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Non-orientable Maps and Hypermaps with Few Faces

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Abstract. A map, or a cellular division of a compact surface, is often viewed as a cellular imbedding of a connected graph in a compact surface. It generalises to a hypermap by replacing "graph" with "hypergraph". In this paper we classify the non-orientable regular maps and hypermaps with size a power of 2, the non-orientable regular maps and hypermaps with 1, 2, 3, 5 faces and give a sufficient and necessary condition for the existence of regular hypermaps with 4 faces on non-orientable surfaces. For maps we classify the non-orientable regular maps with a prime number of faces. These results can be useful in classifications of non-orientable regular hypermaps or in non-existence of regular hypermaps in some non-orientable surface such as in [5].

Key Words: Maps, hypermaps, graphs imbeddings, non-orientable surfaces

MSC 2000: 05C25, 05C30, 05C65, 05B45, 52C20, 57M07, 57M15, 57M50, 57M60

1. Introduction

A *surface*, which can be orientable or non-orientable, is a compact, connected 2-manifold (without boundary). A *map* is an embedding of a graph in a surface S so that each connected component of the complement of the graph in the surface is simply-connected (i.e., homeomorphic to an open disk). Such a region is called a *face* of the map. Denote by V, E, F the number of vertices, edges and faces in the map $\mathcal{M}$. The integer $N = N(\mathcal{M}) = E - F - V$ is determined by the surface; that is, it is constant over all maps $\mathcal{M}$ on a fixed surface S, simply because each face in a map is simply connected. This $N = N(S)$ is just the negative of the Euler characteristic of S. Orientable surfaces have $N = -2, 0, 2, 4, 6, \ldots$ while non-orientable surfaces have $N = -1, 0, 1, 2, 3, 4, \ldots$

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Others convenient ways of coding maps besides the purely topological will be considered, such as the algebraic notion of a map as a *connection group* acting on a set, and the related notion of an *edge-coloured graph* of a certain kind. Both are motivated by the following construction:

For each face in the map $\mathcal{M}$, choose a point in its interior to call its center. Similarly choose a midpoint in each edge. Subdivide the map by joining each face-center to each vertex and edge-midpoint surrounding it (this is often called the *barycentric* subdivision). This divides the faces into subregions called *flags*, each of which we can think of a "right triangle" with the right angle at the edge-midpoint. Fig. 1 illustrates the subdivision and shows the neighbours of one flag f.

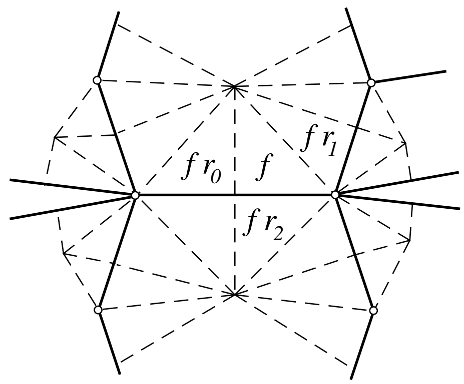

Figure 1: Neighbours of a flag

For any flag f, its three neighbours are fr_0, fr_1, and fr_2; fr_1 is its neighbour across its hypotenuse, fr_2 across the leg which is along an edge and fr_0 across its other leg. If we let Ω stand for the collection of flags, then each of the r_i's is a permutation on Ω. Let $C = C(\mathcal{M})$ be the subgroup of S_Ω generated by r_0, r_1, and r_2. C is the *connection group* (also called the *monodromy group*) of the map $\mathcal{M}$, and from Fig. 1 we can see that for any flag f, fr_0r_2 must be the same flag as fr_2r_0. Thus $(r_0r_2)^2$ must be the identity I in C.

Conversely, suppose that C is any group of permutations on a set Ω which is generated by three involutions r_0, r_1 and r_2 which satisfy $(r_0r_2)^2 = I$. We take right triangles and label them with the elements of Ω. Then joining these triangles in the appropriate ways by the r_i's we get a map on a surface (though the map might be degenerate in one of the ways described below). Such a group C can be considered to be the algebraic equivalent of a map.

Consider again the subdivision of the map into flags. We form a graph (we will use the language of "nodes" and "arcs" for this graph to distinguish them from the vertices and edges of the map) by first placing one new node in each flag, and then by joining each node to those in the three neighbouring flags. Let us assign the colour 0 to all arcs from a flag f to the flag fr_0 and similarly for colours 1 and 2. This gives us a coloured graph which is a combinatorial equivalent of the map [2]. Again reversing our point of view, suppose we have a graph Γ whose arcs are coloured 0, 1, 2 so that every node meets exactly one arc of each colour. Then the subgraph consisting of arcs numbered 0 and 1 only has 2-valent nodes and so is a union of disjoint cycles. We can call these the "faces" of the map; similarly the 0-2 cycles are the "edges" and the 1-2 cycles are the "vertices". If we want Γ to have the structure of a map we must require the 0-2 cycles to have length 4.

We then have three distinct ways to look at a map: as a topological map $\mathcal{M}$, as a connection group C on a set Ω, or as a tri-coloured graph Γ. These are equivalent objects and we can expect properties of one to be reflected by properties of another. An excellent example of this correspondence of properties is given by the following theorem, which we present without proof:

Theorem 1 *These three conditions are equivalent:*

(1) the map $\mathcal{M}$ is orientable

(2) the tri-coloured graph Γ is bipartite

(3) the group C^+ generated by r_0r_1 and r_1r_2 is not transitive on Ω.

1.1. Hypermaps

An *hypermap* is a natural generalisation of *map* and this generalisation is easiest to see in the algebraic and combinatorial settings. If we remove the restriction on C that $(r_0r_2)^2 = I$ or the property in Γ that 0-2 cycles have length 4, the resulting group or graph corresponds to a hypermap. In the topological viewpoint, we may think of an edge of a map, together with its midpoint, to be a *2-star*, i.e. a point with two half-edges emanating from it. If we allow edges to become 3-stars, 4-stars, etc, we have a topological hypermap. The hypermap becomes a map in which faces are surrounded by two kinds of points: one called "vertices" and one called "edges". This map is the *Walsh map* [3] of the hypermap.

Once we have made this generalization, the special role that edges play in maps disappears. By swapping indices among r_0, r_1, r_2 in C, or by a consistent re-assignment of colours 0, 1, 2 in Γ, we can produce a superficially new hypermap on the same surface. There may be as many as 6 different hypermaps we can construct in this way from one group or graph, and we will refer them all as *duals* of one another.

