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# Convex Closures, Weak Topologies and Feeble Approach Spaces*

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In this paper we develop a setting within the realm of approach theory to be able to deal with the non-topological notion of closed convex closure in an isometric way. We work out the necessary machinery and show how closed convex and weak* closed convex closure and the weak and weak* topologies in normed spaces fit into the framework.

## 1. Introduction

There are several types of closure operations that neither topology nor approach theory, see Lowen [15], [16], can deal with. One type of closure which is not topological is the so-called pre-topological closure which, when compared to topological closure, looses the idempotency property. This has been dealt with also in the setting of approach theory in several papers by Colebunders, Lowen and Verbeeck [5], [6], [7].

A closure of a totally different type is e.g. closed-convex closure in a normed space. This type of closure which, unlike topological closure, does not distribute over finite unions is used considerably in the literature. We refer to Aerts, Colebunders, Van der Voorde and Van Steirteghem [1] for an extensive literature list and motivations for studying closure spaces. Further we refer to Aumann [2], Claes, Colebunders and Sonck [4], Dikranjan, Giuli, Tozzi [8], Erné [10], Faure and Frölicher [11], Moore [18], Rockafellar [20] and Singer [21], [22] for more applications.

Following a suggestion from Ivan Singer in a private communication, in this paper we explore the possibility to weaken the concept of approach spaces to incorporate not only topological and metric spaces but also closure spaces. After a short study of the new structures and the categorical situation we develop a suitable setting for closed-convex closures in normed spaces and demonstrate their relation to the

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usual norm structure and in particular to the weak and weak* topology and the weak and weak* approach structures in [17], showing the relevance of these new structures in the realm of functional analysis.

We refer to Dikranjan and Tholen [9] for general information on closure operators. For categorical topological concepts we refer to Herrlich [12], [13] and Preuss [19]. For general categorical notions finally we refer to Herrlich and Strecker [14].

## 2. Preliminaries

Throughout this work, we consider $\mathbb{P} := [0, \infty]$ with its quantale structure basically consisting of the usual order and complete lattice structure and the additive semigroup. Hence we will use the symbols $+$ and $-$ also for the natural extensions of these operators on $\mathbb{P}$. In order to have a "subtraction" interior to $\mathbb{P}$, when required, we will use truncated subtraction which is defined and denoted as $a \ominus b := (a - b) \vee 0$ for all $a, b \in \mathbb{P}$.

On approach theory, there are many papers and a basic reference work [15] and we would like to refer the interested reader to those for more in-depth information. In these preliminaries we restrict our attention to the basic concepts required for the paper. Approach spaces can be introduced in various equivalent ways, here we will use the characterizations using distance operator and gauges.

A *distance* $\delta$ on a set $X$ is a function $\delta : X \times 2^X \to \mathbb{P}$ with the following axioms:

(D1) $\forall x \in X, \forall A \in 2^X : x \in A \Rightarrow \delta(x, A) = 0$,
(D2) $\forall x \in X : \delta(x, \emptyset) = \infty$,
(D3) $\forall x \in X, \forall A, B \in 2^X : \delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}$,
(D4) $\forall x \in X, \forall A \in 2^X, \forall \varepsilon \in \mathbb{P} : \delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon$ where $A^{(\varepsilon)}$ is the set of points $x$ for which $\delta(x, A) \leq \varepsilon$.

The category of all quasi-metric spaces equipped with non-expansive maps as morphisms is denoted by $\mathsf{qMet}$. Given a collection $\mathcal{G} \subset \mathsf{qMet}(X)$ and a quasi-metric $d$, we will say that $d$ is locally dominated by $\mathcal{G}$, if

$$\forall x \in X, \forall \varepsilon > 0, \forall \omega < \infty : \exists d_x^{\omega,\varepsilon} \in \mathcal{G} \text{ such that } d(x, .) \wedge \omega \leq d_x^{\omega,\varepsilon}(x, .) + \varepsilon.$$

Further we will say that $\mathcal{G}$ is locally saturated, if any quasi-metric $d$ which is locally dominated by $\mathcal{G}$ already belongs to $\mathcal{G}$. A subset $\mathcal{G}$ of $\mathsf{qMet}(X)$ is called a *gauge* if it is an ideal in $\mathsf{qMet}(X)$ such that

(G) $\mathcal{G}$ is locally saturated.

A function between approach spaces $f : X \to Y$ is called a contraction if, for all $x \in X$ and $A \subset X$,

$$\delta_Y(f(x), f(A)) \leq \delta_X(x, A).$$

The category with objects approach spaces and morphisms contractions is denoted by $\mathsf{App}$. It is a topological category [15] which contains both the categories $\mathsf{Top}$ of topological spaces and continuous maps and of $\mathsf{(q)Met}$ of (quasi-)metric spaces and contractions as full subcategories, both concretely coreflectively embedded and Top also concretely reflectively embedded [15], [14].

We also recall that a *closure space* $(X, c)$ is a set $X$ equipped with a *closure operator* $c : 2^X \to 2^X$ such that

(C1) For all $A \subset X : A \subset c(A)$,
(C2) For all $A, B \subset X : A \subset B \Rightarrow c(A) \subset c(B)$,
(C3) For all $A \subset X : c(c(A)) = c(A)$.
(C4) $c(\emptyset) = \emptyset$.

and that given closure spaces $(X, c)$ and $(X', c')$ a map $f : X \to X'$ is called *continuous* if $f(c(A)) \subset c(f(A))$ for all $A \subset X$. The category consisting of closure spaces and continuous maps is denoted Cls. It was shown in [8] that Cls is a topological category.

We will pay particular attention to concrete coreflections of a category $\mathcal{A}$ to a subcategory $\mathcal{B}$ and we recall that in our context this involves the construction, for each object $A \in \mathcal{A}$, of an object $B_A \in \mathcal{B}$ on the same underlying set as $A$, the structure of which is the coarsest $\mathcal{B}$-structure finer than the $\mathcal{A}$-structure of $A$. Like in the case of a topology underlying a metric, in the present context, it can also be thought of as the construction of a $\mathcal{B}$-structure on the same underlying set as $A$ which underlies the structure of $A$. Thus for instance the Top-coreflection of a metric space in App is nothing else but the topological space equipped with the topology of the given metric. For a precise categorical description of this we refer to [14], [19].

## 3. The new structures

In this section, we describe two basic structures which determine what we will call a *feeble approach space*. We refrain from giving the complete set of axioms for each concept and merely state what is the difference with the analogous structures of App. We require new domination and saturation conditions.

Given a collection $\mathcal{G} \subset \mathsf{qMet}(X)$ and a quasi-metric $d$, we will say that $d$ is *locally weakly dominated* by $\mathcal{G}$, if

$$\forall x \in X, \forall A \subset X : \inf_{a \in A} d(x, a) \leq \sup_{e \in \mathcal{G}} \inf_{a \in A} e(x, a).$$

Further we will say that $\mathcal{G}$ is *locally strongly saturated*, if any quasi-metric $d$ which is locally weakly dominated by $\mathcal{G}$ already belongs to $\mathcal{G}$. It is also easily verified that any quasi-metric which is locally dominated by $\mathcal{G}$ is locally weakly dominated by $\mathcal{G}$ and that local strong saturation implies local saturation. In addition, if $\mathcal{G}$ is a gauge, then it is locally strongly saturated.

