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Basis of Quartic Splines over Triangulation

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Abstract. The modeling of complex shapes usually requires a well-based space of splines. The aim of this work is to give the construction method of such spline space basis over the chosen class of triangulations. This basis has several useful properties — local minimal support, low degree of polynomials. We also present several problems, that arise in lower-degree polynomials.

Key Words: geometric modeling, splines, triangulation, spline basis

MSC 1994: 53A05, 68U05.

1. Introduction

For several years geometric modeling has been in the phase of finding common mathematical background for its constructions. The construction of smooth surfaces over the given triangulation also plays an important role here. Not long ago even the dimension of the space of all piecewise polynomials over given triangulation was not known. Up to now it has only been known for low-degree continuity. For the notion of continuity see [6], [5] or [4].

There are a lot of constructions using higher degree of polynomial pieces. They, however, usually build just one surface or just subspace of spline space over the given triangulation and have a large number of parameters which, on the one hand, provides the tool for controlling the spline, on the other hand is sometimes unpredictable (and usually without geometric interpretation), which requires some experience on the part of the user.

This is the main reason why we seek for some basis. It systematically produces all possible splines by choosing appropriate control points. Moreover, we can precompute basis functions at chosen sample points of the domain and substantially speed up the construction of the final surface.

In recent years not very much successful work has been done. One construction method of local minimal support basis is described in [3]. This, however, works only for the polynomials of at least 5th degree. Also the author is nor very clear about how the control points influence the function. General theory with main results can be found in [5], chapter 8.

We would like to present the construction of the basis for degree four piecewise polynomial surface over a triangulation that is general enough. The splines will be C^1 continuous over this

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domain. The basis for higher-degree piecewise polynomials can be constructed analogously, however, over arbitrary triangulation.

In Section 2 we provide a brief outline of the notions used in geometric modeling and the geometric interpretation of several conditions used later. Section 3 contains the dimension of such space over constrained triangulation and Sections 4 and 5 the construction of the basis. Its independence is proven in Section 6. Finally, the minimality of basis functions' support is considered in Section 7. Several conclusion remarks are put in Section 8.

2. Basic notions

2.1. Description of the domain

Here we would like to give a brief overview of the used notions and concepts. In the following we will construct bivariate spline maps. These are piecewise polynomial maps from a subset Ω of two-dimensional manifold into R. Ω will be divided into finite number of triangles and each triangle will be associated with a polynomial map.

The domain Ω of bivariate spline functions can be a bounded polygon, a bounded polygon with polygonal holes or it can be considered closed without any border (connected like polyhedron or polyhedral tore etc.).

The denotation of the following definition is valid for the whole work.

Definition 2.1 *The domain Ω is a subset of 2-manifold. The triangulation $\mathcal{T}$ of Ω is a set of all vertices, edges and triangles of the triangulated domain. V is the number of vertices, E is the number of edges and F is the number of triangles in the triangulation $\mathcal{T}$. Further notation is:*

N — *number of boundaries (holes),*
V_B — *number of vertices of all boundaries,*
V_I — *number of interior vertices,*
E_B — *number of edges from all boundaries,*
E_I — *number of interior edges.*

Throughout the work we will consider only regular triangulations, where all triangles are proper, i.e. the vertices of each triangle are non-collinear. Moreover, each boundary vertex belongs to the only boundary.

Lemma 2.2 *Let Ω be the triangulated ($\mathcal{T}$) domain of genus 0 with N boundaries (holes) and let V_B denote the number of vertices at all boundaries. Then*

$$\begin{aligned} E &= 3V - V_B + 3(N-2); \\ F &= 2V - V_B + 2(N-2) \end{aligned}$$

Note 2.3: The formulas of Lemma 2.2 could be simply generalised for domains of arbitrary genus. But using such general domain, all further considerations stay the same, only the mathematical formulae are slightly more complicated. Therefore, we restrict our work to the domains of genus 0.

Definition 2.4 *Let Ω be the domain with the triangulation $\mathcal{T}$, n is the degree.*

$$P_n := \text{span}\ \{x^i y^j; i \geq 0, j \geq 0, i + j \leq n\}$$

is the space of bivariate polynomials. Let q be the order of the continuity, $0 \leq q < n$ and $C^q(\Omega)$ be the space of all functions with all partial derivatives up to q continuous.

$$S_n^q(\mathcal{T}) := \{F \in C^q(\Omega); F|_\triangle \in P_n \ \forall \triangle \in \mathcal{T}\}$$

is the space of C^q continuous bivariate polynomial spline functions of degree n defined over Ω.

It is obvious that $S_n^q(\mathcal{T})$ is the linear space. We will treat with Bézier representation of splines. The element of the space is piecewise polynomial function which is composed of triangular Bézier patches; for each triangle $\triangle \in \mathcal{T}$ there is a Bézier patch defined.

2.2. Elements of geometric modeling

First, we would like to give a brief overview of barycentric calculus in plane.

Let $A_0 = [x_0, y_0]$, $A_1 = [x_1, y_1]$, $A_2 = [x_2, y_2]$ be three non-collinear points in plane τ. Then for each $A \in \tau$, $A = [x, y]$ there exist unique three numbers λ_0, λ_1, λ_2 such that

$$\lambda_0 + \lambda_1 + \lambda_2 = 1 \quad \text{and} \quad A = \lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2.$$

We will also write that

$$A = (\lambda_0, \lambda_1, \lambda_2) \text{ w.r.t. } \triangle A_0 A_1 A_2.$$

If we assume that indices will continue cyclically, each λ_i can be computed as follows

$$\lambda_i = \frac{\det(A_{i-1} A A_{i+1})}{\det(A_0 A_1 A_2)},$$

where

$$\det(A_{i-1} A A_{i+1}) = \begin{vmatrix} x_{i-1} & y_{i-1} & 1 \\ x & y & 1 \\ x_{i+1} & y_{i+1} & 1 \end{vmatrix} \quad \text{and} \quad \det(A_0 A_1 A_2) = \begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}.$$

This notion can be easily used in any Euclidean finite dimensional space.

Further, we would like to recall the notion of ratio of three collinear points. Let A, B, C be three collinear points such that $C \neq B$. Then formally we define (ABC) — the desired ratio — as

$$(ABC) = \frac{C - A}{C - B}$$

The famous Ceva theorem deals with ratio. It states the following:

Theorem 2.5 (Ceva) *Let A, B, C be three non-collinear points and A', B', C' be three points on lines BC, CA, AB, respectively and $\{A, B, C\} \cap \{A', B', C'\} = \emptyset$. Then lines AA', BB' and CC' meet in the common point if and only if $(ABC')(BCA')(CAB') = -1$.*

The proof is elementary and can be found in many books concerning plane geometry or affine spaces.

Now, let us to express Ceva theorem in barycentric calculus. Let $\mathrm{d}_{or}(A, B, C)$ be the function of oriented distance of point A to line BC and $\mathrm{d}_{or}(A, B, C) > 0$ if and only if $\triangle ABC$ is oriented counter-clockwise. Let the points be assigned according to Fig. 1 left.

