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Subdivision of Linear Corner Cutting Curves

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Abstract. The most important properties of Bézier and B-spline curves are the convex hull property, the affine invariance, the possibility to subdivide and the variation diminishing property. Therefore it would be of great interest to have a larger class of point controlled curves with the same properties. It is known that all corner cutting curves have the first two properties. In this paper we deal with the subdivision of corner cutting curves, especially of linear corner cutting curves. For uniformly tangent corner cutting curves (a subclass which contains B-spline curves) we present a simple method for computing the control points of the new curves.

Key Words: Corner cutting; Bézier curves; B-spline curves; subdivision.

Preliminaries

Above all, the importance of Bézier and B-spline curves in Computer Aided Geometric Design is based on the following properties.

(a) *Convex hull property*: Each curve is contained in the convex hull of its control points.

(b) *Affine invariance*: One gets the affine image of a curve by using the affine images of its control points.

(c) *Subdivision*: The curve can be subdivided into two curves of the same type.

(d) *Variation diminishing property*: A straight line intersects the curve at most as many times as it intersects its control polygon.

The points of a Bézier curve can be computed by the algorithm of de Casteljau, the points of a B-spline curve by the algorithm of de Boor. Thus a Bézier curve and each restriction of a B-spline curve to an interval limited by two knots is a special *corner cutting curve* (see [1]). This means that there are *control points* $P_0, \ldots, P_N$ and strictly monotone increasing *cutting functions*

$$\alpha_j^i : [a, b] \to [0, 1] \qquad (1 \leq i \leq j \leq N)$$

so that we get each point $X(u)$ of the curve as the terminal point P_N^N of the following *corner cutting algorithm* (see Figure 1).

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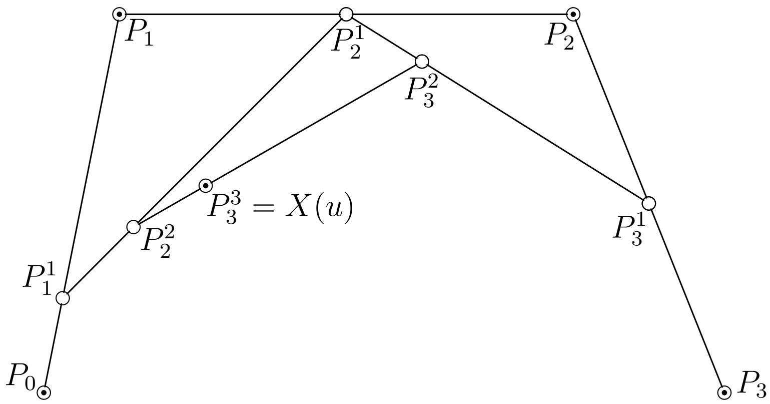

Figure 1: The corner cutting algorithm

$$\begin{aligned} j = 0, \ldots, N: &\quad \mathsf{p}_j^0 := \mathsf{p}_j \\ i = 1, \ldots, N : j = i, \ldots, N: &\quad \mathsf{p}_j^i := (1 - \alpha_j^i(u))\mathsf{p}_{j-1}^{i-1} + \alpha_j^i(u)\mathsf{p}_j^{i-1} \end{aligned} \tag{1}$$

Properties (a) and (b) apply to all corner cutting curves, but generally these curves are not variation diminishing (see [1]). Therefore, we have to search for subsets of corner cutting curves with properties (a)-(d).

We again start with the algorithms of de Casteljau and de Boor. Calling a corner cutting curve *linear* if all cutting functions are polynomials of degree 1, we see that these algorithms generate special linear corner cutting curves. So, the following question arises. Are properties (a)-(d) inherent in all linear corner cutting curves? In this paper we deal with property (c).

Subdivision schemes can be found in several fields of applied mathematics (cf. [15]). In CAGD, subdivision algorithms are used to split point controlled curves or surfaces by generating the control points of the two parts. By this means we get a refinement of the given structure without changing the shape of the curve or surface. On the one hand, this can be used for local modification of curves and surfaces. On the other hand, by repeating the subdivision process a sequence of finer and finer polygonal (when dealing with curves) or polyhedral (when dealing with surfaces) meshes is generated which converges to some limit.

As early as 1984 Boehm, Farin and Kahmann emphasized that subdivision is a central tool in CAGD (see [3]). So, a large number of papers dealt with this topic. Here, only one current example may be given. Subdivision strategies can be used when dealing with the problem of filling an n-sided hole of a surface (see [12] or [13]). The resulting surface should be tangent plane continuous or curvature continuous. The background of the second by far more difficult case was pointed out by Reif in [14]. Also several generalizations of the well-known subdivision algorithms were developed. In [5], Micchelli and Pinkus generalized the scheme known from halving a Bézier curve to define their Matrix Subdivision Scheme. Instead of the usual 2-point subdivision schemes, Dyn, Levin and Gregory studied 4-point schemes (see [4]).