**Definition 3.1.** A function

$$\delta : X \times 2^X \to \mathbb{P}$$

is called a *feeble distance* on $X$, if it satisfies the properties (D1), (D2) and (D4) and the following replacement of (D3)

(D3f) $\forall x \in X, \forall A, B \in 2^X : A \subset B \Rightarrow \delta(x, B) \leq \delta(x, A)$.

Note that analogously to the case for distances, if $\delta$ is a feeble distance then

$$\forall x \in X, \forall A, B \in 2^X : \delta(x, A) \leq \delta(x, B) + \sup_{b \in B} \delta(b, A).$$

A subset $\mathcal{G}$ of $\mathsf{qMet}(X)$ is called a *feeble gauge* if it is a downset (i.e. $e \leq d, d \in \mathcal{G} \Rightarrow e \in \mathcal{G}$) and the following replacement of the saturation condition holds

(Gf) $\mathcal{G}$ is locally strongly saturated.

Now we will point out that the structures given above are actually equivalent. We will do this by giving the transition formulas for going from one structure to another. The proof of the following result is rather lengthy and goes along the same lines, and in most cases is exactly the same, as the proof for approach spaces in [15] and [16]. So we will only indicate those steps involving the new domination and saturation condition.

**Theorem 3.2.** *If $\mathcal{G}$ is a feeble gauge on $X$, then the function*

$$\delta : X \times 2^X \to \mathbb{P} : (x, A) \mapsto \sup_{d \in \mathcal{G}} \inf_{a \in A} d(x, a)$$

*is a feeble distance on $X$ and conversely if $\delta : X \times 2^X \to \mathbb{P}$ is a feeble distance on $X$, then*

$$\mathcal{G} := \{d \in \mathsf{qMet}(X) \mid \forall A \subset X, \forall x \in X : \inf_{a \in A} d(x, a) \leq \delta(x, A)\}$$

*is a feeble gauge on $X$.*

**Proof.** We only mention that for the second part to prove (Gf), if $d \in \mathsf{qMet}(X)$ is such that $d$ is locally weakly dominated by $\mathcal{G}$, then it follows that, for all $x \in X$ and $A \subset X$ :

$$\inf_{a \in A} d(x, a) \leq \sup_{e \in \mathcal{G}} \inf_{a \in A} e(x, a) \leq \sup_{e \in \mathcal{G}} \delta(x, A) = \delta(x, A)$$

and hence $d \in \mathcal{G}$. □

Note that, exactly as in the case of approach spaces, if $\delta : X \times 2^X \to \mathbb{P}$ is a feeble distance on $X$, then, for any $\xi < \infty$ and $Z \subset X$, the function

$$d_Z^\xi : X \times X \to \mathbb{P} : (x, y) \mapsto (\delta(x, Z) \wedge \xi) \ominus (\delta(y, Z) \wedge \xi)$$

is a quasi-metric in the feeble gauge associated with $\delta$ and that these quasi-metrics generate the feeble gauge. Making use of this one can verify that going from a feeble distance to its associated feeble gauge and then to the feeble distance associated with that will give you the original feeble distance and similarly the other way round, so that both descriptions are indeed perfectly equivalent.

**Definition 3.3.** A pair $(X, \mathfrak{S})$, where $\mathfrak{S}$ is a feeble distance or a feeble gauge is called a *feeble approach space.*

## 4. The categorical setup

In the case of closure spaces, of which topological spaces are a special case, we have seen that the morphisms are defined in the same way as for topological spaces. For feeble approach spaces we reason analogously and define the morphisms as in the approach case.

**Definition 4.1.** A function $f : X \to X'$ between feeble approach spaces is called a *contraction* if for all $x \in X$ and $A \subset X$: $\delta'(f(x), f(A)) \leq \delta(x, A)$.

**Theorem 4.2.** *For a function $f : X \to X'$ between feeble approach spaces the following are equivalent:*

(1) *$f$ is a contraction,*
(2) *$\forall d' \in \mathcal{G}' : d' \circ (f \times f) \in \mathcal{G}$.*

**Proof.** This proof goes along the same lines as the approach case and we leave this to the reader. □

Feeble approach spaces and contractions are the objects and morphisms of a category, which we will denote by fApp. Clearly also App is a full subcategory of fApp. Moreover, as App, fApp is a topological category (see e.g. [12] and [19]). In order to show this we prove that unique initial structures exist. Here, there is considerable deviation from the approach case, initial structures are quite different.

**Theorem 4.3.** fApp *is a topological category. In particular, suppose given the feeble approach spaces $(X_j)_{j\in J}$ and the source*

$$(f_j : X \to X_j)_{j\in J}$$

*in* fApp*. If, for each $j \in J$, $\delta_j$ is the feeble distance on $X_j$, then the initial feeble distance is given by*

$$\delta(x, A) := \sup_{j\in J} \delta_j(f_j(x), f_j(A))$$

*for all $x \in X$ and $A \subset X$.*

**Proof.** Clearly fApp is concrete, small fibered and fulfills the terminal separator property. First we have to show that $\delta$ is a feeble distance on $X$. It immediately follows from the definitions that $\delta$ satisfies (D1), (D2) and (D3f). To prove (D4) let $x \in X, A \subset X$ and $\varepsilon \in \mathbb{P}$, then it is easily verified that for all $j \in J$ we have $f_j(A^{(\varepsilon)}) \subset f_j(A)^{(\varepsilon)}$ and hence

$$\begin{aligned}\sup_{j\in J} \delta_j(f_j(x), f_j(A)) &\leq \sup_{j\in J} \delta_j(f_j(x), f_j(A)^{(\varepsilon)}) + \varepsilon \\ &\leq \sup_{j\in J} \delta_j(f_j(x), f_j(A^{(\varepsilon)})) + \varepsilon\,.\end{aligned}$$

By construction, all the functions $f_j, j \in J$, are contractions.

To prove that $\delta$ is an initial structure, let $(Z, \delta')$ be a feeble approach space and let $g : Z \to X$ be such that all compositions $f_j \circ g : (Z, \delta') \to (X_j, \delta_j)$ are contractions. Further, let $z \in Z$ and $B \subset Z$. Since $\delta_j(f_j(g(z)), f_j(g(B))) \leq \delta'(z, B)$ for all $j \in J$, we obtain, again immediately from the definition, that $\delta(g(z), g(B)) \leq \delta'(z, B)$ which completes the proof. Uniqueness is clear. □

The characterization of initial structures in $\mathsf{fApp}$ with feeble gauges can also be determined. We give it here but leave the verifications to the reader.