Let $\tilde{t}_i = (\lambda_{i0}, \lambda_{i1}, \lambda_{i2})$, $i = 0,1,2$ be a point in barycentric coordinates according to $\triangle t_0t_1t_2$. Further, let $t_it_j\tilde{t}_i$; $i,j = 0,1,2$, $i \neq j$ be non-collinear to have the presumptions of Ceva theorem satisfied. Then we can write for $\tilde{t}_i$

$$\frac{\lambda_{i,i+1}}{\lambda_{i,i-1}} = \frac{\det(\tilde{t}_i t_{i-1} t_i)}{\det(t_{i+1}\tilde{t}_i t_i)} = \frac{\mathrm{d}_{or}(\tilde{t}_i t_{i-1} t_i)}{\mathrm{d}_{or}(t_{i+1}\tilde{t}_i t_i)} = -\frac{(\bar{t}_i - t_{i-1})}{(\bar{t}_i - t_{i+1})} = -(t_{i-1}t_{i+1}\bar{t}_i).$$

Thus, in Ceva theorem the ratio condition can be rewritten as

$$\frac{\lambda_{0,1}}{\lambda_{0,2}}\frac{\lambda_{1,2}}{\lambda_{1,0}}\frac{\lambda_{2,0}}{\lambda_{2,1}} = 1. \tag{1}$$

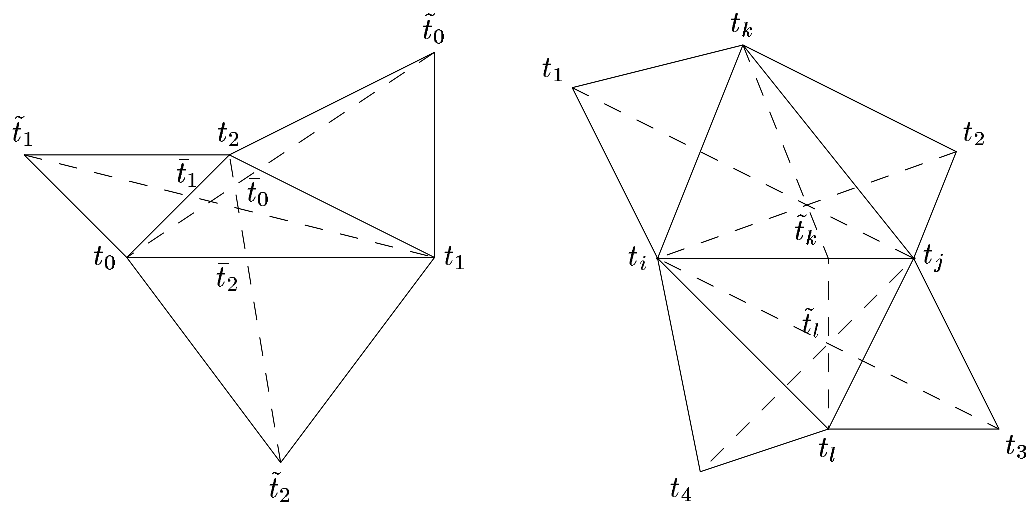

Figure 1: The first (left) and the second (right) Ceva condition. While the triangle on the left does not satisfy the first Ceva condition, on the right the second Ceva condition is satisfied.

Definition 2.6 *We will say that $\triangle t_0t_1t_2 \in \mathcal{T}$ satisfies the first Ceva condition if and only if there exist 3 neighbouring triangles in $\mathcal{T}$: $\triangle\tilde{t}_0t_1t_2$, $\triangle t_0\tilde{t}_1t_2$, $\triangle t_0t_1\tilde{t}_2$ with $t_it_j\tilde{t}_i$ non-collinear for all $i \neq j$, $i,j = 0,1,2$ and the lines $t_0\tilde{t}_0$, $t_1\tilde{t}_2$, $t_2\tilde{t}_2$ coinciding in common point. It is equivalent to algebraic condition (1).*

Definition 2.7 *We will say that two neighbouring triangles $\triangle t_it_jt_k$, $\triangle t_it_jt_l \in \mathcal{T}$ satisfy the second Ceva condition if and only if there exist 4 neighbouring triangles in $\mathcal{T}$ (see Fig. 1 right): $\triangle t_it_kt_1$, $\triangle t_jt_kt_2$, $\triangle t_jt_lt_3$, $\triangle t_it_lt_4$ such that triples of points $t_it_jt_n$; $n = 1,2,3,4$, $t_jt_kt_1$, $t_it_kt_2$, $t_it_lt_3$, $t_jt_lt_4$ are non-collinear and for which the following holds: if $\tilde{t}_k$ is the intersection of t_it_2, t_jt_1 and $\tilde{t}_l$ is the intersection of t_it_3, t_jt_4, then lines $t_k\tilde{t}_k$, $t_l\tilde{t}_l$, t_it_j coincide in common point.*

The second Ceva condition can also be expressed in barycentric calculus, as is presented in the following theorem.

Theorem 2.8 *Let $\triangle t_it_jt_k$, $\triangle t_it_jt_l$, t_1, t_2, t_3, t_4 be as in Definition 2.7. Let $(\lambda_{m0}, \lambda_{m1}, \lambda_{m2})$, $m = 1,2$ are the barycentric coordinates of t_m w.r.t. $\triangle t_it_jt_k$ and $(\lambda_{m0}, \lambda_{m1}, \lambda_{m2})$, $m = 3,4$ are the barycentric coordinates of t_m w.r.t. $\triangle t_it_jt_l$. Then $\triangle t_it_jt_k$, $\triangle t_it_jt_l$ satisfy the second Ceva condition if and only if*

$$\frac{\lambda_{12}}{\lambda_{10}}\frac{\lambda_{21}}{\lambda_{22}} = \frac{\lambda_{42}}{\lambda_{40}}\frac{\lambda_{31}}{\lambda_{32}}. \tag{2}$$

Proof: Let (2) hold. Let $\bar{t}_k, \bar{t}_l \in t_i t_j$ be such points that lines $t_i t_2$, $t_j t_1$, $t_k \bar{t}_k$ coincide in a common point $\tilde{t}_k$ and similarly $t_i t_3$, $t_j t_4$, $t_l \bar{t}_l$ coincide in $\tilde{t}_l$. Let $\bar{t}_k = \lambda_{k0} t_i + \lambda_{k1} t_j$ and $\bar{t}_l = \lambda_{l0} t_i + \lambda_{l1} t_j$. Then from (1)

$$\frac{\lambda_{12}}{\lambda_{10}} \frac{\lambda_{21}}{\lambda_{22}} = \frac{\lambda_{k1}}{\lambda_{k0}} \quad \text{and} \quad \frac{\lambda_{42}}{\lambda_{40}} \frac{\lambda_{31}}{\lambda_{32}} = \frac{\lambda_{l1}}{\lambda_{l0}}$$

and from (2)

$$\frac{\lambda_{k1}}{\lambda_{k0}} = \frac{\lambda_{l1}}{\lambda_{l0}}.$$

Since $\lambda_{k0} + \lambda_{k1} = \lambda_{l0} + \lambda_{l1} = 1$, we have $\bar{t}_k = \bar{t}_l$ and therefore $\triangle t_i t_j t_k$, $\triangle t_i t_j t_l$ satisfy the second Ceva condition.