In most papers studying subdivision in CAGD the convergence of the infinite subdivision process is analyzed (cf. [6, 7, 8, 10, 11]. In the present paper we proceed to the following questions. Which corner cutting curves are subdivisible at any point? And: Is there a simple method of computing the new control points? In a sense, this is the counterpart to a method described by Prautzsch in [9]. Prautzsch considers a fixed transformation of control points and searches for the corresponding cutting functions. In the present paper the cutting functions are given and the corresponding transformation of control points is derived.

The paper is organized as follows. In the first section we recapitulate the most important definitions about corner cutting curves. Further we note some results about Bernstein polynomials. Then we study the subdivision of linear corner cutting curves. In Theorem 1 we give a sufficient condition for a linear corner cutting curve to be subdivisible and we show how the control points of the two parts depend on the control points of the entire curve. In section 3 we specialize the obtained results on uniformly tangent corner cutting curves. An example will show the applicability of the proved Theorems. We conclude with some remarks.

1. Introduction

We start with a concise summary of results about corner cutting curves which will be needed in the following sections. Details can be found in [1]. Further, we give two Lemmata about Bernstein polynomials.

1.1. Corner cutting curves

First of all we need a parametric representation of the curve resulting from algorithm (1). To get such a representation we have to accumulate the single steps of this algorithm. Proceeding so and denoting the differentiation with respect to u by a dot we get the following definition of a corner cutting curve (short: cc curve).

Let $P_0, \ldots, P_N$ be points of the Euclidean 3-space E^3 and let α_j^i $(i = 1, \ldots, N; j = i, \ldots, N)$ be C^r-functions $(r \geq 1)$ with

$$\forall u \in [a,b] : \alpha_j^i(u) \in [0,1] \text{ and } \dot{\alpha}_j^i(u) > 0 \ (1 \leq i \leq j \leq N). \tag{2}$$

Further, let the $(N+2-i, N+1-i)$ matrices $A_N^i = A_N^i(u)$ for $i = 1, \ldots, N$ be defined by

$$A_N^i = \begin{pmatrix} 1-\alpha_i^i & 0 & \cdots & 0 \\ \alpha_i^i & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1-\alpha_N^i \\ 0 & \cdots & 0 & \alpha_N^i \end{pmatrix}.$$

Then the C^r-curve

$$c : \mathsf{x}(u) = \begin{pmatrix} \mathsf{p}_0 & \ldots & \mathsf{p}_N \end{pmatrix} \underbrace{A_N^1(u) A_N^2(u) \cdots A_N^N(u)}_{(N+1,1)\text{-matrix}} =: (\mathsf{p}_0 \ldots \mathsf{p}_N) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix}, \ u \in [a,b]$$

is called a *corner cutting curve* or *cc curve* of *degree* N with *cutting functions* α_j^i and *blending functions* f_k. The points $P_0, \ldots, P_N$ are the *control points* of c. If all functions α_j^i are polynomials of degree 1, c is called *linear*.

For $u \in [a,b]$ the matrices $A_N^i = A_N^i(u)$ have nonnegative entries and their columns sum to one. We call such matrices *stochastic*.

Linear cc curves will be the main topic of the next sections. It should be mentioned that each linear cc curve has a "Casteljau-type evaluation algorithm" as defined in [2] by Barry and Goldman.

An interesting subset of linear cc curves is given by the *uniformly tangent* cc curves, which are defined by certain contact conditions (see [1]). Examples of uniformly tangent cc curves are Bézier curves and restrictions of B-spline curves to an interval limited by two knots. A cc curve is uniformly tangent iff

$$A_N^{i-1}(u)\dot{A}_N^i(u) = \dot{A}_N^{i-1}(u)A_N^i(u) \quad (i = 2, \ldots, N). \tag{3}$$

1.2. Something about Bernstein polynomials

The following results about the Bernstein polynomials

$$B_i^N(u) = \binom{N}{i}(1-u)^{N-i}u^i \qquad (i = 0, \ldots, N)$$

will be used in the next sections. The proofs are straightforward.