**Theorem 4.4.** *Consider the source $(f_j : X \to X_j)_{j \in J}$ where $X_j$ is a feeble approach space for each $j \in J$.*

*If for each $j \in J$, $\mathcal{G}_j$ is the feeble gauge of $X_j$ and*

$$\mathcal{B} = \{d_j \circ (f_j \times f_j) \mid d_j \in \mathcal{G}_j, j \in J\}$$

*then the initial feeble gauge is given by*

$$\mathcal{G} = \{d \in \mathsf{qMet}(X) \mid d \text{ is locally weakly dominated by } \mathcal{B}\}.$$

**Proof.** This is left to the reader. □

In order to see that initial structures in $\mathsf{fApp}$ are quite different to the initial structures in $\mathsf{App}$ it suffices to look at a simple example.

**Example 4.5.** Let $X$ be the real line $\mathbb{R}$ equipped with the Euclidean distance. Since in $\mathsf{App}$ finite products of metric spaces are again metric we know that the product $X \times X$ in $\mathsf{App}$ is equipped with the gauge generated (in the $\mathsf{App}$-sense, see [15]) by $\{d\}$ where $d(x, a) := \max\{|x_1 - a_1|, |x_2 - a_2|\}$. The distance hence is

$$\delta^{\mathsf{App}}(x, A) := \inf_{a \in A} d(x, a).$$

Now if we take $A := \{(a, a) \mid a \in \mathbb{R}\}$ and $x \in \mathbb{R} \times \mathbb{R}$ arbitrary, then it follows that

$$\delta^{\mathsf{App}}(x, A) = \left| \frac{x_1 - x_2}{2} \right|.$$

However if we take the product in $\mathsf{fApp}$ then it follows from 4.3 that with the same set $A$ and for an arbitrary point $x \in \mathbb{R} \times \mathbb{R}$ we have $\delta^{\mathsf{fApp}}(x, A) = 0$.

Nevertheless, $\mathsf{App}$ is actually very nicely embedded in $\mathsf{fApp}$. In the following, given a set $A \subset X$, $\mathfrak{P}(A)$ will stand for the collection of all finite covers of $A$ with subsets of $A$.

**Theorem 4.6.** $\mathsf{App}$ *is concretely coreflectively embedded in* $\mathsf{fApp}$ *with coreflection given by*

$$\mathrm{id}_X : (X, \delta_{\mathsf{App}}) \to (X, \delta)$$

*where*

$$\delta_{\mathsf{App}} : X \times 2^X \to \mathbb{P} : \delta_{\mathsf{App}}(x, A) := \sup_{\mathcal{P} \in \mathfrak{P}(A)} \min_{P \in \mathcal{P}} \delta(x, P).$$

**Proof.** First we prove that $\delta_{\mathsf{App}}$ is indeed a distance. (D1) and (D2) are clear. For (D3), we first prove that $\forall x \in X, \forall A, B \in 2^X : \delta_{\mathsf{App}}(x, A \cup B) \leq \delta_{\mathsf{App}}(x, A) \wedge \delta_{\mathsf{App}}(x, B)$. To do this it suffices to show that $C \subset D \subset X$ implies $\delta_{\mathsf{App}}(x, D) \leq \delta_{\mathsf{App}}(x, C)$. Clearly, for all $\mathcal{P} \in \mathfrak{P}(D)$ we have $\mathcal{P}' = \{P \cap C \mid P \in \mathcal{P}\} \in \mathfrak{P}(C)$. Consequently

$$\begin{aligned}\delta_{\mathsf{App}}(x, D) &= \sup_{\mathcal{P}\in\mathfrak{P}(D)} \min_{P\in\mathcal{P}} \delta(x, P)\\ &\leq \sup_{\mathcal{P}\in\mathfrak{P}(D)} \min_{P\in\mathcal{P}} \delta(x, P \cap C)\\ &\leq \delta_{\mathsf{App}}(x, C).\end{aligned}$$

Conversely, if $\mathcal{P} = \{A_k \mid k = 1, 2, ..., n\} \in \mathfrak{P}(A)$ and $\mathcal{Q} = \{B_j \mid j = 1, 2, ..., m\} \in \mathfrak{P}(B)$, then $\mathcal{P} \cup \mathcal{Q} \in \mathfrak{P}(A \cup B)$. It follows that

$$\begin{aligned}\delta_{\mathsf{App}}(x, A) \wedge \delta_{\mathsf{App}}(x, B) &= \sup_{\mathcal{P}\in\mathfrak{P}(A)} \sup_{\mathcal{Q}\in\mathfrak{P}(B)} (\min_{P\in\mathcal{P}} \delta(x, P) \wedge \min_{Q\in\mathcal{Q}} \delta(x, Q))\\ &= \sup_{\mathcal{P}\in\mathfrak{P}(A)} \sup_{\mathcal{Q}\in\mathfrak{P}(B)} \min_{R\in\mathcal{P}\cup\mathcal{Q}} \delta(x, R)\\ &\leq \sup_{\mathcal{R}\in\mathfrak{P}(A\cup B)} \min_{R\in\mathcal{R}} \delta(x, R)\\ &= \delta_{\mathsf{App}}(x, A \cup B).\end{aligned}$$

To prove (D4), first, we point out that for all $\varepsilon \in \mathbb{P}$ and $A \subset X$, we will denote $A^{(\varepsilon)} = \{y \mid \delta(y, A) \leq \varepsilon\}$ and $A^{[\varepsilon]} = \{y \mid \delta_{\mathsf{App}}(y, A) \leq \varepsilon\}$. Now let $x \in X$ and $\varepsilon \in \mathbb{P}$, then we have

$$\begin{aligned}\delta_{\mathsf{App}}(x, A) &= \sup_{\mathcal{P}\in\mathfrak{P}(A)} \min_{P\in\mathcal{P}} \delta(x, P)\\ &\leq \sup_{\mathcal{P}\in\mathfrak{P}(A)} \min_{P\in\mathcal{P}} \delta(x, P^{(\varepsilon)}) + \varepsilon\\ &\leq \sup_{\mathcal{P}\in\mathfrak{P}(A)} \min_{P\in\mathcal{P}} \delta(x, P^{(\varepsilon)} \cap A^{[\varepsilon]}) + \varepsilon.\end{aligned}$$