Now, let the second Ceva condition be satisfied. Let $t \in t_i t_j$, $t = \lambda_0 t_i + \lambda_1 t_j$ be such point that

$$\frac{\lambda_{12}}{\lambda_{10}} \frac{\lambda_{21}}{\lambda_{22}} \frac{\lambda_0}{\lambda_1} = 1 \quad \text{and} \quad \frac{\lambda_{42}}{\lambda_{40}} \frac{\lambda_{31}}{\lambda_{32}} \frac{\lambda_0}{\lambda_1} = 1,$$

which directly implies (2). □

3. Constraints

As mentioned above, we will focus on $S_4^1(\mathcal{T})$. We require the triangulation $\mathcal{T}$ to satisfy three following conditions:

1. Let $[t_i, t_j]$ be an interior edge and the triangles $\triangle t_i t_j t_k \in \mathcal{T}$ and $\triangle t_i t_j t_l \in \mathcal{T}$. We require that the points t_k, t_i, t_l are non-collinear and also the points t_k, t_j, t_l are non-collinear (see Fig. 2).
2. No triangle of $\mathcal{T}$ satisfies the first Ceva condition.
3. No two neighbouring triangles of $\mathcal{T}$ satisfy the second Ceva condition.

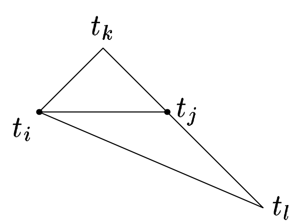

Figure 2: Unacceptable situation of two neighbouring triangles (constraint 1).

Note 3.1: Let us have a brief look at barycentric coordinates in $\mathcal{T}$. Let $\triangle t_0 t_1 t_2, \triangle \tilde{t}_0 t_1 t_2 \in \mathcal{T}$ have a common edge $[t_1 t_2]$. Let $\lambda_0, \lambda_1, \lambda_2$ be barycentric coordinates of $\tilde{t}_0$ w.r.t. $\triangle t_0 t_1 t_2$:

$$\tilde{t}_0 = \lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2.$$

Then

$$\lambda_0 \neq 0, \quad \lambda_1 \neq 0, \quad \lambda_2 \neq 0.$$

Actually, if $\lambda_0 = 0$, then $\tilde{t}_0$ lies on the line determined by $[t_1t_2]$ and $\triangle\tilde{t}_0t_1t_2$ is not a proper triangle. If $\lambda_1 = 0$, then $\tilde{t}_0, t_1, t_2$ are collinear and the situation does not respect the first constraint put on the triangulation. The case $\lambda_2 = 0$ is similar to $\lambda_1 = 0$.

Theorem 3.2 *The dimension of $S_4^1(\mathcal{T})$ satisfies*

$$\dim(S_4^1(\mathcal{T})) = 6V + 3(N-2) \tag{3}$$

For the proof see [5], or [7].

Definition 3.3 *Let $dim = \dim(S_4^1(\mathcal{T}))$. Let $E_i : \Omega \to R;\ i = 1, \ldots, dim$ is a basis of $S_4^1(\mathcal{T})$. Each $F \in S_4^1(\mathcal{T})$ can be uniquely expressed as a linear combination of the basis elements:*

$$F = d_1E_1 + d_2E_2 + \ldots + d_{dim}E_{dim}.$$

Points $d_i \in R;\ i = 1, \ldots, dim$ are called control points of F w.r.t. the basis $E_1, \ldots, E_{dim}$.

For each E_j from the basis of $S_4^1(\mathcal{T})$ we will try to get as small support as possible. The smaller the support of E_j, the "more local" alteration of the function $F = \sum d_iE_i$ when the control point d_j is moved.

Now, we are looking for the basis of $S_4^1(\mathcal{T})$. We will proceed as follows:

First we will ask if $\Omega_j \subset \Omega$ is a suitable support for a basis function E_j. It has to be large enough to be able to define E_j over it. For instance, E_j can not be defined such that its support consists of the only triangle whose all 3 vertices are interior w.r.t. Ω, because the $E_j(u) = 0\ \forall u \in \Omega$; thus E_j is not basis function.

In the second step, when the suitable support Ω_j for the basis function is found, we will define E_j such that $\operatorname{supp} E_j = \Omega_j$.

4. Basic basis functions

In this section we formulate the construction of basis functions.

Let $\Omega_j \subset \Omega$ and $\mathcal{T}_j \subset \mathcal{T}$ be the corresponding triangulation of Ω_j. Then $V^j, V_B^j, V_I^j, \ldots$ denote the number of vertices from $\mathcal{T}_j$, boundary (w.r.t. $\mathcal{T}_j$) vertices from $\mathcal{T}_j$, interior vertices from $\mathcal{T}_j$ and so on. Let Ω_j have just one boundary.

Further, let $S_4^1(\mathcal{T}_j)$ be defined similarly to $S_4^1(\mathcal{T})$:

$$S_4^1(\mathcal{T}_j) := \{F \in C^1(\Omega_j); F|_\triangle \in P_n\ \forall \triangle \in \mathcal{T}_j\}$$

Now, let us look at a special subspace of $S_4^1(\mathcal{T}_j)$.

Definition 4.1 *Let $\mathcal{T}_j$ be the triangulation of Ω_j. Then let $\partial\Omega_j$ be the boundary of Ω_j. $\tilde{S}_4^1(\mathcal{T}_j)$ is such subspace of $S_4^1(\mathcal{T}_j)$ that the following conditions hold for each $F \in \tilde{S}_4^1(\mathcal{T}_j)$ and $u \in \partial\Omega_j$:*

$$\begin{aligned} F(u) &= 0 \\ D_eF(u) &= 0 \quad \textit{for each vector } e, \end{aligned}$$

where D_eF is the directional derivative of F in the direction of vector e.

Lemma 4.2 *Let $\mathcal{T}_j \subset \mathcal{T}$ be the triangulation of $\Omega_j \subset \Omega$. If $\dim(\tilde{S}_4^1(\mathcal{T}_j)) > 0$, then there exists a non-zero function $E_j \in S_4^1(\mathcal{T})$ such that $\operatorname{supp} E_j = \Omega_j$ and, moreover, $E_j|_{\Omega_j} \in \tilde{S}_4^1(\mathcal{T}_j)$.*

Proof: Let $\dim(\tilde{S}_4^1(\mathcal{T}_j)) > 0$. Then there exists a non-zero $E'_j \in \tilde{S}_4^1(\mathcal{T}_j)$. Let us define E_j as follows:

$$E_j(u) = \begin{cases} E'_j(u) & \text{if } u \in \Omega_j \\ 0 & \text{otherwise.} \end{cases}$$

Since for each $u \in \partial\Omega_j$ it holds that $E'_j(u) = 0$ and $D_e E'_j(u) = 0$ for each e, the function E_j is C^1 continuous and thus $E_j \in S_4^1(\mathcal{T})$. The properties supp $E_j = \Omega_j$ and $E_j|_{\Omega_j} = E'_j$ follow from the definition of E_j. □

Note 4.3: The reverse implication of the previous lemma does not hold if $\partial\Omega_j \cap \partial\Omega \neq \emptyset$. In this case all functions from $\tilde{S}_4^1(\mathcal{T}_j)$ have to be zero on the common part of the boundary and all its directional derivatives have to be zero here as well. But for the functions from $S_4^1(\mathcal{T})$ this is not required.