Lemma 1: *For the Bernstein polynomials the following is true.*

$$\begin{pmatrix} B_0^N(u_0u) \\ \vdots \\ B_N^N(u_0u) \end{pmatrix} = \begin{pmatrix} B_0^0(u_0) & \cdots & B_0^N(u_0) \\ & \ddots & \vdots \\ 0 & & B_N^N(u_0) \end{pmatrix} \begin{pmatrix} B_0^N(u) \\ \vdots \\ B_N^N(u) \end{pmatrix}, \tag{4}$$

$$\begin{pmatrix} B_0^N(u_0(1-u)+u) \\ \vdots \\ B_N^N(u_0(1-u)+u) \end{pmatrix} = \begin{pmatrix} B_0^N(u_0) & & 0 \\ \vdots & \ddots & \\ B_N^N(u_0) & \cdots & B_0^0(u_0) \end{pmatrix} \begin{pmatrix} B_0^N(u) \\ \vdots \\ B_N^N(u) \end{pmatrix}. \tag{5}$$

Lemma 2: *For the Bernstein polynomials it holds*

$$\frac{d^k B_j^N}{du^k}(u) = N(N-1)\cdots(N-k+1)\sum_{\lambda=0}^{k}(-1)^{k-\lambda}\binom{k}{\lambda}B_{j-\lambda}^{N-k}(u) \tag{6}$$

with $B_j^i \equiv 0$ *if* $i < 0$ *or* $j \notin \{0, \ldots, i\}$.

2. Subdivision

In the following, we consider a *linear* cc curve c. Without loss of generality we take the parameter interval $[a, b] = [0, 1]$, i.e. c may have the parametric representation

$$c : \mathsf{x}(u) = \big(\mathsf{p}_0 \ldots \mathsf{p}_N\big)A_N^1(u)\cdots A_N^N(u) = \big(\mathsf{p}_0 \ldots \mathsf{p}_N\big)\begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix}, \ u \in [0,1]\,. \tag{7}$$

As the Bernstein polynomials B_j^N form a basis of the linear space of the polynomials of degree less or equal N, there is a unique representation

$$A_N^1(u)\cdots A_N^N(u) = \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix} = R\begin{pmatrix} B_0^N(u) \\ \vdots \\ B_N^N(u) \end{pmatrix} \tag{8}$$

with a $(N+1, N+1)$ matrix R which does not depend on u.

In section 2 we demand that R is a regular matrix. Now we split the curve c at $X(u_0), u_0 \in]0,1[$ into two curves

$$c_1 : \mathsf{y}(u) = \mathsf{x}(u_0 u) = \big(\mathsf{p}_0 \dots \mathsf{p}_N\big) \begin{pmatrix} f_0(u_0 u) \\ \vdots \\ f_N(u_0 u) \end{pmatrix}, \quad u \in [0,1] \tag{9}$$

and

$$c_2 : \mathsf{z}(u) = \mathsf{x}(u_0(1-u)+u) = \big(\mathsf{p}_0 \dots \mathsf{p}_N\big) \begin{pmatrix} f_0(u_0(1-u)+u) \\ \vdots \\ f_N(u_0(1-u)+u) \end{pmatrix}, \quad u \in [0,1]\,. \tag{10}$$

We are searching for type (7) representations of c_1 and c_2, i.e. we are searching for control points $Q_0, \dots, Q_N$ with

$$c_1 : \mathsf{y}(u) = \big(\mathsf{q}_0 \dots \mathsf{q}_N\big) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix}, \quad u \in [0,1]$$

and for control points $S_0, \dots, S_N$ with

$$c_2 : \mathsf{z}(u) = \big(\mathsf{s}_0 \dots \mathsf{s}_N\big) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix}, \quad u \in [0,1]\,.$$

We start with c_1. For this curve we have

$$\begin{aligned} c_1 : \mathsf{y}(u) &= \mathsf{x}(u_0 u) = \big(\mathsf{p}_0 \dots \mathsf{p}_N\big) \begin{pmatrix} f_0(u_0 u) \\ \vdots \\ f_N(u_0 u) \end{pmatrix} \\ &\overset{(8)}{=} \big(\mathsf{p}_0 \dots \mathsf{p}_N\big) R \begin{pmatrix} B_0^N(u_0 u) \\ \vdots \\ B_N^N(u_0 u) \end{pmatrix} \\ &\overset{(4)}{=} \big(\mathsf{p}_0 \dots \mathsf{p}_N\big) R \begin{pmatrix} B_0^0(u_0) & \cdots & B_0^N(u_0) \\ & \ddots & \vdots \\ 0 & & B_N^N(u_0) \end{pmatrix} \begin{pmatrix} B_0^N(u) \\ \vdots \\ B_N^N(u) \end{pmatrix} \\ &\overset{(8)}{=} \big(\mathsf{p}_0 \dots \mathsf{p}_N\big) R \begin{pmatrix} B_0^0(u_0) & \cdots & B_0^N(u_0) \\ & \ddots & \vdots \\ 0 & & B_N^N(u_0) \end{pmatrix} R^{-1} \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix} \\ &=: \big(\mathsf{p}_0 \dots \mathsf{p}_N\big) T_1(u_0) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix} =: \big(\mathsf{q}_0 \dots \mathsf{q}_N\big) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix}. \end{aligned} \tag{11}$$