If $z \in A^{[\varepsilon]}$, then

$$\delta_{\mathsf{App}}(z, A) = \sup_{\mathcal{P}\in\mathfrak{P}(A)} \min_{P\in\mathcal{P}} \delta(z, P) \leq \varepsilon$$

and it follows that $\forall \mathcal{P} \in \mathfrak{P}(A) : \min_{P\in\mathcal{P}} \delta(z, P) \leq \varepsilon$. Consequently, $\forall \mathcal{P} \in \mathfrak{P}(A) : \exists P_0 \in \mathcal{P}$ such that $\delta(z, P_0) \leq \varepsilon$ and so $z \in \cup_{P\in\mathcal{P}} P^{(\varepsilon)}$ and thus we obtain that $\mathcal{Q} =\{P^{(\varepsilon)} \cap A^{[\varepsilon]} \mid P \in \mathcal{P}, \mathcal{P} \in \mathfrak{P}(A)\} \in \mathfrak{P}(A^{[\varepsilon]})$ from which it follows that

$$\begin{aligned}\delta_{\mathsf{App}}(x, A) &\leq \sup_{\mathcal{R}\in\mathfrak{P}(A^{[\varepsilon]})} \min_{R\in\mathcal{R}} \delta(x, R) + \varepsilon\\ &= \delta_{\mathsf{App}}(x, A^{[\varepsilon]}) + \varepsilon.\end{aligned}$$

Second, obviously, $\delta \leq \delta_{\mathsf{App}}$ from which it follows that $\mathrm{id}_X : (X, \delta_{\mathsf{App}}) \to (X, \delta)$ is a contraction. To complete the proof, let $(Y, \delta')$ be an approach space and let

$$f : (Y, \delta') \to (X, \delta)$$

be a contraction. Then, for all $z \in Y$ and $B \subset Y$, we have

$$\begin{aligned}\delta_{\mathsf{App}}(f(z), f(B)) &= \sup_{\mathcal{P} \in \mathfrak{P}(f(B))} \min_{P \in \mathcal{P}} \delta(f(z), P) \\ &\leq \sup_{\mathcal{P} \in \mathfrak{P}(f(B))} \min_{P \in \mathcal{P}} \delta(f(z), P \cap f(B)) \\ &\leq \sup_{\mathcal{P} \in \mathfrak{P}(f(B))} \min_{P \in \mathcal{P}} \delta'(z, f^{-1}(P) \cap B) \\ &= \sup_{\mathcal{P} \in \mathfrak{P}(f(B))} \delta'(z, B) \\ &= \delta'(z, B)\end{aligned}$$

from which it follows that $f : (X, \delta') \to (X, \delta_{\mathsf{App}})$ is a contraction. □

A closure space can be viewed as a special type of feeble approach space in a trivial way. We leave all verifications to the reader. Given a closure space $(X, c)$ we associate with it a natural feeble distance $\delta_c : X \times 2^X \to [0, \infty]$ given by

$$\delta_c(x, A) := \begin{cases} 0 & x \in c(A) \\ \infty & x \notin c(A). \end{cases}$$

The associated feeble gauge is then given by

$$\mathcal{G}_c := \{d \in \mathsf{qMet}(X) \mid \mathcal{T}_d \subset \mathcal{T}_c\}$$

where $\mathcal{T}_c$ is the set of all open sets of $c$ and $\mathcal{T}_d$ is the topology of $d$. Recall that open (and closed) sets in a closure space are defined in exactly the same way as in topology but that the collection of open sets (respectively closed sets) need not be stable under finite intersections (respectively finite unions).

If $(X, c)$ and $(X', c')$ are closure spaces and $f : X \to X'$ is a function then it follows at once from the definitions that $f$ is continuous as a map between the closure spaces if and only if it is a contraction as a map between the associated feeble approach spaces.

All together this means that

$$\begin{gathered}\mathsf{Cls} \to \mathsf{fApp} \\ (X, c) \mapsto (X, \delta_c) \\ f \mapsto f\end{gathered}$$

is a full concrete embedding of $\mathsf{Cls}$ into $\mathsf{fApp}$.

However, we are able to show more. Note that a feeble approach space $(X, \delta)$ is (derived from) a closure space if and only if

$$\delta(X \times 2^X) = \{0, \infty\}.$$

**Theorem 4.7.** Cls *is both concretely reflectively and concretely coreflectively embedded in* fApp.

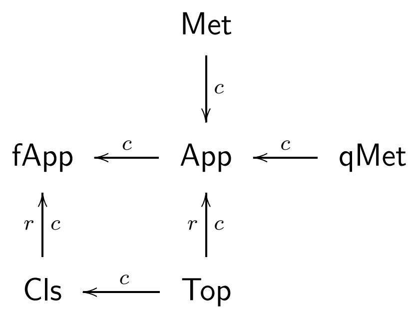
**Proof.** The first claim is a consequence of the fact that in order to be concretely reflective it suffices to be initially closed. Indeed, the result then follows from the remark preceding this theorem and from 4.3. For the second claim note that, as the reader can easily verify, for any feeble approach space $(X, \delta)$ its Cls-coreflection is given by

$$\mathrm{id}_X : (X, \delta_c) \to (X, \delta),$$

where $c$ is the closure operator given by $c(A) := \{x \in X \mid \delta(x, A) = 0\}$ for all $A \subset X$ $\square$

As is the case for approach spaces it is in particular the coreflection which is of importance since it generalizes the concept of a topology underlying a metric.

The following diagram gathers the categorical results from this section.




Note that, if $X$ is a feeble approach space, the (quasi-)metric coreflection is the (quasi-)metric coreflection of its approach coreflection. For any $x, a \in X$ we have $\delta_{\mathsf{App}}(x, \{a\}) = \delta(x, \{a\})$ .

## 5. Feeble approach structures for closed convex closures

Let $(E, \| \|)$ be a normed space. We denote by $\delta_{\| \|}$ the distance determined by the norm and by $\mathrm{conv}(A)$ the convex hull of an arbitrary set $A \subset E$. Taking the closed convex closure is a well known operation used in e.g. functional analysis and optimization theory given by

$$c_{co} : 2^E \to 2^E : A \mapsto \overline{\mathrm{conv}(A)}$$

where the overline denotes closure in the norm structure. This is a closure which is not topological as it does not distribute over finite unions.

**Theorem 5.1.** *The operator*

$$\delta^{co} : E \times 2^E \to \mathbb{P} : (x, A) \mapsto \delta_{\| \|}(x, \mathrm{conv}(A))$$

*is a feeble distance on* $E$.