When we need more preciseness from our notation, we will give names such as D_{01} or D_{021} to the different dualities; $D_{01}(\mathcal{M})$, for example is the hypermap that results from switching labels 0 and 1 in the coloured graph version of $\mathcal{M}$, and $D_{021}(\mathcal{M})$ is $D_{01}(D_{02}(\mathcal{M}))$. If $\mathcal{M}$ is a map, we will continue to use D for D_{02}, the usual duality in maps.

1.2. Degenerate cases

There are two kinds of situations that can occur in C or Γ which correspond to undesirable situations in the topological viewpoint:

I: It is possible that in Γ, some two nodes are joined by more than one arc, as, for example if $fr_2 = fr_0$ for some flag f. This corresponds to a map-like object which has a "free" edge or "half-edge". This can be regarded as an edge which has a vertex at only one of its two ends, or as a loop which has been pinched at the mid-point to identify its two halves.

II: It is possible that in Γ, some node is joined to itself by some arc, as, for example, if $fr_2 = f$ for some flag f. We can model this on a surface by allowing the surface to have boundary. Then the flag f must have its corresponding side on this boundary.

1.3. Regularity

We give three definitions for a *symmetry* (often called an *automorphism*) of a map:

Definition 1 A *symmetry* of a hypermap $\mathcal{M}$ on a surface S is a rearrangement of its vertices, edges and faces which can be accomplished by a homeomorphism of S onto itself.

Definition 2 A *symmetry* of a hypermap (Ω, C) is an element of S_Ω which commutes with every element of C.

Definition 3 A *symmetry* of a hypermap Γ is an automorphism of the graph Γ which preserves edge-colour.

The symmetries of $\mathcal{M}$ form a group $G = G(\mathcal{M})$ under composition. Our general term for a map with a sufficiently large symmetry group is "regular", but there are two kinds of regularity to be considered:

Definition 4 A hypermap $\mathcal{M}$ is *rotary* provided that $G(\mathcal{M})$ contains symmetries R and S which act on the map as rotations one step about some face and some vertex incident to that face, respectively.

Definition 5 $\mathcal{M}$ is *reflexible* provided that (1) it is rotary and (2) $G(\mathcal{M})$ contains a symmetry which acts as a reflection in some face.

One easily sees that if $\mathcal{M}$ is rotary, then $G(\mathcal{M})$ is transitive on faces, on edges, and on vertices. If $\mathcal{M}$ is rotary and non-orientable (and these are the maps of interest in this paper) then $\mathcal{M}$ must be reflexible. And if $\mathcal{M}$ is reflexible, then $G(\mathcal{M})$ is transitive on flags.

Suppose that $\mathcal{M}$ is reflexible, and choose some flag I of $\mathcal{M}$ to be a *root*. Then because $G = G(\mathcal{M})$ is transitive on flags, there must exist symmetries α_0, α_1, α_2 such that each α_i sends I to Ir_i. Then G is generated by α_0, α_1, α_2, and G is isomorphic to C; indeed, the correspondence $\alpha_i \leftrightarrow r_i$ generates an anti-isomorphism of G onto C. If $\mathcal{M}$ is orientable, then $G^+(\mathcal{M}) = \langle R = \alpha_0\alpha_1,\ S = \alpha_2\alpha_1 \rangle$ is the group of orientation-preserving symmetries of $\mathcal{M}$, while if $\mathcal{M}$ is non-orientable, then $G^+ = G$. Conversely, any group G generated by α_0, α_1, α_2 such that each ${\alpha_i}^2$ is the identity is the group of some reflexible hypermap $\mathcal{M}$. $\mathcal{M}$ is non-degenerate if and only if each α_i has order 2 and each $\alpha_i\alpha_j$ has order at least 2. We can form this $\mathcal{M}$ uniquely from G by letting the elements of G be the flags of $\mathcal{M}$, and defining gr_i to be $\alpha_i g$ for each flag g.

A non-degenerate rotary hypermap $\mathcal{M}$ has a group which is transitive on faces, on edges and on vertices. Thus all of the faces have the same number of flags, as do the vertices and edges. We say $\mathcal{M}$ is of type $\{e, p, q\}$ provided that e is the order of r_0r_2, p is the order of r_0r_1 and q is the order of r_1r_2. Then e, p, q are also the orders of $\alpha_0\alpha_2$, $R = \alpha_0\alpha_1$, $S = \alpha_2\alpha_1$ respectively, in G. In Γ, then, edges are $2e$-cycles, faces are $2p$-cycles and vertices are $2q$-cycles. If, as before, V, E, F stand for the number of vertices, edges and faces, and if we let $2J$ be the number of flags, then an easy counting argument shows that $J = eE = pF = qV$. It is not hard to show then that $N(\mathcal{M}) = J - (E + F + V)$. When $\mathcal{M}$ is a map, of course, then $e = 2$, $J = 2E$ and $N = E - F - V$, as usual. We will call $[N, \{e, p, q\}, \{E, F, V\}, |G| = 2J]$ the *data line* for $\mathcal{M}$. To generalize that slightly, if e, q, p, E, V, F, J are any integers satisfying $J = eE = pF = qV$ and $N = J - (E + F + V)$, we will call $[N, \{e, p, q\}, \{E, F, V\}, 2J]$ a *viable line* for N.

If a degeneracy occurs at one flag, it occurs at all flags. If the degeneracy is of the first kind (say, $fr_0 = fr_2$), then the map must be $*_k$; this is a map of k half-edges meeting at a single vertex on the sphere. Its group is D_k of order $2k$, and its data line is $[-2, \{1, k, k\}, \{k, 1, 1\}, 2k]$, though some might quibble over whether k half-edges should give $E = k$ or $E = k/2$. If the degeneracy is of the second kind ($fr_2 = f$), then the "map" consists of a single k-gon, with the edges forming the border of the bordered surface. We call this degenerate structure P_k (P

is for Polygon); its group is D_k of order $2k$. In this paper we will use "$\mathcal{M}$ is non-orientable" as short for "$\mathcal{M}$ is a reflexible hypermap on a non-orientable surface (without boundary)"; this excludes the degenerate cases.