For c_2 we get from (5) and (8)

$$\begin{aligned} c_2 : \mathsf{z}(u) &= \mathsf{x}(u_0(1-u)+u) \\ &= (\mathsf{p}_0 \ldots \mathsf{p}_N) R \begin{pmatrix} B_0^N(u_0) & & 0 \\ \vdots & \ddots & \\ B_N^N(u_0) & \cdots & B_0^0(u_0) \end{pmatrix} R^{-1} \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix} \\ &=: (\mathsf{p}_0 \ldots \mathsf{p}_N) T_2(u_0) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix} =: (\mathsf{s}_0 \ldots \mathsf{s}_N) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix}. \end{aligned} \tag{12}$$

We summarize these results in

Theorem 1: *Let*

$$c : \mathsf{x}(u) = (\mathsf{p}_0 \ldots \mathsf{p}_N) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix}, \; u \in [0,1]$$

be a linear cc curve with

$$\begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix} = R \begin{pmatrix} B_0^N(u) \\ \vdots \\ B_N^N(u) \end{pmatrix}, \; \det R \neq 0,$$

and let $u_0 \in]0,1[$. *Then the curve* c_1 *given by* (9) *has the parametric representation*

$$c_1 : \mathsf{y}(u) = (\mathsf{p}_0 \ldots \mathsf{p}_N) T_1(u_0) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix}, \; u \in [0,1]$$

with

$$T_1(u_0) = R \begin{pmatrix} B_0^0(u_0) & \cdots & B_0^N(u_0) \\ & \ddots & \vdots \\ 0 & & B_N^N(u_0) \end{pmatrix} R^{-1} =: R\Delta_1(u_0)R^{-1}.$$

The curve c_2 *given by* (10) *has the parametric representation*

$$c_2 : \mathsf{z}(u) = (\mathsf{p}_0 \ldots \mathsf{p}_N) T_2(u_0) \begin{pmatrix} f_0(u) \\ \vdots \\ f_N(u) \end{pmatrix}, \; u \in [0,1]$$

with

$$T_2(u_0) = R \begin{pmatrix} B_0^N(u_0) & & 0 \\ \vdots & \ddots & \\ B_N^N(u_0) & \cdots & B_0^0(u_0) \end{pmatrix} R^{-1} =: R\Delta_2(u_0)R^{-1}.$$

The presented subdividing method using Bernstein polynomials seems to be a detour. But this method has several advantages:

	R	$\Delta_i(u_0)$
blending functions of c	yes	NO
parameter value u_0	NO	yes

Table 1: Dependences

- The matrices $T_i(u_0)$ $(i = 1, 2)$ supplying the new control points depend on both the blending functions of c and the parameter value u_0 (see also Equation (1.3) in [9]). The described method resolves these dependences as shown in Table 1.
- For a Bézier curve, R is the $(N+1, N+1)$ unit matrix.
- In the case of degree elevation, in $\Delta_i(u_0)$ only one new column has to be computed.
- The matrices $\Delta_i(u_0)$ can be written down immediately. So we know the matrices $T_i(u_0)$, if we know the matrix R which describes the transformation between the basis $\{f_0, \dots, f_N\}$ and the basis $\{B_0^N, \dots, B_N^N\}$.
- If we have c_1, for c_2 no new computation is necessary. To get $\Delta_2(u_0)$, we only have to rearrange the elements of $\Delta_1(u_0)$.

Two questions remain open.

- Which linear cc curves have linearly independent blending functions, i.e. a regular matrix R?
- How can the matrix R be computed in a simple way?

We will answer these questions for uniformly tangent cc curves in the following section.

3. Uniformly tangent cc curves

Let c be a *uniformly tangent* cc curve. Uniformly tangent cc curves are *linear* cc curves (see Theorem 7 of [1]). So, without loss of generality c may be given by (7) and (8). In section 2 we demanded that R is a regular matrix. In this section, i.e. for the subset of uniformly tangent cc curves, we can drop this assumption and *prove* the regularity of R.