**Proof.** (D1), (D2) and (D3f) are clear. To prove (D4), first note that for all $\varepsilon \in \mathbb{P}$ and $A \subset E$, we let

$$A^{(\varepsilon)} = \{y \in E \mid \delta^{co}(y, A) \leq \varepsilon\} \quad \text{and} \quad A^{[\varepsilon]} = \{y \in E \mid \delta_{\| \, \|}(y, A) \leq \varepsilon\}.$$

Now let $x \in E, A \in 2^E$ and $\varepsilon \in \mathbb{P}$. Then we have

$$\begin{aligned} \delta^{co}(x, A) &= \delta_{\| \, \|}(x, \operatorname{conv}(A)) \\ &\leq \delta_{\| \, \|}(x, (\operatorname{conv}(A))^{[\varepsilon]}) + \varepsilon. \end{aligned}$$

Therefore, in order to complete the proof, it suffices to show that $\operatorname{conv}(A^{(\varepsilon)}) \subset (\operatorname{conv}(A))^{[\varepsilon]}$. If $z \in \operatorname{conv}(A^{(\varepsilon)})$, then there exists $z_1, \dots, z_m \in A^{(\varepsilon)}$ and $\alpha_1, \dots, \alpha_m$ positive such that $\sum_{k=1}^m \alpha_k = 1$ and $z = \sum_{k=1}^m \alpha_k z_k$. That $z_k \in A^{(\varepsilon)}$ means that

$$\delta_{\| \, \|}(z_k, \operatorname{conv}(A)) = \inf_{y \in \operatorname{conv}(A)} \|z_k - y\| \leq \varepsilon.$$

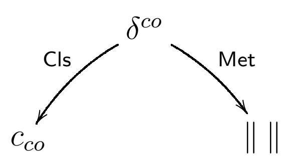
Hence, for any $k = 1, 2, ..., m$ and $\theta > 0$ choose $y_k \in \operatorname{conv}(A)$ such that $\|z_k - y_k\| \leq \varepsilon + \theta$. Then we have

$$\begin{aligned} \delta_{\| \, \|}(z, \operatorname{conv}(A)) &= \inf_{y \in \operatorname{conv}(A)} \|z - y\| \\ &\leq \left\| z - \sum_{k=1}^m \alpha_k y_k \right\| \\ &= \left\| \sum_{k=1}^m \alpha_k z_k - \sum_{k=1}^m \alpha_k y_k \right\| \\ &\leq \sum_{k=1}^m \alpha_k \|z_k - y_k\| \\ &\leq \sum_{k=1}^m \alpha_k (\varepsilon + \theta) = \varepsilon + \theta \end{aligned}$$

from which it follows that $\delta_{\| \, \|}(z, \operatorname{conv}(A)) \leq \varepsilon$ by the arbitrariness of $\theta$. □

We will refer to the feeble approach structure defined in the foregoing result as the *convexity (feeble approach) structure* and to the distance as the *convexity feeble distance.*

**Theorem 5.2.** *The* Cls*-coreflection of the convexity structure is the structure of closed convex closure and the* Met*-coreflection is the norm structure*



**Proof.** The first claim follows at once from 4.7 since $\delta_{\| \|}(x, \text{conv}(A)) = 0$ means that $x \in \overline{\text{conv}(A)}$ and the second one follows at once from the definition. □

Theorem 4.6 tells us what the App-coreflection is in general. However in order to describe this App-coreflection of the convexity structure more concretely, we recall the following from [17]. Given a normed space $E$ we consider its dual $E'$ consisting of all real-valued continuous linear maps. We denote the closed ball of a space $E$ by $B_E$. The dual $E'$ is a Banach space equipped with the dual norm $\|f\| := \sup_{x\in B_E} |f(x)|$. This dual creates on $E$ the so-called weak topology $\sigma(E, E')$ which is generated by the neighborhoods of the origin $V(F, \varepsilon) := \{x \in E \mid \sup_{f\in F} |f(x)| < \varepsilon\}$ where $F$ is a finite subset of $B_{E'}$ and $\varepsilon > 0$.

The fact that the original setup is a normed space (hence equipped with valuable numerical information) and that, not only, all numerical information is lost in the auxiliary, but equally important weak topology, but that moreover this weak topology is never metrizable, was the reason in [17] to try and do better by taking the naturally created approach structure on $E$ generated by $E'$ in the following way. The weak topology, by construction is the coarsest topology making all functions in $B_{E'}$ (and consequently also in $E'$) continuous as maps to $\mathbb{R}$ equipped with the Euclidean topology. The weak approach structure, with distance $\delta_w$, is defined as the coarsest approach structure on $E$ making all the functions in $B_{E'}$ contractions when viewed as maps between $E$ and $\mathbb{R}$ equipped with the Euclidean norm, i.e. such that

$$(f : (E, \delta_w) \longrightarrow (\mathbb{R}, \delta_{\mathbb{E}}))_{f\in B_{E'}}$$

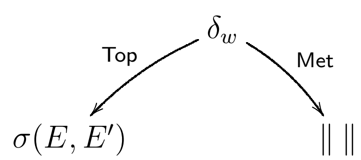
is initial. This means that the gauge for this structure is given by the collection of metrics

$$d_F(x, y) := \sup_{f\in F} |f(x) - f(y)|$$

where $F$ is a finite subset of $B_{E'}$. The *weak distance* $\delta_w$ is then given by

$$\delta_w(x, A) = \sup_{F\subset B_{E'} \text{ finite}} \inf_{a\in A} d_F(x, a).$$

In [17] the basic properties of this weak approach structure were studied and we refer the interested reader to that paper. However, we recall the following salient facts. The weak approach structure is a distance lying over the weak topology, in the sense that the weak topology is the topological coreflection of the weak approach structure. In other words, whereas the weak topology is never metrizable, it is in a natural way "distancizable", namely by the weak approach structure. The metric coreflection of the weak approach structure is the norm structure (a consequence of the Hahn-Banach theorem).



One of the many results in [17], and which we will require, is the following which generalizes and implies Mazur's theorem.

**Theorem 5.3 ([17]).** *If $A \subset E$ is convex then for any $x \in E$*

$$\delta_w(x, A) = \delta_{\|\ \|}(x, A).$$

We now prove the following surprising fact.

**Theorem 5.4.** *The* App*-coreflection $\delta^{co}_{\mathsf{App}}$ of the convexity feeble distance coincides with the weak distance $\delta_w$.*

**Proof.** We recall the two distances which come into play. From 4.6 it follows that the App-coreflection of $\delta^{co}$ is given by

$$\delta^{co}_{\mathsf{App}}(x, A) = \sup_{\mathcal{P}\in\mathfrak{P}(A)} \min_{P\in\mathcal{P}} \delta_{\|\ \|}(x, \mathrm{conv}(P))$$

whereas the weak distance $\delta_w$ is given by

$$\delta_w(x, A) = \sup_{F\subset B_{E'} \text{ finite}} \inf_{a\in A} \sup_{f\in F} |f(x) - f(a)|$$

for all $x \in E$ and $A \subset E$.

Suppose that $\delta_w(x, A) \leq \alpha$ and let $\mathcal{P} \in \mathbb{P}(A)$. Then it follows from 5.3 that

$$\begin{aligned}\min_{P\in\mathcal{P}} \delta_{\|\ \|}(x, \mathrm{conv}(P)) &= \min_{P\in\mathcal{P}} \delta_w(x, \mathrm{conv}(P)) \\ &\leq \min_{P\in\mathcal{P}} \delta_w(x, P) \\ &= \delta_w(x, A) \leq \alpha\end{aligned}$$

and hence there exists $P_0 \in \mathcal{P}$ such that $\delta_{\|\ \|}(x, \mathrm{conv}(P_0)) \leq \alpha$ which proves that $\delta^{co}_{\mathsf{App}}(x, A) \leq \alpha$ and hence that $\delta^{co}_{\mathsf{App}} \leq \delta_w$.

