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On Instantaneous Rectilinear Congruences

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Abstract. Two rectilinear congruences are introduced and investigated, which are generated by the instantaneous axes of revolution of the moving Darboux frame along the two families of lines of curvature on a regular non-spherical and non-developable surface.

Key Words: Line congruence, instantaneous axis, lines of curvature
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1. Introduction

The ambient space is the Euclidean space $\mathbb{E}^3$. For our work we used [1, 3, 7] as general references. Let the vector $\mathbf{r} = \mathbf{r}(u, v)$ represent a regular non-spherical and non-developable surface S, and suppose that the u- and v-lines of this parametrization are lines of curvature, i.e., the elements g_{12} and h_{12} of the first and second fundamental forms vanish identically ($g_{12} = h_{12} = 0$).

A rectilinear congruence in the Euclidean space $\mathbb{E}^3$ is represented by the vector equation

$$\mathbf{R}(u, v, \lambda) = \mathbf{r}(u, v) + \lambda\mathbf{d}(u, v),\ \ \lambda \in \mathbb{R}, \tag{1}$$

where $\mathbf{r} = \mathbf{r}(u, v)$ is its base surface (the surface of reference) and $\mathbf{d} = \mathbf{d}(u, v)$ is the unit vector giving the direction of the straight lines of the congruence.

Consider now the unit vectors $\mathbf{e}_1 = \mathbf{e}_1(u, v)$, $\mathbf{e}_2 = \mathbf{e}_2(u, v)$ of the tangents of the parametric curves $v =$ const., $u =$ const. and the unit vector $\mathbf{e}_3 = \mathbf{e}_3(u, v)$ of the normal to the surface S at any regular point, then we have

$$\mathbf{e}_1 = \frac{\mathbf{r}_u}{\sqrt{g_{11}}}, \quad \mathbf{e}_2 = \frac{\mathbf{r}_v}{\sqrt{g_{22}}}, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2\,, \tag{2}$$

which are invariant functions on the surface and $\times$ is the usual vectorial product in $\mathbb{E}^3$.

Using that u, v are lines of curvature on the surface S, we now calculate $ds_1 = \sqrt{g_{11}}\,du$ and $ds_2 = \sqrt{g_{22}}\,dv$, the arc length parameters of the curves $v =$ const., $u =$ const., respectively. The moving frame $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ on the surface S at every regular point is then called the

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Darboux frame. Hence by means of the derivatives with respect to the arc length parameter of the curve $v=$ const., with tangent $\mathbf{e}_1$ on S, the derivative formula with respect to the Darboux frame may be stated as

$$\frac{\partial}{\partial s_1}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}=\begin{pmatrix}0 & q & k\\ -q & 0 & 0\\ -k & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}, \tag{3}$$

where

$$k=\frac{h_{11}}{g_{11}},\quad q=-\frac{(g_{11})_v}{2g_{11}\sqrt{g_{22}}}$$

are the normal and geodesic curvatures of the curves $v=$ const., respectively. Similarly, the derivative formula of the Darboux frame of the curves $u=$ const., with tangent $\mathbf{e}_2$ on the surface S is

$$\frac{\partial}{\partial s_2}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}=\begin{pmatrix}0 & q^* & 0\\ -q^* & 0 & k^*\\ 0 & -k^* & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}, \tag{4}$$

where

$$k^*=\frac{h_{22}}{g_{22}},\quad q^*=\frac{(g_{22})_u}{2g_{22}\sqrt{g_{11}}}$$

have the same meaning as in (3) for the curves $u=$ const. on the surface. Here, g_{ij} and h_{ij} are the coefficients of the first and second fundamental forms of the surface S. We shall denote $\partial/\partial s_1$ and $\partial/\partial s_2$ by the suffixes 1 and 2.