1.4. Projections, coverings and homomorphisms

If $\mathcal{N}$ and $\mathcal{M}$ are hypermaps, a *projection* of $\mathcal{N}$ onto $\mathcal{M}$ is (1) a continuous function from the surface of $\mathcal{N}$ onto that of $\mathcal{M}$ which preserves faces, vertices, and edges; (2) a function φ from $\Omega(\mathcal{N})$ onto $\Omega(\mathcal{M})$ such that $ar_i = b$ in $\mathcal{N}$ implies $\varphi(a)r_i = \varphi(b)$ in $\mathcal{M}$; (3) a function from t he nodes of the coloured graph of $\mathcal{N}$ onto those of $\mathcal{M}$ which sends nodes joined by an arc to nodes joined by an arc of the same colour. We call $\mathcal{M}$ a *projection* of $\mathcal{N}$ and $\mathcal{N}$ a *covering* of $\mathcal{M}$.

If $\mathcal{M}$ and $\mathcal{N}$ are reflexible, the projection induces a homomorphism from $G(\mathcal{N})$ onto $G(\mathcal{M})$. If K is the kernel of this homomorphism then $|K|$ flags are sent to each flag of $\mathcal{M}$ and we say $\mathcal{N}$ is a $|K|$-*fold* covering of $\mathcal{M}$. Given a reflexible hypermap $\mathcal{N}$ and a normal subgroup K of $G(\mathcal{N})$, we can form $\mathcal{M} = \mathcal{N}/K$ from $\mathcal{N}$ by identifying two vertices, faces, edges, flags, points of the surface of $\mathcal{N}$ if some element of K sends one to the other. $\mathcal{M}$ is a reflexible hypermap and $G(\mathcal{M})$ is isomorphic to $G(\mathcal{N})/K$. $\mathcal{M}$ is non-degenerate if none of the α_i's or $\alpha_i\alpha_j$'s is in K.

1.5. Some families of non-orientable reflexible maps

On the projective plane (a surface with negative Euler characteristic $N = -1$), besides the four maps formed from the regular polyhedra by identifying antipodal points, there is an infinite family $\{\delta_k\}$ of regular maps consisting of a single $2k$-gon with opposite edges identified non-orientably. For example, Fig. 2 shows δ_5. The data line for δ_k is $[-1, \{2, 2k, 2\}, \{k, 1, k\}, 4k]$.

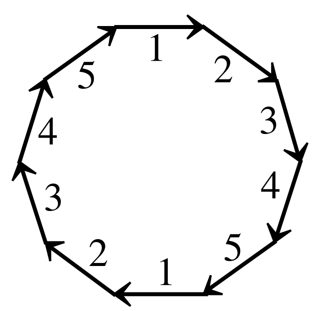

Figure 2: The map δ_5

Two families of non-orientable regular maps are Γ_k and $B^*(k, 2c)$, described in [4]. Γ_k is defined for all positive integers k and has data line $[3k - 4, \{2, 4, 3k\}, \{6k, 3k, 4\}, 24k]$; opp$B^*(k, 2c)$ is defined and non-orientable whenever k and c are both odd. It has data line $[(k-1)(c-1)-1, \{2, 2c, 2k\}, \{kc, k, c\}, 4kc]$. When k is even, $k = 2m$, then $D\Gamma_{2m}$ is bipartite and thus it can be viewed as the Walsh map for a hypermap we will call GW_m; this leads to a family of hypermaps GW_m with data line $[6m - 4, \{4, 3m, 4\}, \{3m, 4, 3m\}, 24m]$.

For future use, let us note now that from [4] we can deduce that the map Γ_k satisfies and is characterized by the relation $I = \alpha_1\alpha_2\alpha_1\alpha_0\alpha_1\alpha_2\alpha_0\alpha_1\alpha_0$. From that, we gather that GW_m satisfies and is characterized by the relation $I = \alpha_0\alpha_2\alpha_0\alpha_2\alpha_1\alpha_2\alpha_1$. Because the choice of which class of vertices of the map $D\Gamma_{2m}$ are to be the vertices and which the edges of the hypermap GW_m is arbitrary, $GW_m = D_{01}(GW_m)$, and so GW_m also satisfies $\alpha_1\alpha_2\alpha_1\alpha_2\alpha_0\alpha_2\alpha_0 = I$.

2. Non-orientable reflexible hypermaps with 1, 2 and 3 faces

To simplify notation, assume we are considering the dual of the hypermap in which $2 \leq e \leq p \leq q$ (the case $e = 1$ happens only in the degenerate hypermap $*_k$). We start by a theorem whose proof is obvious.

Theorem 2 *If $\mathcal{M}$ is a non-orientable hypermap in which $e = p = 2$ then $\mathcal{M}$ must be $D\delta_k$ for some k, with data line* $[-1, \{2, 2, 2k\}, \{k, k, 1\}, 4k]$.

The following two theorems classify non-orientable reflexible hypermaps with 1 and 2 faces. Although one- and two-faced regular hypermaps were classified in [1], for the sake of completeness and illustration of the involvement of the three aspects of hypermaps (surface embedding, connection group, coloured graph) we give here different proofs for the non-orientable versions only.

Theorem 3 *If $\mathcal{M}$ is a non-orientable reflexible hypermap with exactly one face, then $\mathcal{M}$ must be δ_k for some k. It follows that $\mathcal{M}$ must have data line* $[-1, \{2, 2k, 2\}, \{k, 1, k\}, 4k]$.

Proof: In this proof, we illustrate the use of the coloured-graph approach to hypermaps. If $\mathcal{M}$ has just one face of order k, then the nodes of the coloured graph may be labelled f_i, g_i so that arcs coloured 0 join f_i to g_i, arcs coloured 1 join g_i to f_{i+1}, where the indices are numbers $\bmod k$, as in Fig. 3:

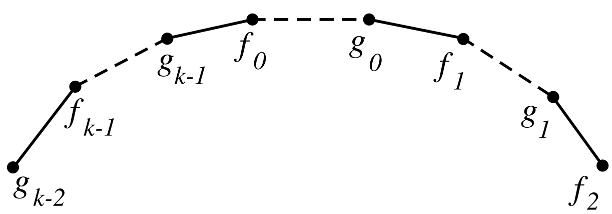

Figure 3: A hypermap with one face

Because $\mathcal{M}$ is non-orientable, this coloured graph must not be bipartite, so some edge coloured 2 must join some f_i to some f_j (i. e., $f_i r_2 = f_j$). Without any loss of generality, assume $f_0 r_2 = f_j$. By symmetry, we must have $f_1 r_2 = f_{j+1}, \ldots$ in general, $f_i r_2 = f_{i+j}$. But then $f_0 r_2 = f_j$, and $f_j r_2 = f_{j+j} = f_{2j}$, and since r_2 is an involution, we conclude that $2j = 0 \bmod k$. Because $\mathcal{M}$ is non-degenerate, $j = 0$ is impossible, so we must have $k = 2j$ and each f_i is connected by edges coloured 2 to $f_{i+j} = f_{i-j}$, and the same must be true for the g_i's. These are precisely the connections for the map δ_k, as required. □

Theorem 4 *If $\mathcal{M}$ is a non-orientable reflexible hypermap with exactly two faces, then $\mathcal{M}$ must be a dual of $D\delta_2$ shown in Fig. 4, having data line* $[-1, \{2, 2, 4\}, \{2, 2, 1\}, 8]$.

Proof: In this proof, we argue from a group-theoretic point of view. Let A be the face of $\mathcal{M}$ which contains the root flag I. Then α_0 and α_1 generate the stabilizer of face A. This must be a dihedral group D_k for some k, and $\alpha_0\alpha_1$ generates the cyclic subgroup C_k of D_k. Because there are only two faces, D_k must have index 2 in G and so must be normal in G. Consider the product $\rho = \alpha_2\alpha_0\alpha_1\alpha_2$. This conjugate of $\alpha_0\alpha_1$ must be in D_k. Can it be in C_k? If so, then α_2 must conjugate C_k to itself and the coset $C_k\alpha_0 = C_k\alpha_1$ to itself, also. Consider a word in G^+; pairs of α_2's enclosing a sequence of α_0's and α_1's can be replaced by another string of α_0's and α_1's with a length of the same parity. Continuing this gives a word of even length in α_0 and α_1, or a word in α_0 and α_1 of odd length followed by α_2. Thus G^+ must be contained in $C_k \cup C_k\alpha_0\alpha_2$. This contradicts the non-orientability of $\mathcal{M}$.

Then ρ must be in the coset $C_k\alpha_0$. Since everything in the coset has order 2, $\alpha_0\alpha_1$, as a conjugate of ρ, must be of order 2 and so $k = 2$. Then $C_k\alpha_0$ has exactly two elements, α_0 and α_1. Thus ρ is one of these two. Suppose it is α_1. Then the relations $I = (\alpha_0)^2 = (\alpha_1)^2 = (\alpha_2)^2 = (\alpha_0\alpha_2)^2 = (\alpha_0\alpha_1)^2 = (\alpha_0\alpha_2)^2 = \alpha_2\alpha_0\alpha_1\alpha_2\alpha_1$ hold in G; but these are exactly the defining relations for $D\delta_2$. If, on the other hand, ρ is α_0, we get the defining relations for $D_{01}(D\delta_2)$. □

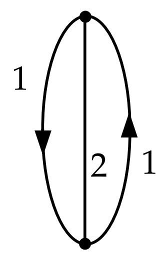

Figure 4: The map $D\delta_2$

Theorem 5 *If $\mathcal{M}$ is a non-orientable reflexible hypermap with exactly three faces, then $\mathcal{M}$ or $D_{01}(\mathcal{M})$ must be*

(1) the hypermap $D_{02}(GW_1)$ whose Walsh map is shown in Fig. 5 or

(2) the map Γ_1 (also called the hemi-cube; this map is shown in Fig. 6) or

(3) the map $\text{opp}B^*(3, 4k+2)$ for some k. [Notice that $k = 0$ gives $D\delta_3$.]

These have data lines $[2, \{3, 4, 4\}, \{4, 3, 3\}, 24]$, $[-1, \{2, 4, 3\}, \{6, 3, 4\}, 24]$ *and* $[4k-1, \{2, 2(2k+1), 6\}, \{3(2k+1), 3, 2k+1\}, 12(2k+1)]$, *respectively.*

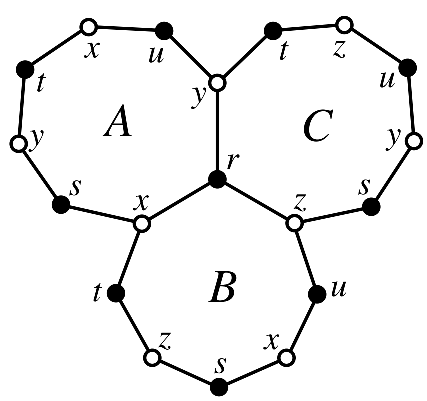

Figure 5: A Walsh map of a dual of GW_1

Fig. 5 shows a Walsh map for the hypermap $D_{02}(GW_1)$. It has three faces: A, B, and C, four vertices: r, s, t and u and three edges: x, y and z.

Proof: We provide an outline of the proof, which relies on the Walsh map presentation for a hypermap. Notation/convention: A Walsh map of a hypermap is bipartite; we will refer to its white nodes as "edges", its black nodes as "vertices" and the connections between nodes as "arcs". Interchanging the roles of white and black nodes is a duality (D_{01}) of the hypermap.

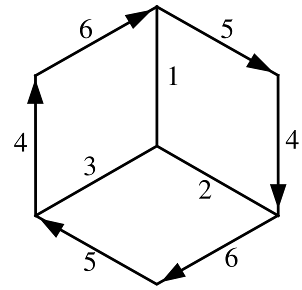

Figure 6: The hemi-cube

Let the faces be labelled A, B, C. The arcs in face A separate it from faces B and C. There are two possibilities for the order in which faces B and C occur as we regard the arcs of A in circular order: $\ldots B, C, B, C, B, C, \ldots$ and $\ldots B, C, C, B, B, C, C, B, \ldots$ In the first case, it is easy to show that the surface must be orientable: we assign "clockwise" at each vertex to be the direction in which the faces appear in the order $A, B, C, A, \ldots$ and at each edge $A, C, B, A, \ldots$ This assignment is consistent throughout the hypermap and so the surface is orientable. In the second case, we choose the dual in which each edge meets two faces and each vertex meets all three, as in Fig. 7