Conversely, suppose that $\alpha < \delta_w(x, A)$, then there exist $f_1, \ldots, f_n \in B_{E'}$ such that

$$A \subset \bigcup_{k=1}^{n} \{z \mid |f_k(x) - f_k(z)| \geq \alpha\}.$$

Hence, if for all $k = 1, \ldots, n$ we put

$$P_k := \{z \mid f_k(z) \geq f_k(x) + \alpha\} \quad \text{and} \quad N_k := \{z \mid f_k(z) \leq f_k(x) - \alpha\}$$

then it follows that

$$A = \bigcup_{k=1}^{n} (P_k \cap A) \cup \bigcup_{k=1}^{n} (N_k \cap A).$$

Moreover, for all $k = 1, \ldots, n$ we have that both $P_k$ and $N_k$ are closed half-spaces and thus convex, hence

$$\mathrm{conv}(P_k \cap A) \subset P_k \quad \text{and} \quad \mathrm{conv}(N_k \cap A) \subset N_k$$

which implies that

$$\forall z \in \mathrm{conv}(P_k \cap A) : |f_k(x) - f_k(z)| \geq \alpha,$$

and

$$\forall z \in \mathrm{conv}(N_k \cap A) : |f_k(x) - f_k(z)| \geq \alpha.$$

Then we find, again making use of 5.3, that

$$\begin{aligned}
\delta^{co}_{\mathsf{App}}(x, A) &= \sup_{\mathcal{P} \in \mathfrak{P}(A)} \min_{P \in \mathcal{P}} \delta_{\| \|}(x, \mathrm{conv}(P)) \\
&= \sup_{\mathcal{P} \in \mathfrak{P}(A)} \min_{P \in \mathcal{P}} \delta_w(x, \mathrm{conv}(P)) \\
&= \sup_{\mathcal{P} \in \mathfrak{P}(A)} \min_{P \in \mathcal{P}} \sup_{F \subset B_{E'} \text{finite}} \inf_{z \in \mathrm{conv}(P)} \sup_{f \in F} |f(x) - f(z)| \\
&\geq \min_{k=1}^{n} \inf_{z \in \mathrm{conv}(P_k \cap A)} \sup_{j=1}^{n} |f_j(x) - f_j(z)| \wedge \min_{k=1}^{n} \inf_{z \in \mathrm{conv}(N_k \cap A)} \sup_{j=1}^{n} |f_j(x) - f_j(z)| \\
&\geq \min_{k=1}^{n} \inf_{z \in \mathrm{conv}(P_k \cap A)} |f_k(x) - f_k(z)| \wedge \min_{k=1}^{n} \inf_{z \in \mathrm{conv}(N_k \cap A)} |f_k(x) - f_k(z)| \\
&\geq \alpha.
\end{aligned}$$

Thus we also have that $\delta_w \leq \delta^{co}_{\mathsf{App}}$ and we are done. □

This result has the following consequence if we look at the topological coreflections of both structures and use the commutativity of the diagram following 4.7.

**Corollary 5.5.** *The topological coreflection of the closure structure of closed convex closure is the weak topology, in particular this structure is entirely determined by the closed convex sets.*

**Proof.** Given any $A \subset E$ its weak closure is given by

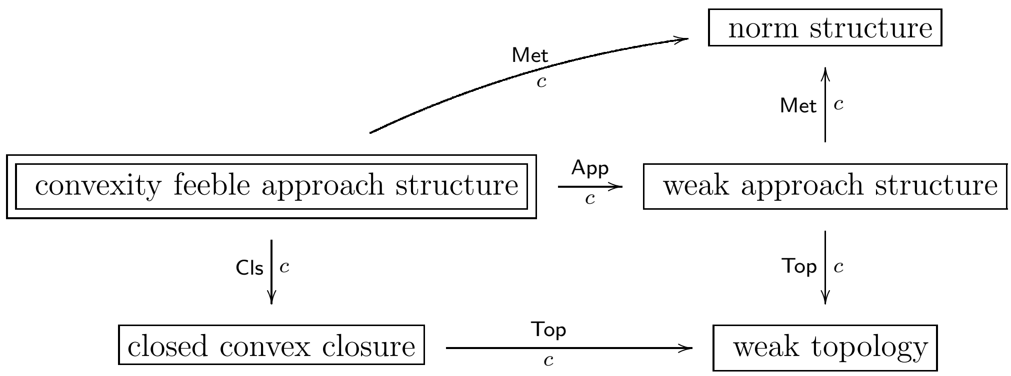
$$\mathrm{cl}_w(A) = \bigcap_{\mathcal{P} \in \mathbb{P}(A)} \bigcup_{P \in \mathcal{P}} c_{co}(P) = \bigcap_{\mathcal{P} \in \mathbb{P}(A)} \bigcup_{P \in \mathcal{P}} \overline{\mathrm{conv}(P)}.$$ □

Note that a consequence of Mazur's theorem also follows immediately from this.

**Corollary 5.6 (Mazur).** *If a sequence $(x_n)_n$ converges weakly to $x$ then there exists a sequence of convex combinations of the $x_n$ which converges to $x$ in the norm structure.*

**Proof.** From 5.5 we have $\mathrm{cl}_w(\{x_n \mid n\}) \subset \overline{\mathrm{conv}(\{x_n \mid n\})}$. □

The following diagram depicts the various structures derived from the convexity feeble approach structure and their relations. Each arrow represents coreflection to the type of structure denoted next to it.




**Theorem 5.7.** *For $x \in E$ arbitrary and $A \subset E$ we have $\delta^{co}_{\mathsf{App}}(x,A) \le \delta_{\|\,\|}(x,A)$ and in case $A$ is totally bounded we have $\delta^{co}_{\mathsf{App}}(x,A) = \delta_{\|\,\|}(x,A)$.*

**Proof.** The inequality in general follows immediately from the results of [17] and from 5.4.

Let $x \in E$, $A \subset E$ totally bounded and $\varepsilon > 0$. Choose a finite collection of sets $A_k$, $k = 1, \dots, m$ such that

$$A = \bigcup_{k=1}^{m} A_k \quad \text{with } \operatorname{diam}(A_k) \le \varepsilon \text{ for all } k = 1, \dots, m.$$

Then it follows that $\operatorname{conv}(A_k) \subset (A_k)^{[\varepsilon]}$ and hence

$$\begin{aligned}
\delta^{co}_{\mathsf{App}}(x,A) &= \sup_{\mathcal{P}\in\mathfrak{P}(A)} \min_{P\in\mathcal{P}} \delta_{\|\,\|}(x, \operatorname{conv}(P)) \\
&\ge \min_{k=1}^{m} \delta_{\|\,\|}(x, \operatorname{conv}(A_k)) \\
&\ge \min_{k=1}^{m} \delta_{\|\,\|}(x, (A^k)^{[\varepsilon]}) \\
&\ge \min_{k=1}^{m} \delta_{\|\,\|}(x, A^k) - \varepsilon \\
&= \delta_{\|\,\|}(x,A) - \varepsilon.
\end{aligned}$$

By the arbitrariness of $\varepsilon$, we obtain $\delta_{\|\,\|}(x,A) \le \delta^{co}_{\mathsf{App}}(x,A)$. □

The following is a variant to Mazur's theorem, not for convex sets but for sets which are totally bounded.