Since k, k^*, q, q^* are the invariant quantities of the lines of curvature on the surface, these invariants and their derivatives must fulfil the Gauss-Codazzi equations:

$$-q^2+q_2-kk^*=q_1^*+q^{*2},\quad q(k^*-k)+k_2=0,\quad q^*(k^*-k)+k_1^*=0. \tag{5}$$

2. The main results

It is known that on every regular non-spherical and non-developable surface there exists an orthogonal net such that the tangents to the surface along one family of this net form a rectilinear congruence, having this surface as one of its *focal surfaces*. Thus, we have the congruences:

$$T_1\colon\ \mathbf{Y}(u,v,\lambda)=\mathbf{r}(u,v)+\lambda\mathbf{e}_1(u,v),\quad \lambda\in\mathbb{R}, \tag{6}$$

$$T_2\colon\ \mathbf{Y}^*(u,v,\lambda)=\mathbf{r}(u,v)+\lambda\mathbf{e}_2(u,v),\quad \lambda\in\mathbb{R}, \tag{7}$$

in view of (1). These tangents of the surface S, which belong to the Darboux frame at any regular point, generate such congruences.

At first the equations of the focal surfaces of T_1 are

$$S_1=S\colon\ \mathbf{r}=\mathbf{r}(u,v),\quad S_2\colon\ \mathbf{x}=\mathbf{r}(u,v)-\frac{1}{q^*}\mathbf{e}_1(u,v),\quad q^*\neq 0. \tag{8}$$

Similarly, the equations of the focal surfaces of T_2 are

$$S_1^*=S\colon\ \mathbf{r}=\mathbf{r}(u,v),\quad S_2^*\colon\ \mathbf{x}^*=\mathbf{r}(u,v)+\frac{1}{q}\mathbf{e}_2(u,v),\quad q\neq 0. \tag{9}$$

From equations (3) and (4) we see that

$$\mathbf{H} = \frac{-k\mathbf{e}_2 + q\mathbf{e}_3}{\sqrt{k^2+q^2}}, \quad \mathbf{H}^* = \frac{k^*\mathbf{e}_1 + q^*\mathbf{e}_3}{\sqrt{k^{*2}+q^{*2}}} \tag{10}$$

are the direction vectors of the instantaneous axes of revolution of the Darboux frame along the curves $v =$ const., $u =$ const., respectively. In view of equations (3), (4) and (5), we have

$$\left.\begin{aligned} \mathbf{H}_1 = \frac{\partial \mathbf{H}}{\partial s_1} &= \left(\frac{q_1 k - q k_1}{k^2+q^2}\right)\left(\frac{q\mathbf{e}_2 + k\mathbf{e}_3}{\sqrt{k^2+q^2}}\right) \\ \mathbf{H}_2 = \frac{\partial \mathbf{H}}{\partial s_2} &= \frac{kq^*}{\sqrt{k^2+q^2}}\,\mathbf{e}_1 + k\left(\frac{q_1^* + q^{*2}}{k^2+q^2}\right)\left(\frac{q\mathbf{e}_2 + k\mathbf{e}_3}{\sqrt{k^2+q^2}}\right). \end{aligned}\right\} \tag{11}$$

In [1], we have obtained the following theorem:

Theorem 1 *For the rectilinear congruences generated by the tangents to the lines of curvature on a regular non-spherical and non-developable surface, the necessary and sufficient conditions that their developable surfaces touch the (non-degenerate) focal surfaces along lines of curvature are*

$$q^2 - q_2 = 0, \quad q^{*2} + q_1^* = 0 \quad (q \neq 0,\ q^* \neq 0). \tag{12}$$

It is known that the consecutive normals along a line of curvature intersect, the points of intersection being the corresponding centers of curvature. The set of all these normals at the points of S constitute a rectilinear congruence. The locus of the centers of curvature for all points of the surface S is called the *surface of centers* or *centro-surface* of S. In general it consists of two sheets, corresponding to the two families of lines of curvature, and these are called the focal surfaces. The equations of those surfaces are given by

$$F_1\colon\ \mathbf{y}(u,v) = \mathbf{r}(u,v) + \frac{1}{k(u,v)}\,\mathbf{e}_3(u,v), \quad k \neq 0, \tag{13}$$

$$F_2\colon\ \mathbf{y}^*(u,v) = \mathbf{r}(u,v) + \frac{1}{k^*(u,v)}\,\mathbf{e}_3(u,v), \quad k^* \neq 0, \tag{14}$$

A physical interpretation of the functions k, k^*, q, q^* is gained by considering the Darboux frame as moving space with respect to a fixed one. Then the rotation matrix is defined by $A(u,v) = \{\mathbf{e}_i = \mathbf{e}_i(u,v),\ i = 1,2,3\}$, and the translation vector is the surface itself. For zero position, we suppose that $u = v = 0$, then $A(0,0)$ is the identity matrix. Hence, along the curves $v =$ const., the angular velocity matrix is

$$\Omega_0 = \begin{pmatrix} 0 & -q_0 & -k_0 \\ q_0 & 0 & 0 \\ k_0 & 0 & 0 \end{pmatrix}. \tag{15}$$