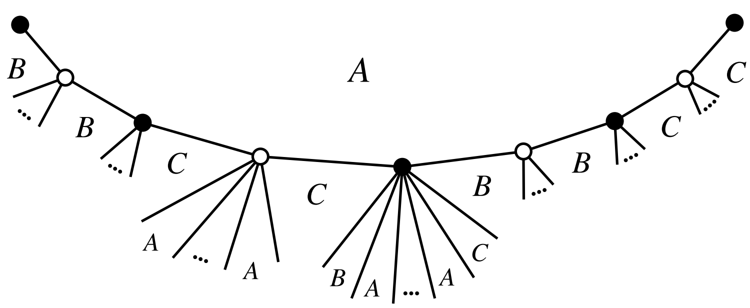

Figure 7: Faces around face A in the Walsh map of a hypermap having 3 faces

Can it happen that an AB-edge x which meets face A more than once does so with opposite orientations? If that is the case, then the rotation of A which sends x to x acts as a reflection at x and so has order 2. Thus x meets A exactly twice, and so $e = 4$. That rotation acts as a reflection about B. Then the first rotation about A which sends B to B must be a reflection about B and so $p = 4$. The map must then be as in Fig. 8:

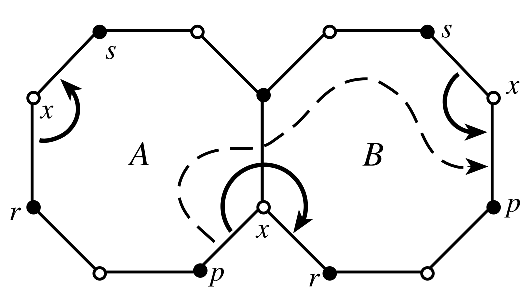

Figure 8: Part of a map with three faces

The path shown in dotted lines passes through flags linked by $r_0r_2r_1r_0r_1r_0r_2$ back to its start. Thus $\alpha_2\alpha_0\alpha_2\alpha_0\alpha_1\alpha_0\alpha_1$ must evaluate to the identity in this map. Then the only possibility is $D_{02}(GW_1)$.

Can it happen that there are edges x and y which meet A and B so that a path from x through A to y through B to x is orientation reversing? If so then we can choose orientations about x and y so that the hypermap must be as in Fig. 9:

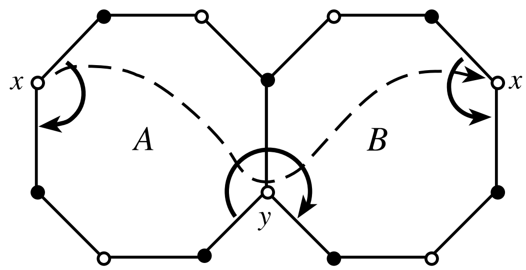

Figure 9: Faces A, B in which edges x and y do and do not agree in orientation

Then let R be any rotation about A which sends y to x. Because edges x and y meet faces A and B only, R must move B to B. Because R, as an action on B, sends y to x with orientations not matching, it must be a reflection about some axis of B. Then the first rotation about A which sends B to B is a reflection about B and so has order 2. Thus, p must be 4 as in Fig. 9. Then x and y are the only edges joining A to B and each of these edges occurs exactly once in each face. Thus $e = 2$; i.e. $\mathcal{M}$ is a map. Tracing the path from edge to edge, we have $I = r_1r_0r_1r_2r_0r_1r_0r_1r_2$ and so $I = \alpha_2\alpha_1\alpha_0\alpha_1\alpha_0\alpha_2\alpha_1\alpha_0\alpha_1$ holds also. Thus, $\mathcal{M}$ is the map Γ_1, the hemi-cube.

If neither of those things happen, then we can assign an orientation arrow to each of the edges so that when seen from the center of A or B these arrows match but when seen from the center of C, the AC edges do not match the BC edges in orientation. Then a rotation one step about an edge joining A and B must send C onto itself with reversed orientations on its edges. Thus it must be a reflection, and so of order 2. Then $\mathcal{M}$ is a map and must be opp$B^*(3, 4k + 2)$ for some k by [4]. □

The same proof outline proves a more general theorem:

Theorem 6 *If $\mathcal{M}$ is a non-orientable reflexible hypermap in which each face meets exactly two others, then $\mathcal{M}$ or $D_{01}(\mathcal{M})$ must be*

(1) the hypermap $D_{02}(GW_k)$,

(2) the map Γ_k,

(3) the map opp$B^*(2k + 1, 4c + 2)$ for some k, c, or

(4) $D\delta_k$ for some k.

Proof: First note that if such a map has m faces, we must be able to number them $0, 1, 2, \ldots, m-1$ so that face F_i meets faces F_{i-1} and face F_{i+1} (mod m) only. As in the proof of Theorem 5, the arcs around F_i must separate it from F_{i-1}, F_{i+1}, F_{i+1}, F_{i-1}, F_{i-1}, F_{i+1}, etc, in circular order, and without loss of generality, we can assume that the roles of edges and vertices are chosen so that each edge meets exactly two faces and each vertex meets every face.

As in the proof of Theorem 5, if we assume that some edge meets a face with opposite orientations, then $\mathcal{M}$ must be $D_{02}(GW_k)$ for some k, and if we assume that two edges x and y joining the same pair of faces have orientations which disagree in one face and agree in the other, then $\mathcal{M}$ must be $D\Gamma_k$ for some k.

If neither of those happen then we can choose orientations so that the edges joining F_0 and F_1 agree in both faces, the edges joining F_1 and F_2 agree in both faces, and so on, up to the edges joining $F_{m-1} = F_{-1}$ to F_0 agree in both faces. Then within F_0 the edges joining F_0 to F_1 either all do or all do not agree with the orientations of edges joining F_0 to F_{-1}. If they did agree, then $\mathcal{M}$ could be shown to be orientable.

So the edges around face 0 appear there with alternating clockwise and counterclockwise orientations. We consider two cases based on the parity of m.

I. m is odd, say $m = 2k+1$. Let x be any edge joining F_k to F_{k+1}. Then R, a rotation one step about x, is a symmetry which switches F_k and $F_{k+1} = F_{-k}$ while preserving orientations of edges. R also switches F_{k-1} and $F_{k+2} = F_{-(k-1)}$ while preserving orientation of edges. Finally, R switches F_1 and F_{-1} while preserving orientations, and so R send F_0 to itself, preserving orientations of edges. Thus R acts on F_0 as a reflection, and so must have order 2. Thus $\mathcal{M}$ is a map, and the classification of [4] shows that $\mathcal{M}$ must be opp$B^*(2k+1, 4c+2)$ for some c, or $D\delta_{2k+1}$.