**Corollary 5.8.** *If $A \subset E$ is totally bounded then $\delta_w(x,A) = \delta_{\|\,\|}(x,A)$ and hence the weak and norm closures of $A$ coincide.*

**Proof.** Immediate from 5.3 and 5.7. □

**Corollary 5.9.** *If $E$ is finite-dimensional then the* App*-coreflection $\delta^{co}_{\mathsf{App}}$ of the convexity feeble distance, the weak distance $\delta_w$ and the norm distance $\delta_{\| \|}$ all coincide.*

**Proof.** Immediate from [17] and 5.4. □

The inequality in 5.7 is in general best possible.

**Example 5.10.** Let $E$ be an infinite-dimensional normed space, let $S_E$ be the unit sphere and let $\alpha > 0$. Then obviously

$$\delta_{\| \|}(0, \alpha S_E) = \alpha.$$

However, it is well known that 0 is in the weak closure of $S_E$, hence by 5.4 $\delta^{co}_{\mathsf{App}}(0, S_E) = \delta_w(0, S_E) = 0$, showing that the difference between $\delta^{co}_{\mathsf{App}}$ and $\delta_{\| \|}$ can be made arbitrarily large, and thus not only different but also not Lipschitz-equivalent.

We now look at the situation for the weak* topology. $E$ creates on $E'$ the so-called weak* topology $\sigma(E', E)$ which is generated by the neighborhoods of the origin $V(F, \varepsilon) := \{f \in E' \mid \sup_{x \in F} |f(x)| < \varepsilon\}$ where $F$ is a finite subset of $B_E$ and $\varepsilon > 0$.

For the same reasons as before, in [17] a natural approach structure with distance $\delta^*_w$ on $E'$ generated by $E$ was defined as the coarsest approach structure on $E'$ making all the functions in $B_E$ contractions when viewed as maps between $E'$ and $\mathbb{R}$ equipped with the Euclidean norm, i.e. such that

$$(\hat{x} : (E', \delta^*_w) \longrightarrow (\mathbb{R}, \delta_{\mathbb{E}}) : f \mapsto f(x))_{x \in B_E}$$

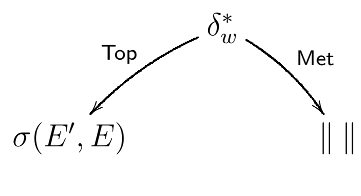
is initial. This means that the gauge for this structure is given by the collection of metrics

$$d_F(f, g) := \sup_{x \in F} |f(x) - g(x)|$$

where $F$ is a finite subset of $B_E$. The *weak* distance* $\delta^*_w$ is then given by

$$\delta^*_w(f, G) = \sup_{F \subset B_E \text{ finite}} \inf_{g \in G} d_F(f, g).$$

In [17] the basic properties of this weak* approach structure too were studied and again we refer the interested reader to that paper. The weak* approach structure is a distance lying over the weak* topology, in the sense that the weak* topology is the topological coreflection of the weak* approach structure. The metric coreflection of the weak* approach structure is the (dual) norm structure.



**Theorem 5.11.** *The operator*

$$\delta^{co^*} : E' \times 2^{E'} \to \mathbb{P} : (f, G) \mapsto \delta^*_w(f, \mathrm{conv}(G))$$

*is a feeble distance on $E'$.*

**Proof.** (D1), (D2) and (D3f) are clear. To prove (D4), first note that for all $\varepsilon \in \mathbb{P}$ and $G \subset E'$, we let

$$G^{(\varepsilon)} = \{g \in E' \mid \delta^{co^*}(g, G) \le \varepsilon\} \quad \text{and} \quad G^{[\varepsilon]} = \{g \in E' \mid \delta^*_w(g, G) \le \varepsilon\}.$$

Now let $f \in E', G \in 2^{E'}$ and $\varepsilon \in \mathbb{P}$. Then, as in 5.1, it suffices to show that $\mathrm{conv}(G^{(\varepsilon)}) \subset (\mathrm{conv}(G))^{[\varepsilon]}$. If $g \in \mathrm{conv}(G^{(\varepsilon)})$, then there exists $g_1, \ldots, g_m \in G^{(\varepsilon)}$ and $\alpha_1, \ldots, \alpha_m$ positive such that $\sum_{k=1}^m \alpha_k = 1$ and $g = \sum_{k=1}^m \alpha_k g_k$. That $g_k \in G^{(\varepsilon)}$ means that

$$\sup_{F \subset B_E \text{ finite}} \ \inf_{h \in \mathrm{conv}(G)} \ \sup_{x \in F} |g_k(x) - h(x)| \le \varepsilon.$$

Hence, for any $k = 1, \ldots, m$, any $F \subset B_E$ finite and any $\theta > 0$ we can find $h_k \in \mathrm{conv}(G)$ such that

$$\sup_{x \in F} |g_k(x) - h_k(x)| \le \varepsilon + \theta.$$

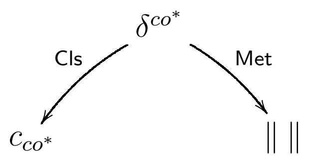
Now fix $F \subset B_E$ finite and $\theta > 0$ and put $h := \sum_{k=1}^m \alpha_k h_k \in \mathrm{conv}(G)$ then it follows that

$$\begin{aligned} \inf_{u \in \mathrm{conv}(G)} \sup_{x \in F} |g(x) - u(x)| &\le \sup_{x \in F} |g(x) - h(x)| \\ &= \sup_{x \in F} \left| \sum_{k=1}^m \alpha_k g_k(x) - \sum_{k=1}^m \alpha_k h_k(x) \right| \\ &\le \sum_{k=1}^m \alpha_k (\varepsilon + \theta) = \varepsilon + \theta \end{aligned}$$

from which it follows that $\delta^*_w(g, \mathrm{conv}(G)) \le \varepsilon$ by the arbitrariness of $F$ and $\theta$. $\square$

We refer to the above feeble distance as the *convexity* feeble distance* or the *convexity* structure.*

**Theorem 5.12.** *The* Cls*-coreflection of the convexity* structure is the structure of weak* closed convex closure and the* Met*-coreflection is the dual norm structure*



**Proof.** The first claim follows at once from 5.11 since $\delta^*_w(f, \mathrm{conv}(G)) = 0$ means that $f \in \overline{\mathrm{cl}^*_w \mathrm{conv}(G)}$ and the second one follows at once from 5.11 and the definition of the dual norm. □

**Theorem 5.13.** *The* App*-coreflection* $\delta^{co^*}_{\mathsf{App}}$ *of the convexity* feeble distance coincides with the weak* distance* $\delta^*_w$.