At first we prove two simple algorithms for the computation of R. Let r_j be column j of R $(j = 0, \dots, N)$ and let the forward differences Δ^i be given by

$$\Delta^0 \mathsf{r}_j = \mathsf{r}_j \quad \text{and} \quad \Delta^i \mathsf{r}_j = \Delta^{i-1}\mathsf{r}_{j+1} - \Delta^{i-1}\mathsf{r}_j \,.$$

To get shorter formulations we define for $i = 0, \dots, N$

$$C_i(u) := \underbrace{\dot{A}_N^1(u) \cdots \dot{A}_N^i(u)}_{\text{empty if } i=0} \underbrace{A_N^{i+1}(u) \cdots A_N^N(u)}_{\text{empty if } i=N} \,.$$

Then we have

$$C_i(u) \stackrel{(3)}{=} \frac{1}{N(N-1)\cdots(N-i+1)} \frac{d^i C_0}{du^i}(u)$$
$$\stackrel{(8)}{=} \frac{1}{N(N-1)\cdots(N-i+1)} \sum_{j=0}^{N} \mathsf{r}_j \frac{d^i B_j^N}{du^i}(u) \tag{13}$$
$$\stackrel{(6)}{=} \sum_{j=0}^{N} \mathsf{r}_j \sum_{\lambda=0}^{i} (-1)^{i-\lambda} \binom{i}{\lambda} B_{j-\lambda}^{N-i}(u)\,.$$

Using the arguments 0 and 1 we get the following results.

Theorem 2: *Let c be a uniformly tangent cc curve given by (7) and (8). Then for $i = 0, \ldots, N$ it holds*

$$C_i(0) = \dot{A}_N^1(0) \cdots \dot{A}_N^i(0) A_N^{i+1}(0) \cdots A_N^N(0) = \Delta^i \mathsf{r}_0\,.$$

So matrix R can be computed by

$$\mathsf{r}_i = C_i(0) - \sum_{j=0}^{i-1} (-1)^{i-j} \binom{i}{j} \mathsf{r}_j \quad (i = 0, \ldots, N)\,. \tag{14}$$

Proof: For $i = 0$ we get from (8)

$$C_0(0) = \sum_{j=0}^{N} \mathsf{r}_j B_j^N(0) = \mathsf{r}_0\,.$$

For $i \geq 1$ we have

$$C_i(0) \stackrel{(13)}{=} \sum_{j=0}^{N} \mathsf{r}_j \sum_{\lambda=0}^{i} (-1)^{i-\lambda} \binom{i}{\lambda} B_{j-\lambda}^{N-i}(0)$$
$$= \sum_{j=0}^{N} \mathsf{r}_j (-1)^{i-j} \binom{i}{j} B_0^{N-i}(0) = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \mathsf{r}_j = \Delta^i \mathsf{r}_0\,. \qquad \square$$

Theorem 3: *Let c be a uniformly tangent cc curve given by (7) and (8). Then for $i = 0, \ldots, N$ it holds*

$$C_i(1) = \dot{A}_N^1(1) \cdots \dot{A}_N^i(1) A_N^{i+1}(1) \cdots A_N^N(1) = \Delta^i \mathsf{r}_{N-i}\,.$$

So matrix R can be computed by

$$\mathsf{r}_i = (-1)^{N-i} C_{N-i}(1) - \sum_{j=1}^{N-i} (-1)^j \binom{N-i}{j} \mathsf{r}_{i+j} \quad (i = N, \ldots, 0)\,. \tag{15}$$

Proof: For $i = 0$ we get from (8)

$$C_0(1) = \sum_{j=0}^{N} \mathsf{r}_j B_j^N(1) = \mathsf{r}_N\,.$$

For $i \geq 1$ we have

$$\begin{aligned} C_i(1) &\overset{(13)}{=} \sum_{j=0}^{N} \mathsf{r}_j (-1)^{N-j} \binom{i}{i+j-N} B_{N-i}^{N-i}(1) = \sum_{j=N-i}^{N} (-1)^{N-j} \binom{i}{i+j-N} \mathsf{r}_j \\ &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \mathsf{r}_{j+(N-i)} = \Delta^i \mathsf{r}_{N-i} \,. \end{aligned}$$

This gives

$$\mathsf{r}_{N-i} = (-1)^i C_i(1) - \sum_{j=1}^{i} (-1)^j \binom{i}{j} \mathsf{r}_{N-i+j} \,. \tag{16}$$

Replacing i by $N-i$, algorithm (16) becomes (15). □

Now we are able to prove the following important result.