II. m is even, say $m = 2k$ for some k. Then similar considerations show that rotation one step about F_k must be a reflection about F_0 and so have order 2. Since $p = 2$, no edge meets any face twice, so $e = 2$ as well. Thus $\mathcal{M}$ must be $D\delta_{2k}$. □

3. Non-orientable reflexible hypermaps of size a power of 2

Theorem 7 *The only reflexible non-orientable hypermaps $\mathcal{M}$ for which the order of $G(\mathcal{M})$ is a power of 2 are duals of δ_k where k is a power of 2.*

Proof: We begin this proof by proving two lemmas:

Lemma 8 *The only reflexible 2-fold coverings of the degenerate hypermap P_k are P_{2k}, ϵ_k (the map formed by embedding a cycle of k vertices and edges around the equator of a sphere) and δ_k.*

Proof: Consider the coloured graph form of P_k. It has flag-nodes $f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_k$ and arcs 0 link f_i to g_i, arcs 1 link g_i to f_{i+1}, arcs 2 link each node to itself. If $\mathcal{N}$ is a 2-fold cover of P_k, we can describe it in terms of nodes f'_i, f''_i, g'_i, g''_i, where the projection of $\mathcal{N}$ onto P_k merely erases the prime marks (dashes). With no loss of generality, we can assume that arcs 0 link f'_i to g'_i and f''_i to g''_i for all i, and that arcs 0 link g'_i to f'_{i+1} and g''_i to f''_{i+1} for $i = 1, 2, 3, \ldots, k-1$. We then have two cases:

I: Arcs of colour 0 link g'_k to f'_1 and g''_k to f''_1. Then $\mathcal{N}$ has two faces, and connectivity requires some (and hence every) arc of colour 2 to join the one face to the other; more precisely, colour 2 joins f'_i to f''_i and g'_i to g''_i. This is exactly the structure of the map ϵ_k.

II: Arcs of colour 0 link g'_k to f''_1 and g''_k to f'_1. Then $\mathcal{N}$ has exactly one face of order $2k$ and there are two possibilities for the arcs of colour 2:

A: Color 2 links each node to itself. This gives P_{2k}.

B: Color 2 links f'_i to f''_i and g'_i to g''_i. This gives precisely the connections for the map δ_k, as required. ♦

Lemma 9 *The only 2-fold reflexible covering of δ_k is ϵ_{2k}.*

Proof: Consider the coloured graph form of δ_k. It has flag-nodes $f_1, f_2, \ldots, f_{2k}$, $g_1, g_2, \ldots, g_{2k}$ and arcs 0 link f_i to g_i, arcs 1 link g_i to f_{i+1}, arcs 2 link f_i to f_{i+k}, g_i to g_{i+k}. If $\mathcal{N}$ is a 2-fold cover of δ_k, we can describe it in terms of nodes f_i', f_i'', g_i', g_i'', where the projection of $\mathcal{N}$ onto δ_k merely erases the prime marks (dashes). With no loss of generality, we can assume that arcs 0 link f_i' to g_i' and f_i'' to g_i'' for all i, and that arcs 0 link g_i' to f_{i+1}' and g_i'' to f_{i+1}'' for $i = 1, 2, 3, \ldots, 2k-1$. We then have two cases:

I: Arcs of colour 0 link g_{2k}' to f_1' and g_{2k}'' to f_1''. Then $\mathcal{N}$ has two faces, and connectivity requires some (and hence every) arc of colour 2 to join the one face to the other; more precisely, colour 2 joins f_i' to f_{i+k}'' and g_i' to g_{i+k}''. By re-labelling f_{i+k}'' as $\hat{f}_i$ and g_{i+k}'' as $\hat{g}_i$, we see that this is exactly the structure of the map ϵ_k.

II: Arcs of colour 0 link g_{2k}' to f_1'' and g_{2k}'' to f_1'. Then $\mathcal{N}$ has exactly one face of order $4k$. Because the colour 2 arcs link f_i' to f_{i+k}' or f_{i+k}'', $\mathcal{N}$ is non-orientable. By the one-face theorem (Theorem 3), $\mathcal{N}$ must be δ_{2k}. But δ_{2k} is not a covering of δ_k; the identification of antipodal nodes in δ_{2k} sends it onto P_k. This contradiction eliminates Case II, proving the lemma. ♦

Returning to the proof of theorem, suppose that $\mathcal{M}$ is a reflexible, non-orientable hypermap and $|G(\mathcal{M})| = 2^{r+1}$ for some r. Any two-group must contain a central involution t; factoring out t induces a projection of $\mathcal{M}$ onto a reflexible hypermap $\mathcal{M}_1$ for which $|G(\mathcal{M}_1)| = 2^r$. Continuing this process gives a sequence of reflexible hypermaps $\mathcal{M} = \mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \ldots$ where $\mathcal{M}_{i-1}$ is a 2-fold covering of $\mathcal{M}_i$. After $r+1$ steps the group will be trivial, and so after some point, all the hypermaps must be degenerate. Let i be the first such that $\mathcal{M}_i$ is degenerate. Because $\mathcal{M}$ is non-orientable, $\mathcal{M}_i$ cannot be $*_k$, so it must be P_k, where $k = 2^{r-i}$. From the first lemma, $\mathcal{M}_{i-1}$ must be δ_{2k}, and from the second lemma, if $i > 2$, then $\mathcal{M}_{i-2}$ must be ϵ_{2k}. But this orientable map cannot be a projection of the non-orientable $\mathcal{M}$, and so i is 1, and so $\mathcal{M} = \mathcal{M}_0$ is δ_{2k}, proving the theorem. □

Even more generally useful is this theorem:

Theorem 10 *Suppose that $\mathcal{M}$ is a non-orientable, reflexible hypermap of type $\{a, b, c\}$ whose group has order $2^k P$, where P is an odd prime. If 2 is a primitive root $\bmod P$, or if $P = 7$, then exactly one of a, b, c, is divisible by P.*

Proof: Let H be a 2-Sylow subgroup of G. Then G acts on the P right cosets of H by right multiplication and this action gives a homomorphism φ of G into S_P (often called the right regular representation of G on the cosets of H). Let $J = \varphi(G)$. Then J is transitive on $\{1, 2, \ldots, P\}$, so the order of J is $2^r P$ for some r. J has a P-Sylow subgroup F, which we can assume is generated by the P-cycle $\sigma = (1\,2\,3\, \ldots\, P)$. We wish to show that F is normal in J.