To avoid awkward situations occurring on the boundary of Ω, first we will restrict our consideration to the domains with no boundary. Boundary conditions will be added later.

Definition 4.4 *Let t be the vertex from $\mathcal{T}$. Then the star of the vertex t is the subspace $\Omega(t) \subset \Omega$, which is defined as follows:*

$$\begin{aligned} \mathcal{T}(t) = \{t\} \cup \{t_i;\ [t,t_i] \in \mathcal{T}\} \cup \\ \{[t,t_i];\ [t,t_i] \in \mathcal{T}\} \cup \{[t_i,t_j];\ [t,t_i] \in \mathcal{T}, [t,t_j] \in \mathcal{T}\} \cup \\ \{\triangle t t_i t_j;\ \triangle t t_i t_j \in \mathcal{T}\} \end{aligned}$$

is the triangulation of the star of t and

$$\Omega(t) = \bigcup P; P \in \mathcal{T}(t).$$

Lemma 4.5 *Let Ω_j be the star of the interior vertex $t_j \in \mathcal{T}$. Then*

$$\dim(\tilde{S}_4^1(\mathcal{T}_j)) = 3.$$

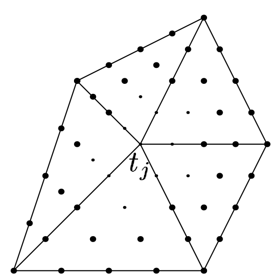

Figure 3: The support of $F \in \tilde{S}_4^1(\mathcal{T}_j)$. The zero Bézier control points are sketched as black circles.

Proof: According to (3), $\dim(S_4^1(\mathcal{T}_j)) = 6V^j - 3 = 6(1+m_j) = 6m_j + 3$, where m_j is $\deg t_j$. The dimension of $\tilde{S}_4^1(\mathcal{T}_j)$ is smaller, because of the constraints put on the boundary of $\tilde{S}_4^1(\mathcal{T}_j)$.

For $F \in \tilde{S}_4^1(\mathcal{T}_j)$ it holds that $F(u) = 0;\ u \in \partial\Omega_j$. It implies that all Bézier control points of F defined over the boundary of Ω_j are zero. Moreover, since $D_e F(u) = 0$ for each $u \in \partial\Omega_j$, also the second line of control points next to the boundary contains only zero Bézier control points (see Fig. 3). The dimension of $\tilde{S}_4^1(\mathcal{T}_j)$ can be expressed as a number of Bézier control points whose values can be arbitrarily chosen if the mentioned points are zero.

To construct a function from $S_4^1(\mathcal{T}_j)$ we need to set

$$\dim(S_4^1(\mathcal{T}_j)) = 6m_j + 3$$

suitable values of its Bézier control points, the others are determined by C^1 continuity. But since $F \in \tilde{S}_4^1(\mathcal{T}_j)$, the properties of this subspace have caused some of their values to be zero. Let us count them.

For each vertex from the boundary of Ω_j let us consider four Bézier control points: one defined exactly over it and three defined over the neighbouring abscissae lying in the incident edges from $\mathcal{T}_j$. Let us choose three of them which are non-collinear and set them to zero. Then, because of the C^1 continuity, also the fourth one is zero. We have set $3m_j$ control points.

For each boundary edge from $\mathcal{T}_j$ (boundary w.r.t. Ω_j) let us set the rest of the control points defined over it to zero. We have set m_j control points.

For each boundary edge from $\mathcal{T}_j$ there are two Bézier control points of the neighbouring line that have to be set to zero. Doing it we have set $2m_j$ control points.

Together we have set $6m_j$ Bézier control points. So for each $F \in \tilde{S}_4^1(\mathcal{T}_j)$ there are only $\dim(S_4^1(\mathcal{T}_j)) - 6m_j = 3$ control points left to be arbitrarily set and thus $\dim(\tilde{S}_4^1(\mathcal{T}_j)) = 3$. □

According to Lemma 4.5 we know that the star of the interior vertex is large enough to construct up to three linearly independent functions such that Ω_j is their support.

Let $[t_i t_j]$ be an interior edge from $\mathcal{T}$. Denote by $a_0, \ldots, a_4$ abscissae on $[t_i t_j]$. Let $b_0, \ldots, b_4$ be corresponding Bézier control points. If E is a function whose support is the star of t_i, then $b_2 = b_3 = b_4 = 0$. If E is a function whose support is the star of t_j, then $b_0 = b_1 = b_2 = 0$. Otherwise, if the support of E is the star of any other vertex from $\mathcal{T}$, $b_0 = b_1 = \ldots = b_4 = 0$. We see that $b_2 = 0$ for each interior edge. So, we are looking for such functions that have a non-zero control point assigned as b_2.

Definition 4.6 *Let $[t_i, t_j]$ be the edge of $\mathcal{T}$. Then the star of the edge $[t_i, t_j]$ is the subspace $\Omega([t_i, t_j]) \subset \Omega$ which is defined as follows:*

$$\begin{aligned}
\mathcal{T}([t_i, t_j]) = \{ & t_i, t_j, [t_i, t_j]\} \cup \\
& \{t_k, [t_i, t_k], [t_j, t_k], \triangle t_i t_j t_k;\ \triangle t_i t_j t_k \in \mathcal{T}\} \cup \\
& \{t_l, [t_i, t_l], [t_k, t_l], \triangle t_i t_k t_l;\ \triangle t_i t_j t_k \in \mathcal{T}, \triangle t_i t_k t_l \in \mathcal{T}\} \cup \\
& \{t_m, [k_j, t_m], [t_k, t_m], \triangle t_j t_k t_m;\ \triangle t_i t_j t_k \in \mathcal{T}, \triangle t_j t_k t_m \in \mathcal{T}\} \\
\Omega([t_i, t_j]) = \bigcup & P; P \in \mathcal{T}([t_i, t_j]).
\end{aligned}$$

Let t_i, t_j be interior vertices. Note that if $\deg t_i \geq 5$ and $\deg t_j \geq 5$, then the star of $[t_i, t_j]$ contains no interior vertex with respect to this star. But when $\deg t_i$ or $\deg t_j$ or both of them are at most 4, then the star of $[t_i, t_j]$ may contain one or two such interior vertices (see Fig. 4).