And the angular velocity vector in this position is given by

$$\mathbf{W} = \begin{pmatrix} 0 \\ -k_0 \\ q_0 \end{pmatrix}. \tag{16}$$

Hence, the instantaneous screw axis becomes

$$\mathbf{I} = \lambda \mathbf{H} + \frac{\Omega\, \mathbf{e}_{10}}{\|\mathbf{W}\|}, \quad \lambda \in \mathbb{R}. \tag{17}$$

Since at zero position $\mathbf{e}_{10} = (1,0,0)$, the term $\Omega\, \mathbf{e}_{10}$ becomes $\Omega\, \mathbf{e}_{10} = q\mathbf{e}_2 + k\mathbf{e}_3$; thus $\mathbf{I}$ is lying in the $(\mathbf{e}_2, \mathbf{e}_3)$-plane and the motion is a revolution. The position vectors of the intersection points, where $\mathbf{I}$ intersects $\mathbf{e}_2$ and $\mathbf{e}_3$, are the surfaces S_2^* and F_1. The same argument is valid along the curves $u =$ const.. Thus, the functions k, k^*, q, q^* define the components of the angular velocities and the positions of the instantaneous axes of the Darboux frame at each point on the surface.

A similar equation to (17) along the curves $u =$ const. can be given as

$$\mathbf{I}^* = \lambda \mathbf{H}^* + \frac{\Omega^* \mathbf{e}_{210}}{\|\mathbf{W}^*\|}, \quad \lambda \in \mathbb{R}. \tag{18}$$

From the above discussions, we can give the following

Theorem 2 *During the motion of the Darboux frame along a line of curvature on a regular non-spherical and non-developable surface the instantaneous axis of revolution and the instantaneous tangent to this line of curvature are orthogonal skew lines. Additionally, the axis of revolution satisfies the following:*

(i) it intersects the surface normal at the center of curvature of this line of curvature,

(ii) it intersects the tangent, which is orthogonal to the path of the Darboux frame, at the center of geodesic curvature of this line of curvature.

The problem we are going to investigate in this paper is the following: By the motion of the Darboux frame on the surface S there are two rectilinear congruences generated by the instantaneous axes of revolution in equation (10). Consider now the line joining the points of the surfaces F_1 and S_2^*: If $\mathbf{x}^*$ is designated as the initial point while the other endpoint $\mathbf{y}$ as the terminal point, and we have the oriented segment

$$\mathbf{x}^*\mathbf{y} = \mathbf{y} - \mathbf{x}^*. \tag{19}$$

From equations (9) and (13) the unit vector along this segment is the unit vector $\mathbf{H}$ of the instantaneous axis, given in equation (10). The same can be done for the surfaces F_2 and S_2.

Thereby, we can introduce the following rectilinear congruences:

$$I\colon\ \mathbf{Q}(u,v,t) = \mathbf{r}(u,v) + \frac{1}{k(u,v)}\,\mathbf{e}_3(u,v) + t\mathbf{H}(u,v), \quad t \in \mathbb{R}, \tag{20}$$

$$I^*\colon\ \mathbf{Q}^*(u,v,t) = \mathbf{r}(u,v) + \frac{1}{k^*(u,v)}\,\mathbf{e}_3(u,v) + t\mathbf{H}^*(u,v), \quad t \in \mathbb{R}. \tag{21}$$

3. Properties of the congruences

If e, f, g, and a, b, b', c are the coefficients of the first and second fundamental forms of the congruence I, in Kummer sense, we have from equations (11), (12), and(21) that

$$\left.\begin{aligned}
e &:= \langle \mathbf{H}_u, \mathbf{H}_u \rangle = g_{11} \langle \mathbf{H}_1, \mathbf{H}_1 \rangle = g_{11} \left(\frac{q_1 k - q k_1}{k^2 + q^2} \right)^2, \\
f &:= \langle \mathbf{H}_u, \mathbf{H}_v \rangle = \sqrt{g_{11} g_{22}}\, \frac{k(qk_1 - q_1 k)(q_1^* + q^{*2})}{(k^2+q^2)^2}, \\
g &:= \langle \mathbf{H}_v, \mathbf{H}_v \rangle = g_{22} \langle \mathbf{H}_2, \mathbf{H}_2 \rangle = g_{22}\, \frac{k^2[q^{*2}(k^2+q^2) + (q_1^* + q^{*2})^2]}{(k^2+q^2)^2},
\end{aligned}\right\} \tag{22}$$