First assume that 2 is a primitive root $\bmod P$. Then the smallest power of 2 which is equivalent to one $\bmod P$ is 2^{P-1}. But r is no more than the power of 2 in $P!$, and this is given by $f(P) = P-$(the number of 1's in the binary expansion of P), which in turn is at most $P-2$. So no divisor of 2^r is equivalent to $1 \bmod P$ except $2^0 = 1$, and so by Sylow theory, F is normal in J.

Next, assume that $p = 7$. Then $r = 1$, 2, 3, or $4 = f(7)$. If $r = 1$ or 2 then again, Sylow theory implies that F is normal in J.

Suppose $r = 3$. Then $|J| = 56$ and if F were not normal, there would be 8 disjoint conjugates of F. These would include (6)(8)=48 elements of order 7, leaving only 8 elements to have other orders. Thus the remaining 8 must be a 2-Sylow subgroup T. But G, and hence J, is generated by involutions; this is clearly impossible since all the involutions are in T.

Finally suppose that $r = 4$. Then J contains a 2-Sylow subgroup of S_7, and so it must contain a transposition. Because F together with a transposition generate all of S_7, that is impossible.

Thus in any case, F is normal in J. In other words, J is contained in the normalizer of F. It is not hard to show that all involutions in $N(F)$ belong to the dihedral group D_{P} of symmetries of the P-gon labelled $1, 2, 3, \ldots, P$ in circular order. Thus J must be D_{P}.

Now let $a_i = \varphi(\alpha_i)$, $i = 0, 1, 2$. Because each α_i is an involution, each a_i must be an involution or the identity. At least two must be non-trivial in order to generate all of D_p, but if all three were non-trivial, then all three of a_0a_1, a_1a_2, a_2a_0 would be in the subgroup C_{P}, which is impossible since $\alpha_0\alpha_1$, $\alpha_2\alpha_0$, $\alpha_1\alpha_2$, generate all of G. Thus two of the a_i's are involutions and the third is trivial. Then one of a_0a_1, a_1a_2, a_2a_0 is of order P and the other two of order 2. The theorem follows directly. □

Of the primes less than 50, we can see that 3, 5, 7, 11, 13, 19, 29, 37 satisfy the hypotheses of the theorem.

As a **corollary** to this theorem, we mention that if $\mathcal{M}$ is a map (i.e., $e = 2$) satisfying the hypotheses of the theorem, then in addition, P divides the length of a Petrie path in $\mathcal{M}$. This length is the order of $\alpha_0\alpha_2\alpha_1$ in the group.

4. Classes of faces

A useful technique is to examine the action of the the group on refinements of the map. One such action is in equivalence classes of faces:

Theorem 11 *For each divisor k of p less than $p/2$, the relation $\sim$ defined by $U \sim T$ exactly when rotation by k steps about U stabilizes T, is an equivalence relation on the faces of $\mathcal{M}$.*

Proof: For any face W, let R_{W} be a rotation by one step about W. If $U \sim T$, then R_{U}^k stabilizes T and has order $p/k > 2$. The elements of the stabilizer of T which have order p/k are exactly those of the form R_{T}^j where $(j, p) = k$. These are the generators of $\langle R_{T}^k \rangle$. Thus we have that $U \sim T$ iff $\langle R_{U}^k \rangle = \langle R_{T}^k \rangle$, and from this, it is easy to show that $\sim$ is an equivalence relation. □

Corollary 12 *The group of the hypermap $\mathcal{M}$ acts on the equivalence classes, and the action is transitive.*

Because the group acts transitively on the classes, they must all be the same size and so:

Corollary 13 *Let s_k be the number of faces (including U) fixed by R_{U}^k. Then s_k must divide F and the group has an action on the set of F/s_k classes.*

In a non-orientable regular hypermap other than δ_2, s_1 cannot be F; in other words, rotation by one step about a face must move some other face.

Theorem 14 *Let h be some divisor of p such that the rotation R_{U}^h by h steps about face U stabilizes every face in m. If $h < p/2$ then $\langle R_{U}^h \rangle$ is normal in G. If $h = p/2$ and neither $\mathcal{M}$ nor $D_{01}(\mathcal{M})$ is: (1)Γ_k, (2) $D_{12}(GW_k)$ or (3) $D_{02}(GW_k)$ for some k, then also $\langle R_{U}^h \rangle$ is normal in G.*

Proof: Suppose that $h < p/2$. If g is an element of G such that $Ug = T$, then $(R_U)^g = R_T$. Thus every conjugate of $\langle R_U^h \rangle$ is some $\langle R_T^h \rangle$, and by the proof of Theorem 11, these are all equal to $\langle R_U^h \rangle$.

To address the case $h = p/2$, we need to introduce a lemma.

Lemma 15 *Suppose that a reflection about some face in a non-orientable hypermap $\mathcal{M}$ fixes every face of $\mathcal{M}$. Then $\mathcal{M}$ or $D_{01}(\mathcal{M})$ must be*

(1) the hypermap $D_{02}(GW_k)$,

(2) the map Γ_k,

(3) the map $\mathrm{opp}B^*(2k+1, 4c+2)$ for some k, c, or

(4) $D\delta_k$ for some k.

Proof: Assume that the reflection is about an axis passing through the face-center and an incident edge, as in Fig. 10, which shows face A being adjacent to faces v, u, t, x, y, z.