Lemma 4.7 *Let t_i, t_j be the interior vertices, $[t_i, t_j] \in \mathcal{T}$. Let $\Omega_{ij} \subset \Omega$ be the star of $[t_i, t_j]$. Then $\dim(\tilde{S}_4^1(\mathcal{T}_{ij})) \geq 1$ and, moreover, there exists $F \in \tilde{S}_4^1(\mathcal{T}_{ij})$ such that its Bézier control point over the midpoint of $[t_i, t_j]$ is non-zero.*

Proof: Let abscissae of $F \in \tilde{S}_4^1(\mathcal{T}_{ij})$ be denoted as in Fig. 5. Let b_i be Bézier control point over a_i for $i = 1, \ldots, 11$. Because of the conditions put on $\tilde{S}_4^1(\mathcal{T}_{ij})$, only $b_1, \ldots, b_{11}$ can have non-zero value, other control points are zero. Because of C^1 continuity there are just 10

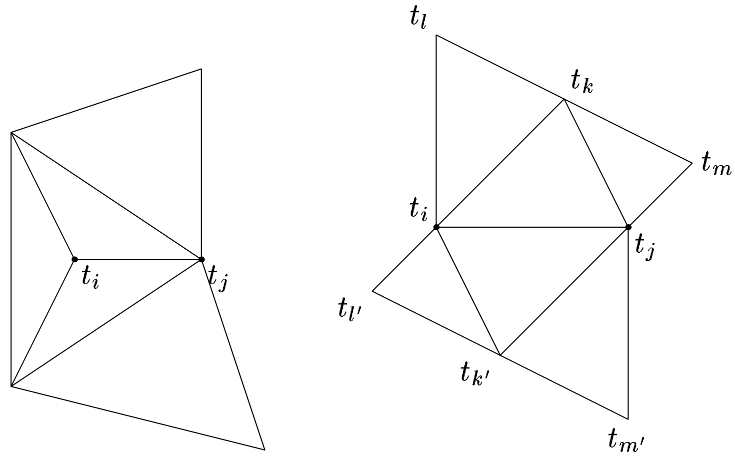

Figure 4: The star of the edge $[t_i t_j]$ — two different shapes: the left one contains an interior vertex while the right one does not.

conditions put on $b_1, \dots, b_{11}$ (they are sketched in Fig. 5). It holds that $\dim(\tilde{S}_4^1(\mathcal{T}_{ij}))$ is equal to the dimension of the solution space of the mentioned system of 10 equations and therefore, $\dim(\tilde{S}_4^1(\mathcal{T}_{ij})) \geq 1$. Now we only need to find out if there exists $F \in \tilde{S}_4^1(\mathcal{T}_{ij})$ such that its Bézier control point b_{11} is non-zero.

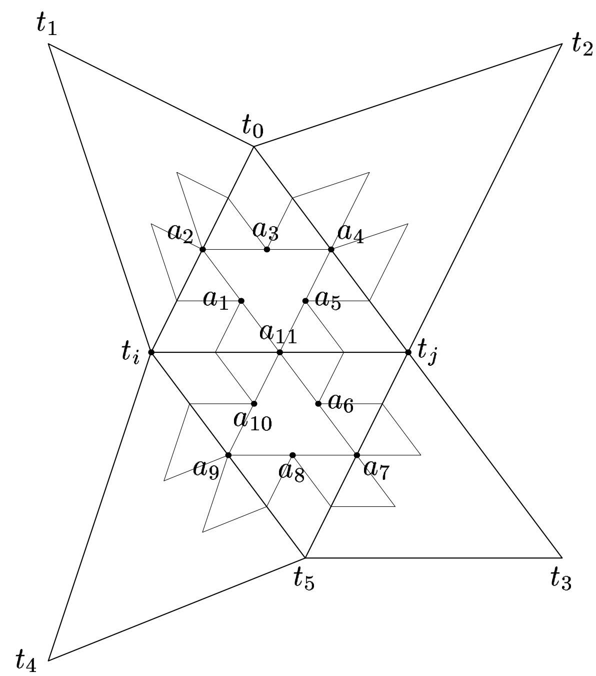

Figure 5: The star of the edge $[t_i t_j]$; a_i is abscissa of Bézier control point b_i.

First we need to assign the barycentric coordinates of several points:

$$
\begin{aligned}
t_1 &= (\lambda_{10}, \lambda_{11}, \lambda_{12}) \text{ w.r.t. } \triangle t_i t_j t_0,\\
t_2 &= (\lambda_{20}, \lambda_{21}, \lambda_{22}) \text{ w.r.t. } \triangle t_i t_j t_0,\\
t_3 &= (\lambda_{30}, \lambda_{31}, \lambda_{32}) \text{ w.r.t. } \triangle t_i t_j t_5,\\
t_4 &= (\lambda_{40}, \lambda_{41}, \lambda_{42}) \text{ w.r.t. } \triangle t_i t_j t_5,\\
t_5 &= (\lambda_{50}, \lambda_{51}, \lambda_{52}) \text{ w.r.t. } \triangle t_i t_j t_0.
\end{aligned}
$$

The solution space S of the system of 10 equations can be expressed as an intersection $S = S_1 \cap S_2 \cap S_3$, where

S_1 is a solution space of 4 equations put on $b_1, \dots, b_5$,
S_2 is a solution space of 4 equations put on $b_6, \dots, b_{10}$,
S_3 is a solution space of 2 equations treating with b_{11}.

Let us find S_1:

$$\begin{aligned} \lambda_{11}b_1+\lambda_{12}b_2 &= 0\\ \lambda_{10}b_2+\lambda_{11}b_3 &= 0\\ \lambda_{20}b_3+\lambda_{21}b_4 &= 0\\ \lambda_{22}b_4+\lambda_{20}b_5 &= 0 \end{aligned}$$

Let b_3 be chosen arbitrarily. Then

$$b_2 = -\tfrac{\lambda_{11}}{\lambda_{10}}b_3, \quad b_1 = \tfrac{\lambda_{12}}{\lambda_{10}}b_3, \quad b_4 = -\tfrac{\lambda_{20}}{\lambda_{21}}b_3, \quad b_5 = \tfrac{\lambda_{22}}{\lambda_{21}}b_3.$$

All coefficients are non-zero (see Note 3.1 after the constraints put on the triangulation) and therefore they can occur in the denominator.

Similarly we find S_2. After choosing b_8 arbitrary

$$b_7 = -\tfrac{\lambda_{30}}{\lambda_{31}}b_8, \quad b_6 = \tfrac{\lambda_{32}}{\lambda_{31}}b_8, \quad b_9 = -\tfrac{\lambda_{41}}{\lambda_{40}}b_8, \quad b_{10} = \tfrac{\lambda_{42}}{\lambda_{40}}b_8.$$

Let us find S_3 and then $S_1 \cap S_2 \cap S_3$:

$$b_{10} = \lambda_{52}b_1 + \lambda_{51}b_{11}, \qquad b_6 = \lambda_{52}b_5 + \lambda_{50}b_{11}$$

implies

$$(\lambda_{50}\frac{\lambda_{31}}{\lambda_{32}} - \lambda_{51}\frac{\lambda_{40}}{\lambda_{42}})b_{11} \quad = \quad \lambda_{52}(\frac{\lambda_{12}\lambda_{40}}{\lambda_{10}\lambda_{42}} - \frac{\lambda_{22}\lambda_{31}}{\lambda_{21}\lambda_{32}})b_3.$$

Since

$$\lambda_{50}\frac{\lambda_{31}}{\lambda_{32}} \neq \lambda_{51}\frac{\lambda_{40}}{\lambda_{42}} \quad \text{and} \quad \frac{\lambda_{12}\lambda_{40}}{\lambda_{10}\lambda_{42}} \neq \frac{\lambda_{22}\lambda_{31}}{\lambda_{21}\lambda_{32}}$$

(see the second and the third constraint), a solution with non-zero b_{11} surely exists. □

The proof of Lemma 4.7 is stronger than its assertion. It also implies the following properties:

Corollary 4.8 *Let F be the function satisfying the conditions of Lemma 4.7 and $b_1, \dots, b_{11}$ be its Bézier control points (assigned as in the proof). Then*

1. $b_i \neq 0;\ i = 1, \dots, 11,$
2. *The values of $b_1, \dots, b_{11}$ can be obtained by setting the value of their arbitrary point and calculating all the others according to C^1 continuity conditions.*

5. Boundary basis functions

As noted after Lemma 4.2, to find a basis function of $S_4^1(\mathcal{T})$ which is non-zero somewhere on the boundary of Ω, we must use a slightly different approach.