and

$$\left.\begin{array}{rcl} a & := & \langle \mathbf{H}_u, \mathbf{y}_u\rangle = g_{11}\langle \mathbf{H}_1, \mathbf{y}_1\rangle = -g_{11}\,\dfrac{k_1(k_1q-q_1k)}{k(k^2+q^2)^{3/2}} \\ b & := & \langle \mathbf{H}_u, \mathbf{H}_v\rangle = \sqrt{g_{11}g_{22}}\,\langle \mathbf{H}_1, \mathbf{y}_2\rangle = 0, \\ b' & := & \langle \mathbf{H}_v, \mathbf{H}_v\rangle = \sqrt{g_{11}g_{22}}\,\langle \mathbf{H}_2, \mathbf{y}_1\rangle = -\sqrt{g_{11}g_{22}}\,\dfrac{k_1(q_1^*+q^{*2})}{(k^2+q^2)^{3/2}} \\ c & := & \langle \mathbf{H}_v, \mathbf{y}_v\rangle = g_{22}\langle \mathbf{H}_2, \mathbf{y}_2\rangle = 0. \end{array}\right\} \tag{23}$$

The same calculations can be valid for the rectilinear congruence I^*. Therefore, if e^*, f^*, g^* and $a^*, b^*, b^{*\prime}, c^*$ are the coefficients of the first and second fundamental forms of I^*, we have

$$\left.\begin{array}{rcl} e^* & := & g_{11}\,\dfrac{k^{*2}[q^2(k^{*2}+q^{*2})+(q^2-q_2^2)^2]}{(k^{*2}+q^{*2})^2}, \\ f^* & := & \sqrt{g_{11}g_{22}}\,\dfrac{k^*(q_2k^*-q^*k_2^*)(q^2-q_2)}{(k^{*2}+q^{*2})^2}, \\ g^* & := & g_{22}\left(\dfrac{q_2^*k^*-q^*k_2^*}{k^2+q^2}\right)^2, \end{array}\right\} \tag{24}$$

and

$$\left.\begin{array}{rcl} a^* & = & 0, \\ b^* & = & -\sqrt{g_{11}g_{22}}\,\dfrac{k_2^*(q^2-q_2)}{(k^{*2}+q^{*2})^{3/2}}, \\ b^{*\prime} & = & 0, \\ c^* & = & -g_{22}\,\dfrac{k_2^*(q_2^*k^*-q^*k_2^*)}{k^*(k^{*2}+q^{*2})^{3/2}}. \end{array}\right\} \tag{25}$$

It is clear now from equations (23) and (25) that $b = b' = 0$ and $b^* = b^{*\prime} = 0$, i.e., the rectilinear congruences I and I^* are of normal type, if and only if condition (12) is satisfied. Hence the following theorem is proved:

Theorem 3 *The instantaneous axes of revolution of the Darboux frame along a set of lines of curvature generate a normal rectilinear congruence if and only if the developable surfaces consisting of tangents to this set touch the (non-degenerate) focal surfaces of the congruence along lines of curvature.*

The equations of the focal surfaces of the congruence I are obtained from equation (20) by substitution with the following values:

$$t = 0, \quad t = \frac{k_1\sqrt{k^2+q^2}}{k(k_1q-kq_1)}. \tag{26}$$

In the same manner for the congruence I^*

$$t = 0, \quad t = \frac{k_2^*\sqrt{k^{*2}+q^{*2}}}{k(k^*q_2^*-k_2^*q^*)}. \tag{27}$$

Hence the following theorem is proved:

Theorem 4 *During the motion of the Darboux frame along a family of lines of curvature on a regular non-spherical and non-developable surface the instantaneous axis of revolution is tangent to the central surface (centro-surface) of this family of lines of curvature.*

We now proceed to show when the instantaneous $\mathbf{H}$ is normal to a surface. For this purpose here, first from equation (8) and the condition in (12), we have the following derivatives:

$$\mathbf{x}_1 = -\frac{1}{q^*}\,(q\mathbf{e}_2 + k\mathbf{e}_3), \quad \mathbf{x}_2 = \frac{q_2^*}{q^{*2}}\,\mathbf{e}_1\,. \tag{28}$$

From this equation, the normal vector of the focal surface S_2 is parallel to $\mathbf{H}$. Thus the surface S_2 (S_1) is normal to the instantaneous axis of revolution $\mathbf{H}$ ($\mathbf{H}^*$).