Theorem 4: *The blending functions $f_0, \dots, f_N$ of a uniformly tangent cc curve are linearly independent.*

Proof: We have

$$\begin{aligned} f_0, \dots, f_N \text{ linearly independent} &\iff \text{rank } R = N+1 \\ &\overset{(14)}{\iff} \det \big(C_0(0) \quad C_1(0) \quad \dots \quad C_N(0) \big) \neq 0 \,. \end{aligned}$$

Further, because of rank $\dot{A}_N^j = N+1-j$ the mappings

$$\begin{cases} \mathbb{R}^{N+1-j} & \longrightarrow \quad \mathbb{R}^{N+2-j} \\ \quad \mathsf{x} & \longmapsto \quad \dot{A}_N^j \mathsf{x} \end{cases}$$

are injective. So, the mappings

$$\begin{cases} \mathbb{R}^{N-i+2} & \longrightarrow \quad \mathbb{R}^{N+1} \\ \quad \mathsf{x} & \longmapsto \quad \dot{A}_N^1 \cdots \dot{A}_N^{i-1} \mathsf{x} \end{cases}$$

are injective, too. Thus, for $i = N, \dots, 1$ we can conclude as follows.

$$\begin{aligned} & \dot{A}_N^1 \cdots \dot{A}_N^{i-1} A_N^i(0) \cdots A_N^N(0) \in \Big[\dot{A}_N^1 \cdots \dot{A}_N^i A_N^{i+1}(0) \cdots A_N^N(0), \ \dots, \ \dot{A}_N^1 \cdots \dot{A}_N^N \Big] \\ \iff \ & \exists \lambda_i, \dots, \lambda_N : \dot{A}_N^1 \cdots \dot{A}_N^{i-1} \Big(A_N^i(0) \cdots A_N^N(0) - \sum_{k=i}^{N} \lambda_k \dot{A}_N^i \cdots \dot{A}_N^k A_N^{k+1}(0) \cdots A_N^N(0) \Big) = \mathsf{o} \\ \iff \ & \exists \lambda_i, \dots, \lambda_N : A_N^i(0) \cdots A_N^N(0) = \sum_{k=i}^{N} \lambda_k \dot{A}_N^i \cdots \dot{A}_N^k A_N^{k+1}(0) \cdots A_N^N(0) \,. \end{aligned}$$

But this is impossible, as the elements of the vector on the left sum to 1, while the elements of each vector on the right sum to 0. □

We note some results and questions.

- The elements of each column of R sum to 1. This follows from (14) because of

$$\text{sum of elements of } \; C_i(0) = \delta_{0i}$$

and

$$-\sum_{j=0}^{i-1}(-1)^{i-j}\binom{i}{j} = (-1)^{i+1}\sum_{j=0}^{i-1}(-1)^{j}\binom{i}{j} = (-1)^{i+1}(-1)^{i+1} = 1\,.$$

Is R a stochastic matrix? The matrix R^{-1} is generally not stochastic (see the following example).

- We know

$$\mathsf{r}_0 = A_N^1(0)\cdots A_N^N(0)\,,\ \mathsf{r}_N = A_N^1(1)\cdots A_N^N(1)\,.$$

- (11) shows

$$\begin{aligned} 1 = \sum_{i=0}^{N} f_i(u_0 u) \;&=\; f_0(u)\times\Big(\text{ sum of the elements of column 1 of } T_1(u_0)\Big) + \cdots + \\ & \quad f_N(u)\times\Big(\text{ sum of the elements of column } N+1 \text{ of } T_1(u_0)\Big)\,. \end{aligned}$$

So, because of $\sum_{i=0}^{N} f_i(u) \equiv 1$ and because of the uniqueness of the basis representation, the elements of each column of $T_1(u_0)$ (and of $T_2(u_0)$) sum to 1. But the matrices $T_i(u_0)$ are generally not stochastic (see the following example).

4. Example

For $[a,b] = [0,1]$ the most general uniformly tangent cc curve of degree 2 is given by

$$\begin{aligned} \alpha_2^2(u) &= \lambda u + \mu \quad (0 < \lambda \le \lambda + \mu \le 1)\,, \\ \alpha_1^1(u) &= d\alpha_2^2(u) + (1-d) \quad (0 < d \le \frac{1}{1-\mu})\,, \\ \alpha_2^1(u) &= c\alpha_2^2(u) \quad (0 < c \le \frac{1}{\lambda+\mu}) \end{aligned}$$

(see [1]). In this case we have

$$\begin{aligned} C_0(0) &= \begin{pmatrix} d(1-\mu)^2 \\ 1 - d(1-\mu)^2 - c\mu^2 \\ c\mu^2 \end{pmatrix}\,, \\ C_1(0) &= \lambda \begin{pmatrix} -d(1-\mu) \\ d(1-\mu) - c\mu \\ c\mu \end{pmatrix}\,, \\ C_2(0) &= \lambda^2 \begin{pmatrix} d \\ -c-d \\ c \end{pmatrix}\,. \end{aligned}$$