Definition 5.1 *Let $\mathcal{T}$ be the triangulation of the given domain Ω. Then the extended triangulation $\mathcal{T}'$ is defined as follows:*

$$\begin{aligned} \mathcal{T}_1 &= \mathcal{T} \cup \{t_{ij}, [t_i, t_{ij}], [t_j, t_{ij}] \triangle t_i t_j t_{ij}; \\ &\quad t_{ij} \text{ is exterior to } \Omega \text{ and not at } t_i t_j \text{ line}; [t_i, t_j] \in \mathcal{T} \text{ is boundary edge w.r.t. } \mathcal{T}\} \\ \mathcal{T}' &= \mathcal{T}_1 \cup \{[t_{ij}, t_{jk}], \triangle t_j t_{ij} t_{jk}; \\ &\quad t_j \in \mathcal{T} \text{ is boundary vertex w.r.t. } \mathcal{T} \text{ and } \deg t_j \geq 5 \text{ w.r.t. } \mathcal{T}_1\} \cup \\ &\quad \{t_{j0}, [t_j, t_{j0}], [t_{ij}, t_{j0}], [t_{j0}, t_{jk}], \triangle t_j t_{ij} t_{j0}, \triangle t_j t_{j0} t_{jk}; \\ &\quad t_j \in \mathcal{T} \text{ is boundary w.r.t. } \mathcal{T} \text{ and } \deg t_j = 4 \text{ w.r.t. } \mathcal{T}_1\} \end{aligned}$$

All added objects are non-degenerated and the objects of the same dimension have non-intersecting relative interiors. The constraints put on the triangulation $\mathcal{T}$ are valid also for $\mathcal{T}'$. Then $\Omega' = \bigcup P; P \in \mathcal{T}'$ is called the extended domain.

Definition 5.2 *Let Ω_j be the star of the boundary vertex $t_j \in \Omega$. Let Ω'_j be the star of the vertex t_j in the extended domain Ω', let $\mathcal{T}'_j$ be the corresponding triangulation of Ω'_j. Then*

$$\bar{S}^1_4(\mathcal{T}_j) = \{F \in S^1_4(\mathcal{T}_j); F = F'|_{\Omega_j} \text{ where } F' \in \tilde{S}^1_4(\mathcal{T}'_j)\}.$$

Lemma 5.3 *Let $\Omega_j \subset \Omega$ be the star of the boundary vertex $t_j \in \mathcal{T}$. Then $\dim(\bar{S}^1_4(\mathcal{T}_j)) = 4$.*

Proof: It is very similar to the proof of Lemma 4.5. □

Definition 5.4 *Let Ω_{ij} be the star of the edge $[t_i, t_j] \in \Omega$, where t_i or t_j or both of them lie on the boundary. Let Ω'_{ij} be the star of the edge $[t_i, t_j]$ in the extended domain Ω'. Then*

$$\bar{S}^1_4(\mathcal{T}_{ij}) = \{F \in S^1_4(\mathcal{T}_{ij}); F = F'|_{\Omega_{ij}} \text{ where } F' \in \tilde{S}^1_4(\mathcal{T}'_{ij})\}.$$

Lemma 5.5 *Let $\Omega_{ij} \subset \Omega$ be the star of $[t_i, t_j] \in \mathcal{T}$, where t_i or t_j or both of them are the boundary vertices (w.r.t. $\mathcal{T}$). Then $\dim(\bar{S}^1_4(\mathcal{T}_{ij})) \geq 1$ and, moreover, there exists $F \in \bar{S}^1_4(\mathcal{T}_{ij})$ such that its Bézier control point over the midpoint of $[t_i, t_j]$ is non-zero.*

Proof: This lemma is a direct corollary of Lemma 4.7. □

6. Independence of functions

This section deals with the independence of the above constructed functions. Actually, we prove that they form a basis of $S^1_4(\mathcal{T})$.

Definition 6.1 *Let Ω_j be the star of $t_j \in \mathcal{T}$. If t_j is the interior (boundary) vertex, then the triplet (quadruplet) of linearly independent functions $E_{ji} \in S^1_4(\mathcal{T})$; $i = 1, \ldots, 3(4)$ such that*

1. supp $E_{ji} = \Omega_j$
2. $E_{ji}|_{\Omega_j} \in \tilde{S}^1_4(\mathcal{T}_j)$ $(\bar{S}^1_4(\mathcal{T}_j))$

is called the basis of the star of t_j.

Let $\mathcal{B}_1 \subset S^1_4(\mathcal{T})$, $\mathcal{B}_1$ contains $3V_I + 4V_B = 3V + V_B$ functions: for each vertex t_j there is a basis of its star.

Lemma 6.2 *$\mathcal{B}_1$ is a linearly independent set of functions.*

Proof: The functions from $\mathcal{B}_1$ are linearly independent if and only if their control nets are linearly independent. The second assertion is true (the considerations are very similar to those you will find in more detail in the proof of Lemma 6.4). □

Definition 6.3 *Let $t_i, t_j \in \mathcal{T}$, $[t_i, t_j] \in \mathcal{T}$ and let Ω_{ij} is the star of $[t_i, t_j]$. Let $E_{ij} \in S_4^1(\mathcal{T})$ be such function that*

1. supp $E_{ij} = \Omega_{ij}$
2. $E_{ij}|_{\Omega_{ij}} \in \tilde{S}_4^1(\mathcal{T}_{ij})(\bar{S}_4^1(\mathcal{T}_{ij})$ *in case t_i or t_j is boundary)*
3. *Bézier control point over the medium of $[t_i, t_j]$ is non-zero.*
4. $\forall t \in \mathcal{T} : E_{ij}(t) = 0$ *and* $D_e E_{ij}(t) = 0$ *for each e.*

Then E_{ij} is called the basis function over the edge $[t_i, t_j]$.

The definition of the basis function over the edge is correct. Each edge possesses such function. Really, according to Lemma 4.7 and Lemma 5.5 it is always possible to find E'_{ij} satisfying the first three conditions from the definition:

If $\deg t_i \geq 5$ and $\deg t_j \geq 5$, then supp E'_{ij} has no local interior edge and thus $E_{ij} = E'_{ij}$ is the basis function over $[t_i, t_j]$ satisfying also the fourth condition of the definition.