**Proof.** Now the two distances which come into play are the following. From 4.6 it follows that the App-coreflection of $\delta^{co^*}$ is given by

$$\delta^{co^*}_{\mathsf{App}}(f, G) = \sup_{\mathcal{P} \in \mathfrak{P}(G)} \min_{P \in \mathcal{P}} \delta^*_w(f, \mathrm{conv}(P))$$

whereas the weak* distance $\delta^*_w$ is given by

$$\delta^*_w(f, G) = \sup_{F \subset B_E \text{ finite}} \inf_{g \in G} \sup_{x \in F} |f(x) - g(x)|$$

for all $f \in E'$ and $G \subset E'$.

Suppose that $\delta^*_w(f, G) \leq \alpha$ and let $\mathcal{P} \in \mathbb{P}(G)$. Then it follows that

$$\min_{P \in \mathcal{P}} \delta^*_w(f, \mathrm{conv}(P)) \leq \min_{P \in \mathcal{P}} \delta^*_w(f, P) = \delta^*_w(f, G) \leq \alpha$$

and hence there exists $P_0 \in \mathcal{P}$ such that $\delta^*_w(x, \mathrm{conv}(P_0)) \leq \alpha$ which proves that $\delta^{co^*}_{\mathsf{App}}(f, G) \leq \alpha$ and hence that $\delta^{co^*}_{\mathsf{App}} \leq \delta^*_w$.

Conversely, suppose that $\alpha < \delta_w(f, G)$, then there exist $x_1, \ldots, x_n \in B_E$ such that

$$G \subset \bigcup_{k=1}^{n} \{h \mid |f(x_k) - h(x_k)| \geq \alpha\}.$$

Hence, if for all $k = 1, \ldots, n$ we put

$$P_k := \{h \mid h(x_k) \geq f(x_k) + \alpha\} \quad \text{and} \quad N_k := \{h \mid h(x_k) \leq f(x_k) - \alpha\}$$

then it follows that

$$G = \bigcup_{k=1}^{n} (P_k \cap G) \cup \bigcup_{k=1}^{n} (N_k \cap G).$$

Moreover, for all $k = 1, \ldots, n$ we have that both $P_k$ and $N_k$ are weak*-closed half-spaces and thus convex, hence

$$\mathrm{conv}(P_k \cap G) \subset P_k \quad \text{and} \quad \mathrm{conv}(N_k \cap G) \subset N_k$$

which implies that

$$\forall h \in \mathrm{conv}(P_k \cap G) : |f(x_k) - h(x_k)| \geq \alpha,$$

and

$$\forall h \in \mathrm{conv}(N_k \cap G) : |f(x_k) - h(x_k)| \geq \alpha.$$

Then we find that

$$
\begin{aligned}
&\delta^{co^*}_{\mathsf{App}}(f,G)\\
&= \sup_{\mathcal{P}\in\mathfrak{P}(G)} \min_{P\in\mathcal{P}} \delta^*_w(f,\mathrm{conv}(P))\\
&= \sup_{\mathcal{P}\in\mathfrak{P}(G)} \min_{P\in\mathcal{P}} \sup_{F\subset B_E \text{finite}} \inf_{g\in\mathrm{conv}(P)} \sup_{x\in F} |f(x)-g(x)|\\
&\geq \min_{k=1}^{n} \inf_{g\in\mathrm{conv}(P_k\cap G)} \sup_{j=1}^{n} |f(x_j)-g(x_j)| \wedge \min_{k=1}^{n} \inf_{g\in\mathrm{conv}(N_k\cap G)} \sup_{j=1}^{n} |f(x_j)-g(x_j)|\\
&\geq \min_{k=1}^{n} \inf_{g\in\mathrm{conv}(P_k\cap G)} |f(x_k)-g(x_k)| \wedge \min_{k=1}^{n} \inf_{g\in\mathrm{conv}(N_k\cap G)} |f(x_k)-g(x_k)| \geq \alpha.
\end{aligned}
$$

Thus we also have that $\delta^*_w \leq \delta^{co^*}_{\mathsf{App}}$ and we are done. □

An immediate consequence is the following.

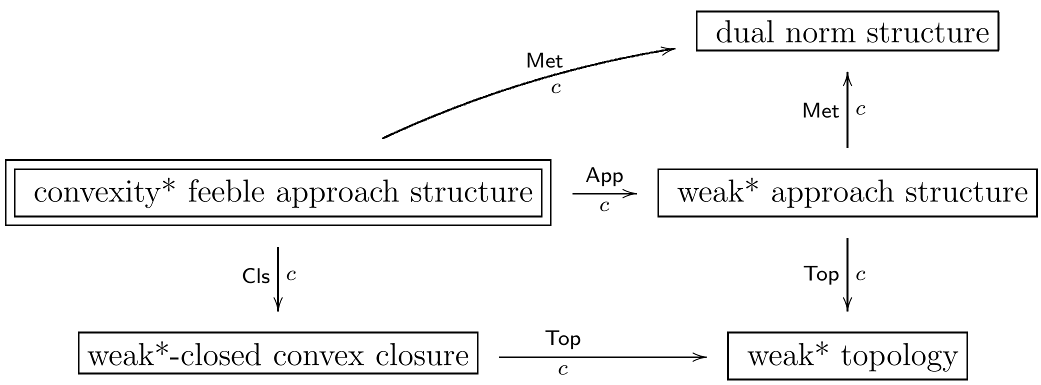
**Corollary 5.14.** *The topological coreflection of the closure structure of weak*-closed convex closure is the weak* topology, in particular this structure is entirely determined by the weak*-closed convex sets by the formula*

$$\mathrm{cl}^*_w(G) = \bigcap_{\mathcal{P}\in\mathbb{P}(G)} \bigcup_{P\in\mathcal{P}} \mathrm{cl}^*_w\mathrm{conv}(P).$$

Note that it is not possible to use a formula for the convexity* feeble distance similar to the one for the convexity feeble distance making use of the dual norm.

**Example 5.15.** If $E$ is a non-reflexive space and $\varphi \in E'' \setminus E$ then $G := \{g \in E' \mid |\varphi(g)| \leq 1\}$ is a closed convex set which is not weak* closed and hence there exists $f \in E'$ such that $0 = \delta^*_w(f,G) < \delta_{\|\,\|}(f,G)$. Since for any $\beta > 0$ we have $\delta^*_w(\beta f, \beta G) = \beta\delta^*_w(f,G) = 0$ and $\delta_{\|\,\|}(\beta f, \beta G) = \beta\delta_{\|\,\|}(f,G) = \beta$ it again follows that we can make the difference between $\delta^*_w$ and $\delta_{\|\,\|}$ arbitrarily large showing not only that they are different but that they are also not Lipschitz-equivalent.

The following diagram also depicts the various structures derived from the convexity* feeble approach structure and their relations. Each arrow again represents coreflection to the type of structure denoted next to it.



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