So, with $\nu := \lambda + \mu$ we get from (14)

$$R = \begin{pmatrix} d(1-\mu)^2 & d(1-\mu)(1-\nu) & d(1-\nu)^2 \\ 1-d(1-\mu)^2 - c\mu^2 & 1-d(1-\mu)(1-\nu) - c\mu\nu & 1-d(1-\nu)^2 - c\nu^2 \\ c\mu^2 & c\mu\nu & c\nu^2 \end{pmatrix}$$

and from this

$$R^{-1} = \frac{1}{cd\lambda^2} \cdot \begin{pmatrix} c\nu\big(1-d(1-\nu)\big) & -cd\nu(1-\nu) & d(1-\nu)(1-c\nu) \\ cd(\mu+\nu-2\mu\nu) - c(\mu+\nu) & cd(\mu+\nu-2\mu\nu) & cd(\mu+\nu-2\mu\nu) - d(2-\mu-\nu) \\ c\mu\big(1-d(1-\mu)\big) & -cd\mu(1-\mu) & d(1-\mu)(1-c\mu) \end{pmatrix}.$$

For $\lambda = 1$ and $\mu = 0$ (which gives $\nu = 1$) these equations become

$$R = \begin{pmatrix} d & 0 & 0 \\ 1-d & 1 & 1-c \\ 0 & 0 & c \end{pmatrix}, \quad R^{-1} = \frac{1}{cd} \begin{pmatrix} c & 0 & 0 \\ c(d-1) & cd & (c-1)d \\ 0 & 0 & d \end{pmatrix}$$

$(0 < c, d \leq 1)$. In this case we get

$$T_1(u_0) = \frac{1}{c} \begin{pmatrix} c\big(u_0 + d(1-u_0)\big) & cd(1-u_0) & d(1-u_0)(c-u_0) \\ c(1-u_0)(1-d) & c\big(1-d(1-u_0)\big) & (1-u_0)\big(u_0(c+d) + c(1-d)\big) \\ 0 & 0 & cu_0^2 \end{pmatrix} \tag{17}$$

and

$$T_2(u_0) = \frac{1}{d} \begin{pmatrix} d(1-u_0)^2 & 0 & 0 \\ u_0\big((1-u_0)(c+d) + (1-c)d\big) & d(1-cu_0) & du_0(1-c) \\ cu_0(u_0+d-1) & cdu_0 & d\big(1-u_0(1-c)\big) \end{pmatrix}. \tag{18}$$

Figures 2 and 3 show the resulting curves and control points for different parameters. There we use the notation of (11) and (12). From (17) and (18) we get

- $Q_0, Q_1 \in \overline{P_0P_1}$,
- $S_1, S_2 \in \overline{P_1P_2}$,
- $Q_2 \in \operatorname{conv}\{P_0, P_1, P_2\} \iff u_0 \in [0, c]$,
- $S_0 \in \operatorname{conv}\{P_0, P_1, P_2\} \iff u_0 \in [1-d, 1]$.

5. Conclusion

We have shown that the blending functions of each uniformly tangent cc curve are linearly independent. So, uniformly tangent cc curves have property (c) of the beginning of the paper. In a forthcoming paper it will be shown that these curves also have property (d). Altogether, the uniformly tangent cc curves form a subset of corner cutting curves which have - as Bézier and B-spline curves - all properties (a)-(d).