If $\deg t_i \leq 4$ then t_i is the interior vertex of supp E'_{ij}. Let E_{i0}, E_{i1}, E_{i2} be a basis of the star of t_i. Let $E_i = d_0 E_{i0} + d_1 E_{i1} + d_2 E_{i2}$ be such function that $E_i(t_i) = E'_{ij}(t_i)$ and $D_e E_i(t_i) = D_e E'_{ij}(t_i)$ $\forall e$. According to the definition of the basis of the star it is always possible to find such function. Then $E_{ij} = E'_{ij} - E_i$ is the desired basis function over the edge $[t_i, t_j]$.

The case of $\deg t_j \leq 4$ is similar.

Let $\mathcal{B}_2$ be such subset of $S_4^1(\mathcal{T})$ that for each edge $[t_i, t_j]$ there exists one of basis functions over $[t_i, t_j]$ in $\mathcal{B}_2$.

Lemma 6.4 *$\mathcal{B}_2$ is a linearly independent set of functions.*

Proof: If $\mathcal{B}_2$ is not a linearly independent set, then $\exists F_i \in \mathcal{B}_2$ such that it is possible to omit it, i.e., the space generated by $\mathcal{B}_2$ and the space generated by $\mathcal{B}_2 \setminus \{F_i\}$ are the same. We will show that no function can be omitted and so $\mathcal{B}_2$ is a linearly independent set.

Let $\triangle t_0 t_1 t_2 \in \mathcal{T}$. We will treat $\triangle t_0 t_1 t_2$ and the neighbouring triangles. The whole situation is illustrated in Fig. 6. Let F_0 be the basis function over the edge $[t_1, t_2]$, F_1 over $[t_2, t_0]$ and F_2 over $[t_0, t_1]$. Further, let $\tilde{t}_i = \lambda_{i0} t_0 + \lambda_{i1} t_1 + \lambda_{i2} t_2$; $i = 0, 1, 2$.

Let $F = \sum d_k F_k$; $F_k \in \mathcal{B}_2$. The same equations hold for control nets of F, F_k.

Let b_1, b_3, b_5 be Bézier control points of F over abscissae a_1, a_3, a_5 (see Fig. 6). Let b_1^i, b_3^i, b_5^i be Bézier control points of F_i over abscissae a_1, a_3, a_5, $i = 0, 1, 2$. The control points b_1, b_3, b_5 of F are affected only by corresponding points of F_0, F_1 and F_2, i.e.

$$b_j = d_0 b_j^0 + d_1 b_j^1 + d_2 b_j^2; \quad j = 1, 3, 5.$$

We will prove that for each triplet of values b_1, b_3, b_5 $\exists F; F = d_0 F_0 + d_1 F_1 + d_2 F_2$ such that b_1, b_3, b_5 are the values of Bézier control points over a_1, a_3, a_5 and therefore no function of F_0, F_1, F_2 can be omitted.

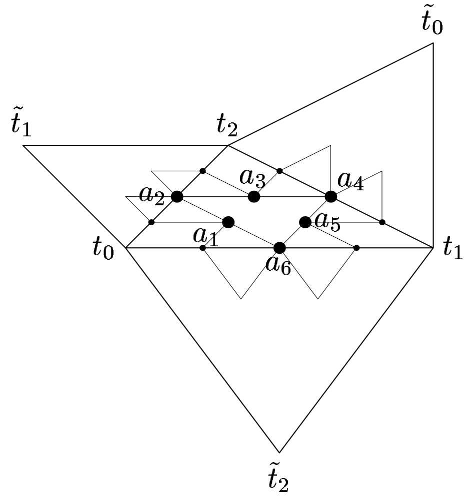

Figure 6: Triangle $t_0t_1t_2$ and its neighbours; a_i is the abscissa of Bézier control point b_i.

C^1 continuity makes Bézier control points of F_0 to satisfy:

$$\begin{aligned}0 &= \lambda_{20}b_6^0 + \lambda_{21}0 + \lambda_{22}b_5^0\\ 0 &= \lambda_{20}0 + \lambda_{21}b_6^0 + \lambda_{22}b_1^0\\ 0 &= \lambda_{10}0 + \lambda_{11}b_1^0 + \lambda_{12}b_2^0\\ 0 &= \lambda_{10}b_2^0 + \lambda_{11}b_3^0 + \lambda_{12}0\end{aligned}$$

Let b_1^0 be the control point of all non-zero ones of F_0, which is arbitrarily chosen, others are determined by C^1 continuity (see Corollary 4.8):

$$b_3^0 = \frac{\lambda_{10}}{\lambda_{12}}b_1^0, \quad b_5^0 = \frac{\lambda_{20}}{\lambda_{21}}b_1^0.$$

Similar situation holds also for the functions F_1, F_2. Let b_5^1 be the arbitrarily chosen control point of F_1. Then

$$b_1^1 = \frac{\lambda_{21}}{\lambda_{20}}b_5^1, \quad b_3^1 = \frac{\lambda_{01}}{\lambda_{02}}b_5^1.$$

And for F_2 let b_3^2 be arbitrarily chosen. Then

$$b_5^2 = \frac{\lambda_{02}}{\lambda_{01}}b_3^2, \quad b_1^2 = \frac{\lambda_{12}}{\lambda_{10}}b_3^2.$$

The conditions $b_j = d_0b_j^0 + d_1b_j^1 + d_2b_j^2;\ j = 1, 3, 5$ give a system of three equations with d_0, d_1 and d_2 as unknowns:

$$\begin{aligned}d_0b_1^0 + d_1\frac{\lambda_{21}}{\lambda_{20}}b_5^1 + d_2\frac{\lambda_{12}}{\lambda_{10}}b_3^2 &= b_1\\ d_0\frac{\lambda_{10}}{\lambda_{12}}b_1^0 + d_1\frac{\lambda_{01}}{\lambda_{02}}b_5^1 + d_2b_3^2 &= b_3\\ d_0\frac{\lambda_{20}}{\lambda_{21}}b_1^0 + d_1b_5^1 + d_2\frac{\lambda_{02}}{\lambda_{01}}b_3^2 &= b_5.\end{aligned}$$

The determinant of the system is

$$D = b_1^0 b_5^1 b_3^2 (2 - \lambda - \frac{1}{\lambda}), \qquad \text{where } \lambda = \frac{\lambda_{01}}{\lambda_{02}} \frac{\lambda_{12}}{\lambda_{10}} \frac{\lambda_{20}}{\lambda_{21}}.$$

$$D = 0 \quad \text{iff} \quad \lambda - 2 + \frac{1}{\lambda} = 0,$$

but this means that $\lambda = 1$ which, according to (1), is impossible. □

Lemma 6.5 *$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a linearly independent set of functions.*

Proof: span $\mathcal{B}_1 \cap$ span $\mathcal{B}_2$ contains only zero function. Really, all functions in $\mathcal{B}_1$ have zero Bézier control point over the midpoint of each edge in its domain. On the other hand every non-zero function in $\mathcal{B}_2$ has at least one of these points non-zero, thus it cannot belong to span $\mathcal{B}_1$. Similarly all functions in $\mathcal{B}_2$ have zero Bézier point and zero directional derivatives associated with each vertex of the domain unlike the non-zero functions in $\mathcal{B}_1$. □

Theorem 6.6 *$\mathcal{B}$ is a basis of $S_4^1(\mathcal{T})$.*

Proof: From the previous lemma, $\mathcal{B}$ is linearly independent. For the cardinality of $\mathcal{B}$ holds

$$|\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2| = 3V + V_B + E = \dim(S_4^1(\mathcal{T})).$$

□

7. Minimality

Definition 7.1 *The subset Ω_j of Ω is called minimal support if $\exists F_j \in S_4^1(\mathcal{T})$ such that* supp $F_j = \Omega_j$ *and for each non-zero $F \in S_4^1(\mathcal{T})$ such that* supp $F \subset \Omega_j$ *holds that* supp $F = \Omega_j$. *We will say that the function F_j has minimal support.*

In the final part of the paper we would like to say a few words about the minimality of our basis support. Again, we will return to the domain with no boundary. Moreover, for all $t \in \mathcal{T}$ we require that $\deg t \geq 5$.

