclosing ball can belong to some of the regions K_0, K_1, K_2 or K_3 shown in Figure 4.1; these are bounded by the lines meeting the midpoints of $\overline{t_1, t_2}$, $\overline{t_1, t_3}$ and $\overline{t_2, t_3}$ (see [1] and [2]). If $x \in K_0$, the proof is finished. Let us assume that x belongs to K_1 (the cases $x \in K_2$ or $x \in K_3$ are similar to this one). The situation is like in Figure 4.2 (see again [1] and [2]), where t_i' ($i \in \{1,2,3\}$) is symmetric to t_i with respect to x. Therefore $\operatorname{conv}(t_1, t_2, t_3, t_1', t_2', t_3')$ belongs to $B(x, \lambda_K)$.

Let x' be the translate of x in direction $t_3 - t_2'$ such that x' lies on $\langle t_2, t_3 \rangle$. It is clear that $\|t_1 - t_2\| \le 2\lambda_K$. K would be centrable if equality holds. In the other case, $B(x', \lambda_K)$ contains S and there exists a ball of radius $\lambda < \lambda_K$ which also contains S. Therefore, λ_K is not the minimal enclosing radius of S, a contradiction. □

As an immediate consequence of Theorem 4.1 we have a statement which completely clarifies the location of Chebyshev sets of non-centrable planar sets.

Corollary 4.2. *If K is a non-centrable set in a normed plane, then the Chebyshev set of K belongs to the relative interior of the convex hull of K.*

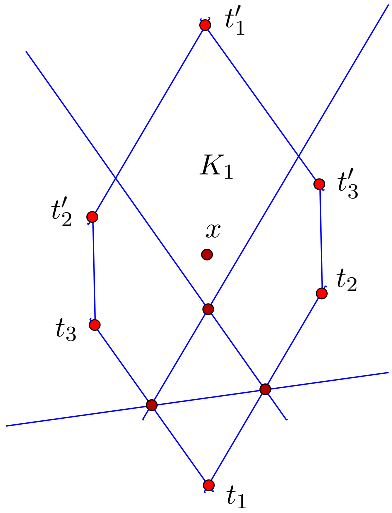

Figure 4.2

The following notion from the combinatorial geometry of convex bodies is also related to our investigations:

A subset P of the boundary of a convex body K is called an *inner illuminating system* of K if for every $x \in \operatorname{bd} K$ there exists at least one $p \in P$ such that the line segment $\overline{px}$ meets the interior of K; cf. [10, § 34].

As Figure 4.3 shows, Theorem 4.1 is not true in dimensions at least three. Moreover, it is not true that every convex body has a minimal enclosing ball with the property described in Theorem 4.1. Otherwise, every bounded set would have a Chebyshev center belonging to its convex hull. This contradicts Theorem 1 in [14] saying that this is only possible when the underlying normed space is of dimension 2 or, for any dimension $d \geq 3$, when it is Euclidean. But if for a Chebyshev center points $t_1, \ldots, t_m$ exist as in Theorem 4.1, then these points form an inner illuminating system of the respective convex body. Since such systems are well studied in the combinatorial geometry of convex bodies, we are motivated for the next definition. Let K be a convex body in a normed space $\mathbb{M}^d$ with Chebyshev center x having the following property: for some $m \in \{2, \ldots, d+1\}$ there exist m points $t_1, \ldots, t_m \in \operatorname{cr}(K, x)$ being the vertices of an $(m-1)$-simplex such that

$$x \in \operatorname{relint}(\operatorname{conv}\{t_1, \ldots, t_m\}).$$

Then x is called a *meanstream Chebyshev center* of K. Our Theorem 4.1 shows that all Chebyshev centers in the planar case are meanstream. If x is a meanstream Chebyshev center of K, then the smallest integer m with the above property is called the *degree* of x, and the system $\{t_1, \ldots, t_m\}$ is said to be a *base system* of x.

The next proposition follows directly from the definition of the degree of a meanstream Chebyshev center.

Proposition 4.3. *Let K be a bounded set in a normed space $\mathbb{M}^d$. Then K is*

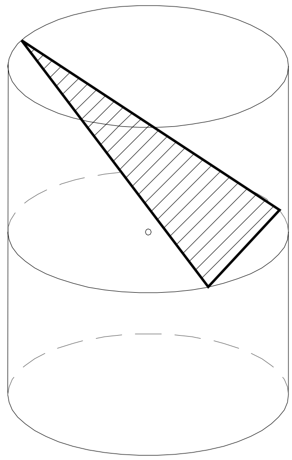

Figure 4.3: $\mathrm{Ch}(K)$ does not belong to $\mathrm{relint}(K)$

centrable if and only if it has a meanstream Chebyshev center of degree 2.