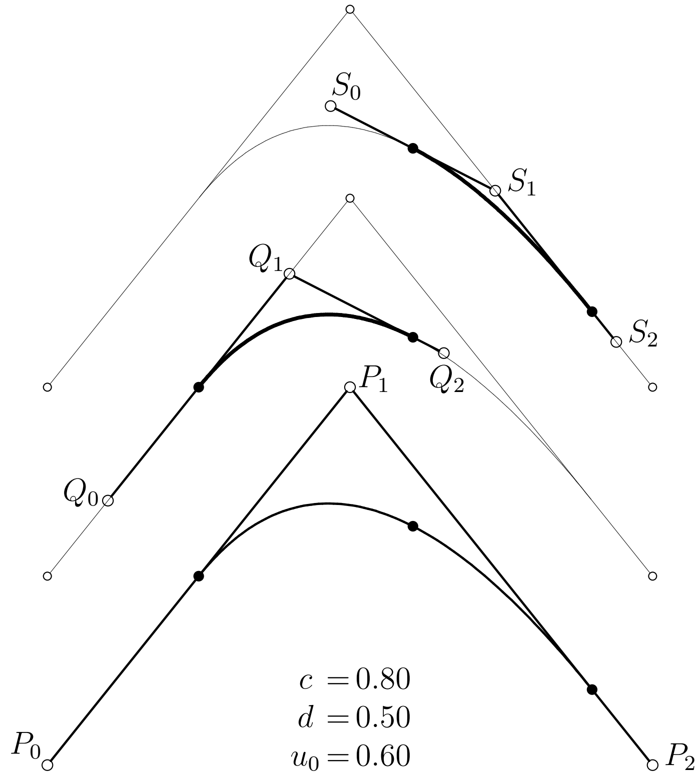

Figure 2: Subdivision of a uniformly tangent cc curve - Example 1

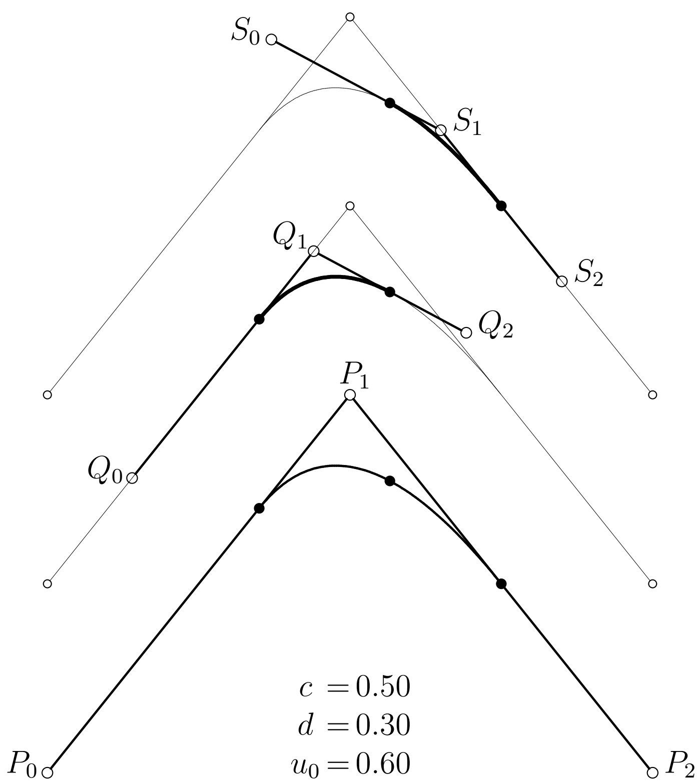

Figure 3: Subdivision of a uniformly tangent cc curve - Example 2

I conjecture that each linear corner cutting curve has these properties. The following examples will show that the cutting functions α_j^i of a linear cc curve have to meet both assumptions

$$\forall u \in [a,b] : \alpha_j^i(u) \in [0,1] \tag{19}$$

and

$$\dot{\alpha}_j^i(u) > 0 \tag{20}$$

($1 \le i \le j \le N$; see (2)) to get linearly independent blending functions f_k. Let $[a,b] = [0,1]$ and let c be a linear cc curve of degree 2 with cutting functions $\alpha_1^1(u) = \lambda u + \mu$ and $\alpha_2^1(u) = \alpha_2^2(u) = u$. Then the blending functions of c are linearly dependent iff $\mu = \lambda + 1$. In Figure 4 (a) where (20) is not true we have $\lambda = -0.1$ and $\mu = 0.9$. In Figure 4 (b) where (19) is not true we have $\lambda = 0.1$ and $\mu = 1.1$. Further, we *cannot* conclude as follows: Let $a < b$ and let

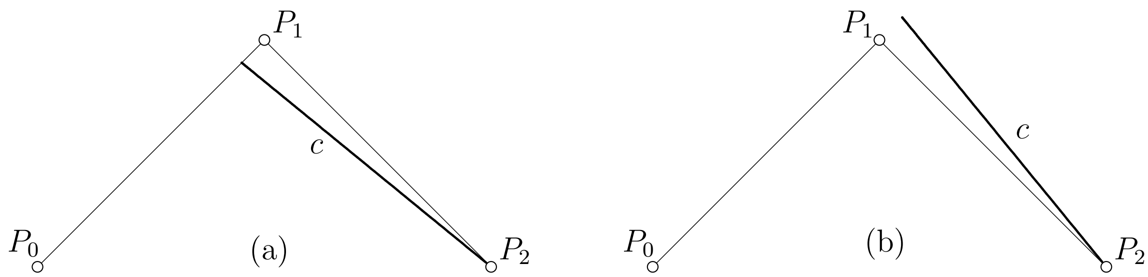

Figure 4: Counterexamples

$f_0, \ldots, f_N$ be polynomials of degree N with the property that f_i has i zeros $\le a$ and $N - i$ zeros $\ge b$. Then $f_0, \ldots, f_N$ are linearly independent. We give the following counterexample. For $[a,b] = [0,1]$ and

$$\begin{aligned} f_0(x) &= (x-3)(x-6)\,, \\ f_1(x) &= (x-1)(x+2)\,, \\ f_2(x) &= x^2 \end{aligned}$$

it holds $f_0 + 9f_1 - 10f_2 = \Omega$.

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