Lemma 7.2 *Let Ω_j be the star of interior vertex $t_j \in \mathcal{T}$. Then Ω_j is the minimal support.*

Proof: Let $E \in S_4^1(\mathcal{T})$; supp $E \subset \Omega_j, E|_{\Omega_j} \in \tilde{S}_4^1(\mathcal{T}_j)$. Let $\triangle_i = \triangle t_j t_i t_{i+1}$ not belong to the support of E_j, i.e. $E_j(t) = 0$ for $t \in \triangle_i$. Then all Bézier control points over $\triangle_i$ are zero. Then, because of C^1 continuity along the edge $[t_j t_i]$ also all control points over the triangle $\triangle_{i+1}$ of the line next to the edge $[t_j t_{i+1}]$ are zero. Also the control points over the boundary of Ω_j and of the neighbouring line are zero. The remaining control points over the center of $[t_j t_{i+2}]$ are zero because of C^1 continuity along this edge. So $\triangle_{i+1}$ does not belong to the support of E either, and so on. Therefore, E is zero function. □

Lemma 7.3 *Let $t_0, t_1, t_2 \in \mathcal{T}$ are interior of at least 5th degree. Let $\triangle t_0 t_1 t_2 \in \mathcal{T}$. Let Ω_{012} consist of triangles $\triangle t_0 t_1 t_2$, $\triangle \tilde{t}_0 t_1 t_2$, $\triangle t_0 \tilde{t}_1 t_2$, $\triangle t_0 t_1 \tilde{t}_2$. Then* $\dim(\tilde{S}_4^1(\mathcal{T}_{012})) = 0$.

Proof: Let us try to find a function F from $S_4^1(\mathcal{T})$ such that supp $F = \Omega_{012}$ and $F|_{\Omega_{012}} \in \tilde{S}_4^1(\mathcal{T}_{012})$.

Because of the condition put on $\tilde{S}_4^1(\mathcal{T}_{012})$, two rows of Bézier control points along $\partial\Omega_{012}$ contain only zero points. There are only six points left: $b_1, \ldots, b_6$ such that we do not know whether they can be non-zero (see Fig. 6).

Let $\tilde{t_i} = \lambda_{i0} t_0 + \lambda_{i1} t_1 + \lambda_{i2} t_2$. Because of C^1 continuity along the edge $[t_2, t_0]$ the points b_1, b_2, b_3 satisfy:

$$\begin{aligned} 0 &= \lambda_{10} 0 + \lambda_{11} b_1 + \lambda_{12} b_2 \\ 0 &= \lambda_{10} b_2 + \lambda_{11} b_3 + \lambda_{12} 0 \end{aligned}$$

and so on. The matrix of this set of equations is as follows:

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{10} & \lambda_{11} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{00} & \lambda_{01} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{02} & \lambda_{00} & 0 \\ 0 & 0 & 0 & 0 & \lambda_{22} & \lambda_{20} \\ \lambda_{22} & 0 & 0 & 0 & 0 & \lambda_{21} \end{pmatrix}.$$

The matrix is singular if and only if

$$\frac{\lambda_{12}}{\lambda_{10}} \frac{\lambda_{01}}{\lambda_{02}} \frac{\lambda_{20}}{\lambda_{21}} = 1.$$

But the last equation is never valid according to our assumption (see the second constraint put on the triangulation). □

Lemma 7.4 *Let t_i, t_j be interior vertices from $\mathcal{T}$, $[t_i t_j] \in \mathcal{T}$ of at least 5th degree. Let $\Omega_{ij} \subset \Omega$ be the star of $[t_i t_j]$. Then Ω_{ij} is the minimal support for such functions whose Bézier control point over the midpoint of $[t_i, t_j]$ is non-zero.*

Proof: Let the vertices of the star of $[t_i, t_j]$ be denoted as in Fig. 5. Let $F \in S_4^1(\mathcal{T})$ be such function that supp $F \subset \Omega_{ij}$, supp $F \neq \Omega_{ij}$ and $F|_{\Omega_{ij}} \in \tilde{S}_4^1(\mathcal{T}_{ij})$. First, let $\triangle t_i t_j t_0$ not belong to the support of F. Then all Bézier control points over $\triangle t_i t_j t_0$ are zero. We can simply check that then, because of C^1 continuity, all control points over all the remaining triangles from $\mathcal{T}_{ij}$ are zero and so F is a zero function.

Now, let $\triangle t_i t_0 t_1$ not belong to supp F. Then due to C^1 continuity $\triangle t_j t_2 t_0$ does not belong to supp F either. What remains is the triangle $\triangle t_j t_i t_5$ with 3 neighbouring triangles. But according to Lemma 7.3, the domain is too small to be a support for such function and so F is zero. □

8. Conclusion and future work

We have presented several facts about spline spaces and the dimension of the fourth degree piecewise polynomial space over the given triangulation, which we would like to extend to a wider class of triangulations containing also more specific relations of triangles. This is directly connected to the construction of basis over such triangulation.

We also try to construct a basis for lower degree polynomials. This seems to be a tougher task. The question also remains open about higher continuity spline space over the given triangulation. The first step would be to find the dimension of such space.

References

[1] H.P. Seidel: *Polar forms and triangular B-spline Surfaces*. Computing in Euclidean Geometry 1992, 235–286.

[2] P. Kashyap: *Geometric interpretation of continuity over triangular domains*. Comp. Aided Geom. Design **15** (1998), 773–786.

[3] O. Davidov: *Locally lineary independent basis for C^1 bivariate splines of degree $q \geq 5$*. in: Mathematical Methods for Curves and Surfaces II, M. Dæhlen, T. Lyche, and L. L. Schumaker (eds.), Vanderbilt University Press, Nashville, 1998. pp. 71–78

[4] J. Hoschek, D. Lasser: *Fundamentals of Computer Aided Geometric Design*. A. K. Peters, 1993.

[5] D. Cox, J. Little, D. O'Shea: *Using Algebraic Geometry*. Springer-Verlag, New York, 1998.

[6] M. P. do Carmo: *Differentialgeometrie von Kurven und Flächen*. Vieweg, Wiesbaden, 1992.

[7] J. Pílniková: *Basis of Quartic Splines over Triangulation*. Diploma Thesis, MFFUK Bratislava, 1999.

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