If K is centrally symmetric, then a subset of $\mathrm{bd}\, K$ is said to be *global* if it is not contained in any closed halfspace whose bounding hyperplane passes through the center of K; see [19].

Proposition 4.4. *Let $K \subset (\mathbb{R}^d, \|\cdot\|)$ be a convex body having a meanstream Chebyshev center x of degree m with base system $\{p_1, \dots, p_m\}$. Then the set $\{p_1, \dots, p_m\}$ is an inner illuminating system of the minimal enclosing ball $B(x, \lambda_K)$ of K.*

Proof. We fix some point p_i. The only points of $\mathrm{bd}\, B(x, \lambda_K)$ not inner illuminated by p_i are those which belong to a proper face of $B(x, \lambda_K)$. But not all $p_1, \dots, p_m$ belong to such a proper face. □

Analogously we get

Proposition 4.5. *Let $K \subset (\mathbb{R}^d, \|\cdot\|)$ be a convex body with meanstream Chebyshev center x of degree $d+1$ and base system $\{p_1, \dots, p_{d+1}\}$. Then $\mathrm{cr}(K, x)$ is global, and $\{p_1, \dots, p_{d+1}\}$ is an inner illuminating system of K.*

It is now our aim to clarify when, for a given centrable set K, $\mathrm{Ch}(K)$ is a singleton. For that reason we need some basic notions referring to the face structure of a convex body. First we recall that a boundary point q of a convex body K is called an *extreme point* if it is not in the relative interior of any line segment belonging to K.

It should be noticed that if x is a meanstream Chebyshev center of a centrable set K with base $\{p_1, p_2\}$, then $x = \frac{1}{2}(p_1 + p_2)$ and $\lambda_K = \frac{1}{2}\|p_1 - p_2\|$.

Proposition 4.6. *The Chebyshev set of two different points p_1 and p_2 is a singleton if and only if p_1 (and therefore p_2) is an extreme point of $B(\frac{1}{2}(p_1 + p_2), \frac{1}{2}\|p_1 - p_2\|)$.*

Proof. Let p_1 be an extreme point. Assume that there exists a Chebyshev center $x+v$, $v \neq 0$, of $\{p_1, p_2\}$ different to x. Then

$$2\lambda_K = \|p_1 - p_2\| \leq \|p_1 - (x+v)\| + \|p_2 - (x+v)\| \leq 2\lambda_K,$$

where $K = \{p_1, p_2\}$. This is only possible when $\|p_1 - (x+v)\| = \|p_2 - (x+v)\| = \lambda_K$. Hence the points $p_1 - v$ and $p_1 + v$ are from $\operatorname{bd} B(x, \lambda_K)$, a contradiction. The proof of the converse can be omitted. □

Proposition 4.7. *Let K be a centrable set in a normed space $\mathbb{M}^d$. The Chebyshev set of K is a singleton if and only if K has a Chebyshev center x with base system $\{p_1, p_2\}$ such that p_1 (and therefore p_2) is an extreme point of $B(x, \lambda_K)$.*

Proof. Since K is centrable, by Proposition 4.3 there exist a Chebyshev center x and two points $\{p_1, p_2\} \in \operatorname{cr}(K, x)$ such that $x \in \operatorname{relint}(\operatorname{conv}\{p_1, p_2\})$. Therefore, $x = \frac{p_1+p_2}{2}$, $\lambda_K = \frac{\|p_1-p_2\|}{2}$ and p_1 is the point symmetric to p_2 with respect to x.

Let us assume that $\operatorname{Ch}(K)$ is a singleton. If p_1 and p_2 are not extreme points of $B(x, \lambda_K)$, then both p_1 and p_2 belong to parallel segments with direction v which are contained in the boundary of $B(x, \lambda_K)$. Let us consider parallel support hyperplanes of $B(x, \lambda_K)$ at p_1 and at p_2 which contain the direction v. Moving the center x in the direction v, it is possible to find a new Chebyshev center of K, a contradiction.