The surface S: $\mathbf{r} = \mathbf{r}(u, v)$, chosen to have the families of the lines of curvature as parametric curves, should obey to the condition $g_{11}g_{22} \neq 0$ at each point to become regular. Now, from equations (22) and (24) for the cylindrical conditions of I and I^* we have, respectively:

$$\sqrt{eg - f^2} = 0, \quad \sqrt{e^*g^* - f^{*2}} = 0, \tag{29}$$

or

$$kq^*(k_1 q - kq_1) = 0, \quad k^*q(q_2^*k^* - q^*k_2^*) = 0\,. \tag{30}$$

For a surface on which the parametric net is an orthogonal net of lines of curvature we have [1, 3, 7]:

(i) if $k = 0$ or $k^* = 0$, then the surface is a developable ruled surface;

(ii) if $q = 0$ or $q^* = 0$, then the surface is a moulding surface;

(iii) if $k_1 q - kq_1 = 0$ or $q_2^*k^* - q^*k_2^* = 0$, then the parametric curves $v = \text{const.}$ or $u = \text{const.}$, respectively, of the surface are plane curves;

(iv) if $k_1 = 0$, $k_2^* \neq 0$ or $k_2^* = 0$, $k_1 \neq 0$, then the surface is called canal surface.

It is known that for the developable ruled surfaces one of the families of the lines of curvature consists of the generators of the surface, and since the Darboux frame is only translated along the generator, then the instantaneous axis of revolution does not exist. Consequently, $\mathbf{H}$ or $\mathbf{H}^*$ are not defined. Therefore we have to impose that $k \neq 0$ or $k^* \neq 0$. Since one of the families of lines of curvature on a moulding surface are plane curves, we need $q \neq 0$ or $q^* \neq 0$. (In fact, we have imposed $q \neq 0$ or $q^* \neq 0$ in equations (8) and (9).) One of the families of lines of curvature of a canal surface are planar curves since they are circular. Moreover, the corresponding central surface of the canal surface becomes a curve and the congruence I or I^* degenerates into a ruled surface. Finally, the condition $k_1 q - kq_1 = 0$ or $q_2^*k^* - q^*k_2^* = 0$ infers the surface S to have a family of lines of curvature $v = \text{const.}$, or $u = \text{const.}$ formed from plane curves. Thus, we have to impose $k_1 q - kq_1 \neq 0$ or $q_2^*k^* - q^*k_2^* \neq 0$.

We record the following theorem:

Theorem 5 *The rectilinear congruence generated by the instantaneous axes of revolution, belonging to the motion of the Darboux frame along a family of non-plane lines of curvature of a non-spherical and non-developable surface, is a non-cylindrical congruence.*

References

[1] R.A. Abdel-Baky: *On the Congruences of the Tangents to a Surface.* Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **136**, 9–18 (1999).

[2] W. Blaschke: *Vorlesungen über Differentialgeometrie.* 4. Aufl., Springer-Verlag, Berlin 1945.

[3] L.P. Eisenhart: *A Treatise in the Differential Geometry of curves and surfaces*. Ginn Comp., New York 1969.

[4] J. Hoschek: *Liniengeometrie*. B.I. Hochschultaschenbuch, Mannheim 1971.

[5] R. Koch: *Zur Geometrie der zweiten Grundform der Geradenkongruenzen des E^3*. Acad. V. Wetensch. V. Belgie, Brüssel (1981).

[6] G. Turken: *On the rectilinear congruences generated by the instantaneous secrewing axes connected with a surface*. Comm. Fac. Sci. Univ. Ank. Series A_1 **44**, 67–80 (1995).

[7] M.A. Weatherburn: *Differential Geometry of three dimensions, vol. 1*. Cambridge University Press 1960.

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