Let us assume that p_1 and p_2 are extreme points of $B(x, \lambda_K)$. Since $x = \frac{p_1+p_2}{2}$ and $\lambda_K = \frac{\|p_1-p_2\|}{2}$, Proposition 4.6 yields $\operatorname{Ch}(\{p_1, p_2\}) = x$ and $\lambda_{\{p_1,p_2\}} = \lambda_K$. As $\{p_1, p_2\} \subseteq K$, by (5) we obtain that

$$x = \operatorname{Ch}(\{p_1, p_2\}) = \operatorname{bi}(\{p_1, p_2\}, \lambda_K) \supseteqq \operatorname{bi}(K, \lambda_K) = \operatorname{Ch}(K)$$ □

5. On ball hulls of finite sets in the planar case

As Proposition 3.8 shows, the ball hull of a set occurs when studying minimal enclosing balls, giving also motivations for more precise descriptions of ball hulls of planar finite sets. Some authors make use of the ball hull in the Euclidean plane (also known as the α-hull or the circular hull) for developing algorithms in combinatorial geometry (e.g., [12, p. 309]) and cluster analysis (e.g., [20], [21], [31]). In this section, we hope to present the foundations of similar algorithms for general norms. For our considerations we need the following lemma which goes back to Grünbaum [18] and Banasiak [5]; see also [24, § 3.3].

Lemma 5.1. *Let $\mathbb{M}^2$ be a normed plane. Let $C \subset \mathbb{M}^2$ be a compact, convex disc whose boundary is the closed curve γ; v be a vector in $\mathbb{M}^2$; $C + v$ be a translate of C with boundary γ'. Then $\gamma \cap \gamma'$ is the union of two segments, each of which may degenerate to a point or to the empty set.*

Suppose that this intersection consists of two connected non-empty components A_1, A_2. Then the two lines of translation supporting $C \cap C'$ intersect $C \cap C'$ exactly in A_1 and A_2.

Choose a point p_i from each component A_i and let $c_i = p_i - v$ and $c_i' = p_i + v$ for $i = 1, 2$. Let γ_1 be the part of γ on the same side of the line $\langle p_1, p_2\rangle$ as c_1 and c_2; let γ_2 be the part of γ on the side of $\langle p_1, p_2\rangle$ opposite to c_1 and c_2; and similarly for γ, γ_1', and γ_2'.

Then $\gamma_2 \subseteq \mathrm{conv}(\gamma_1')$ and $\gamma_2' \subseteq \mathrm{conv}(\gamma_1)$.

Let p and q be two points of the circle $S(x, \lambda)$. In the following, the *minimal circular arc of $B(x, \lambda)$ meeting p and q* is the piece of $S(x, \lambda)$ with endpoints p and q which lies in the half-plane bounded by the line $\langle p, q\rangle$ and not containing the center x. If p and q are opposite in $S(x, \lambda)$, then the two half-circles with endpoints p and q are minimal circular arcs of $S(x, \lambda)$ meeting p and q. In the following, we suppose for simplicity $\lambda = 1$. We denote a minimal circular arc meeting p and q by $\widehat{pq}$.

With Lemma 5.1 we can prove a generalization of Lemma 4.1 in [20].

Lemma 5.2. *Let $\mathbb{M}^2$ be a normed plane with unit disc B and points $p, q \in B$ satisfying $\|p - q\| \leq 1$.*

(*1*) *If $p, q \in S(o, 1)$ and there exists another circle $S(x, 1)$ through p and q, then either the origin o and x are in different half-planes with respect to the line $\langle p, q\rangle$, or the segment $\overline{pq}$ belongs to $S(x, 1) \cap S(o, 1)$.*

(*2*) *Any minimal circular arc of radius 1 meeting p and q also belongs to B.*

(*3*) *If a circular arc meeting p and q is contained in B such that it contains interior points of B, then this arc is a minimal circular arc.*

Proof. (*1*) Let v be the vector $p - q$. Let us assume that x and o are in the same half-plane with respect to the bounding line $p + \lambda v$ (if x and o are in different half-planes defined by the line $p + \lambda v$, $\lambda \in \mathbb{R}$, then nothing has to be proved).

Case 1: The point x *lies between the lines $o + \lambda v$ and $p + \lambda v$, $\lambda \in \mathbb{R}$.* The point x is not from $\overline{oq}$ since $\|q - o\| = \|q - x\| = 1$. For similar reasons, x is not from $\overline{op}$. If x is in the interior of the triangle defined by o, p and q, then $S(p, 1)$ is not a convex curve, since o and x are in $S(p, 1)$ and $q \in B(p, 1)$. If x is not in the triangle defined by o, p and q, let us suppose that x and p are in different half-planes with respect to $\langle o, q\rangle$ (the argument is similar if x and q are in different half-planes with respect to $\langle o, p\rangle$). The points $x + p$, $x + q$, p, and q are in $S(x, 1)$, but the two pairs of points $(x + p, x + q)$ and (p, q) define two parallel segments. Therefore the four points are in the same line, and the segment $\overline{pq}$ is in $S(x, 1)$.

Case 2: The origin o *lies between the lines $x + \lambda v$ and $p + \lambda v$, $\lambda \in \mathbb{R}$.* We can argue like in Case 1, exchanging the role of x and o, because the roles that o and x play in this part of the Lemma are symmetric.

(*2*) Let us consider a minimal circular arc of radius 1 meeting p and q, and let $S(x, 1)$ be the circle that contains this arc. The curves $C = S(o, 1)$ and $C' = S(x, 1)$ verify the hypothesis of Lemma 5.1. Let p_i, γ_i and γ_i' (with $i = 1, 2$) be as in Lemma 5.1. There exist two points p_1 and p_2 such that the component

γ_2' is maximal. Then the points p and q and the minimal circular arc meeting p and q are in γ_2'. By Lemma 5.1 we have $\gamma_2' \subseteq \operatorname{conv}(\gamma_1) \subseteq B(0,1)$.

(3) Let us assume that p and q are in B, $\|p-q\| \leq 1$, and that there exists an arc meeting p and q which has points in $\operatorname{int} B$. Let $B(x,1)$ be the ball such that the arc is in $S(x,1)$. We have $x \neq o$, because there are points of the arc in $\operatorname{int} B$. The set $S(o,1) \cap S(x,1)$ is the union of two segments, each of which may degenerate to a point or to the empty set. If the intersection consists of two connected components A_1 and A_2, we can choose two points $p_1 \in A_1$ and $p_2 \in A_2$ like in Lemma 5.1 such that there is a maximal component γ_2 of $S(x,1)$ defined by p_1 and p_2 and satisfying the following: it is inside B, and the other component of $S(x,1)$ has only the two points p_1 and p_2 in B. The points p and q are in B, and the minimal arc of $S(x,1)$ meeting p and q is also in B, by the above result. Then this minimal arc is in γ_2. Therefore, the other arc defined in $S(x,1)$ by p and q has points outside of B. □

Let p and q be two points such that $\|p-q\| \leq 1$. By part *(1)* of Lemma 5.2, there exist only two minimal arcs meeting p and q (which may degenerate to the segment $\overline{pq}$ if the centers of the unit balls are in the same half-plane).

From part *(2)* of Lemma 5.2 we get immediately

Lemma 5.3. *Let $K = \{p_1, p_2, \ldots, p_n\}$ be a finite set in a normed plane $\mathbb{M}^2$ having diameter 1. Then any possible unit ball $B(x,1)$ which contains K also contains every minimal circular arc of radius 1 meeting p_i and p_j $(i,j = 1,2,\ldots,n)$.*

We continue with a further lemma.

Lemma 5.4. *Let p_1, p_2 and p_3 be three points in a normed plane $\mathbb{M}^2$ such that $\|p_i - p_j\| \leq 1$ $(i,j \in \{1,2,3\})$ and*

- *there exists a point x_{12} and a minimal circular arc $\widehat{p_1p_2}$ contained in $S(x_{12},1)$ such that p_3 is an interior point of $B(x_{12},1)$,*
- *there exists a point x_{23} and a minimal circular arc $\widehat{p_2p_3}$ contained in $S(x_{23},1)$ such that p_1 is an interior point of $B(x_{23},1)$,*
- *p_1, p_2 and p_3 are not in a line.*

Then $p_1 \notin \operatorname{conv}(\widehat{p_2p_3}, \overline{p_2p_3})$ and $p_3 \notin \operatorname{conv}(\widehat{p_1p_2}, \overline{p_1p_2})$.

Proof. Let us assume that $p_3 \in \operatorname{conv}(\widehat{p_1p_2}, \overline{p_1p_2})$. Since p_1 is an interior point of $B(x_{23},1)$, then $S(x_{23},1)$ meets $\overline{p_1p_2}$ or $\widehat{p_1p_2}$ in a point different from p_2. In any case, the arc $\widehat{p_1p_2}$ is not contained in $B(x_{23},1)$, in contradiction to Lemma 5.3. For similar reasons, $p_1 \notin \operatorname{conv}(\widehat{p_2p_3}, \overline{p_2p_3})$. □

Proposition 5.5. *Let $K = \{p_1, p_2, \ldots, p_n\}$ be a finite set in a normed plane $\mathbb{M}^2$ having diameter 1. Then*

$$\operatorname{bh}(K) = \bigcap_{i=1}^{k} B(x_i, 1),$$

where $B(x_i, 1)$, $i = 1, 2, ..., k$, are balls which contain K, and their spheres contain some minimal arcs meeting points of K.

Proof. We fix the clockwise orientation of a closed curve in $\mathbb{M}^2$ as *negative orientation* of that curve. Jung's Theorem (see, e.g., [9]) guarantees the existence of a ball $B(x_1, 1)$ such that $K \subset B(x_1, 1)$. After translating and renaming the points if necessary, we may assume that

- $S(x_1, 1)$ contains two points $p_1, p_2 \in K$,
- the circular arc starting in p_1 with negative orientation and ending in p_2 is a minimal circular arc,
- there is no other minimal circular arc in $S(x_1, 1)$ meeting points of K and being larger than the minimal circular arc meeting p_1 and p_2.

The set K has diameter 1, hence K is contained in $B(p_1, 1) \cap B(p_2, 1) \cap B(x_1, 1)$. Starting in $z = x_1$, we move z along $S(p_2, 1)$ in the negative direction. Let x_2 denote the first position of z such that one of the following conditions is verified:

(1) There is a new point $p_3 \in S(x_2, 1)$ such that
  - the circular arc in $S(x_2, 1)$ starting in p_2 with negative orientation and ending in p_3 is a minimal circular arc,
  - there is no other minimal circular arc in $S(x_2, 1)$ meeting points of K and being larger than the minimal circular arc meeting p_2 and p_3,

(2) $p_1 \in S(x_2, 1)$ with x_2 from the other half-plane defined by the line $\langle p_1, p_2 \rangle$.

In both cases, we consider the set $A = B(x_1, 1) \cap B(x_2, 1)$. Since z moves continuously in $S(p_2, 1)$, A contains K.

If $p_1 \in S(x_2, 1)$ with x_2 in the other half-plane defined by $\langle p_1, p_2 \rangle$, then A is the ball hull of K by

$$\bigcap_{K \subset B(x,1)} B(x, 1) \subset A,$$

since A can be represented as intersection of balls $B(x_i, 1)$ which contain K. Moreover, the boundary of A is generated by minimal arcs meeting points of K. Thus

$$A \subset \bigcap_{K \subset B(x,1)} B(x, 1).$$

If there exists a new point $p_3 \in S(x_2, 1)$, K is contained in $B(p_1, 1) \cap B(p_2, 1) \cap B(p_3, 1) \cap B(x_1, 1) \cap B(x_2, 1)$. Starting in $z = x_2$, we move z along $S(p_3, 1)$ in the negative direction. Let x_3 be the first value of z such that one of the following conditions is verified:

(1) There is a new point $p_4 \in S(x_3, 1)$ such that
  - the circular arc in $S(x_3, 1)$ starting in p_3 with negative orientation and ending in p_4 is a minimal circular arc,
  - there is no other minimal circular arc in $S(x_3, 1)$ meeting points of K and being larger than the minimal circular arc meeting p_3 and p_4.

(2) $p_1 \in S(x_3, 1)$.

In both cases, we consider the set $A = B(x_1, 1) \cap B(x_2, 1) \cap B(x_3, 1)$. Since z moves continuously in $S(p_3, 1)$, A contains K.

If $p_1 \in S(x_3, 1)$, then A is the ball hull of K by

$$\bigcap_{K \subset B(x,1)} B(x, 1) \subset A,$$

since A is an intersection of balls $B(x_i, 1)$ which contain K. Moreover, the boundary of A is generated by minimal arcs meeting points of K. Thus

$$A \subset \bigcap_{K \subset B(x,1)} B(x, 1).$$

If there is a new point p_4 in the above conditions, the process continues in the same way, and it is finite because the number of points p_i is finite. In this process, starting in p_j, it is not possible to get a point p_i with $1 < i < j$, due to Lemma 5.4. At the end of the process, we obtain that the set $A = \bigcap_{i=1}^{k} B(x_i, 1)$ is the ball hull of K because it is the intersection of unit balls which contain K, and its boundary is generated by minimal arcs meeting points of K. □

From Lemma 5.3 and the proof of Proposition 5.5 we obtain

Theorem 5.6. *Let $K = \{p_1, p_2, \ldots, p_n\}$ be a finite set in a normed plane $\mathbb{M}^2$ having diameter 1. Let $\widehat{p_i p_j}$ denote a minimal circular arc of radius 1 meeting p_i and p_j. Let H be the set of all balls of radius 1 such that their boundary contains a circular arc meeting points from K. Then*

$$\mathrm{bh}(K) = \bigcap_{K \subset B(x,1) \subset H} B(x, 1) = \mathrm{conv}\left(\bigcup_{i,j=1}^{n} \widehat{p_i p_j} \right).$$

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