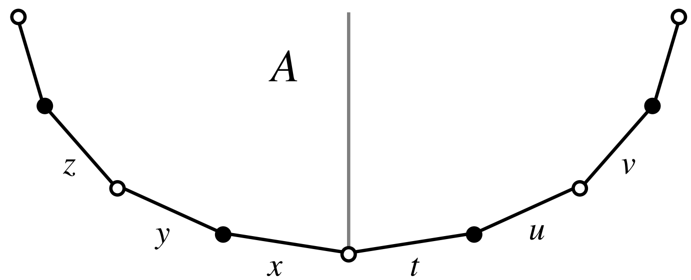

Figure 10: Neighbors of one face of $\mathcal{M}$

Because the reflection fixes all faces, we have that $x = t$, $y = u$, $z = v$. And because $x = t$, the rotation about A which sends x to u must send t to u also; in other words, $u = v$. Thus the face A meets only two others: one is $x = t$, the other is $u = v = z = y$, and so $\mathcal{M}$ satisfies the hypothesis of Theorem 6. By that theorem, $\mathcal{M}$ must be one of the hypermaps mentioned. ♦

Returning to the proof of the theorem, suppose $h = p/2$ and that U is a face such that $R = R_U^h$ fixes all faces. If R acts as a rotation about each face, then the same argument as for $h \neq p/2$ proves the theorem. If not, then R fixes some face T and acts as a reflection about T. Then R satisfies the hypothesis of the lemma, and so $\mathcal{M}$ must be one of the four hypermaps named there. But in the maps $\mathrm{opp}B^*(2k+1, 4c+2)$, rotation by $h = p/2 = 2c+1$ steps about a face does not fix all faces, so that family is excluded from our list of exemptions. □

5. Non-orientable reflexible maps with P faces

Theorem 16 *Suppose that $\mathcal{M}$ is a non-orientable reflexible map (i.e., $e = 2$) having exactly P faces, where P is a prime, $P > 3$. Then $\mathcal{M}$ is $\mathrm{opp}B^*(P, 2k)$ for some odd number k. This has data line $[(k-1)(P-1)-1, \{2, 2k, 2P\}, \{kP, P, k\}, 4kP]$.*

Proof: Let $G = G(\mathcal{M})$ and let $f : G \longrightarrow G' \leq S_P$ be the action of G on the P faces. By Corollary 13, $R = \alpha_0\alpha_1$, acting on the faces, must have one fixed point and move the remaining faces in cycles of one fixed length. This length, h, must divide $P-1$. The image of $\langle \alpha_0, \alpha_1 \rangle$ in G' must have order $2h$ and have P cosets, so $|G'| = 2hP$. Now, $h \mid (P-1)$,

$P > 3$ implies that P divides neither $h-1$ nor $2h-1$. Then a P-Sylow subgroup of G' must be normal. Assume the P-Sylow subgroup is $T = \langle(1\ 2\ 3\ \ldots\ P)\rangle$. Then G' is contained in the normalizer N of T in S_P. But the only transitive subgroup of N generated by involutions is D_P, and so $h = 2$. So rotation R about one face moves all the others in pairs, and so the one face can meet only two others; in other words, $\mathcal{M}$ is bicontactual. But the only non-orientable BCT maps with a prime number of faces are opp$B^*(P, 2k)$ for some odd number k, as required. □

Corollary 17 *There is a reflexible non-degenerate map with exactly P faces (with P odd prime) on the non-orientable surface S if and only if $N(S) = -1 \bmod 2(P-1)$.*

Proof: If $P = 3$ then by Theorem 5, the only non-orientable regular maps with 3 faces are Γ_1 and opp$B^*(3, 4k+2)$ with data lines $[-1, \{2,4,3\}, \{6,3,4\}, 24]$ and $[4k-1, \{2, 2(2k+1), 6\}, \{3(2k+1), 3, 2k+1\}, 12(2k+1)]$, respectively. Both have $N(S) = -1 \bmod 4$. If $P > 3$ then by Theorem 16, there is only the family of non-orientable regular maps opp$B^*(P, 2k)$, for k odd and data line $[(k-1)(P-1)-1, \{2, 2k, 2P\}, \{kP, P, k\}, 4kP]$. Writing $k-1 = 2n$ we get $N(S) = -1 \bmod 2(P-1)$. □

6. Non-orientable reflexible hypermaps with 4 and 5 faces

While a complete classification of 4-faced hypermaps still eludes us, we can establish the following theorem.

Theorem 18 *There is a reflexible non-degenerate hypermap with exactly four faces on the non-orientable surface S if and only if $N(S) = -1$ mod 3.*

Proof: First, if $N(S)$ is of the form $3k-4$, then the map $D\Gamma_k$ is a regular map on S. Now suppose that $\mathcal{M}$ is a non-orientable hypermap with 4 faces which lies on the surface S. Let the faces be labelled A, B, C, D. What faces can A meet? If A meets only B, then B meets only A and thus the map would be disconnected. If A meets only B and D, then B meets only A and C, etc, so $\mathcal{M}$ satisfies the hypotheses of theorem 6. The only possibility for $\mathcal{M}$ from that theorem is $D\delta_4$ which lies on the projective plane, where $N = -1$.

So each face must meet all three other faces. Is it possible for each vertex to meet exactly three of the four faces? If that is the case, assign the direction "clockwise" to each vertex so that about each vertex the faces appear in clockwise order ABC, ACD, ADB, or BDC. This assignment, we claim, is consistent in every face. In face A for example, rotation which moves $B \to C \to D \to B$ will send a vertex ABC to ACD to ADB, and similar considerations hold in each face. Therefore the map would be orientable.

Then each vertex (and, similarly, each edge) must meet two or four of the faces. If four, then there are three kinds of vertices: those in which the order is $ABCD$, $ABDC$ or $ACBD$ in one direction or the other. If two, then there are 6 kinds of vertices: those that meet AB, AC, AD, BC, BD, or CD. If g is the number of each kind of vertex and h is the number of each kind of edge, then there are $3g$ or $6g$ vertices, $3h$ or $6h$ edges and the number $2J$ of flags is also divisible by three, $J = 3a$. Then $N = J - (E + F + V) = 3a - (3 \text{ or } 6)g - (3 \text{ or } 6)h - 4$, which is of the form $3k - 4$, as required. □

Lemma 19 *If $\mathcal{M}$ is a non-orientable reflexible hypermap with exactly five faces, then $\mathcal{M}$ or $D_{01}(\mathcal{M})$ must cover map $D\delta_5$, which has data line $[-1, \{2,2,10\}, \{5,5,1\}, 20]$. Thus $\mathcal{M}$ must be of type $\{2a, 2b, 10c\}$ or $\{10a, 2b, 2c\}$ for some a, b, c.*

Proof: First note that the only map mentioned in Theorem 14 which has five faces is opp$B^*(5, 4c+2)$, and this map is a $(2c+1)$-fold covering of $D\delta_5$, as required. Thus, we can assume in what follows that any rotation which fixes all faces generates a normal subgroup of G.

If $\mathcal{M}$ has 5 faces, then by the divisibility criterion, R, rotation around one face, must fix one and move all of the others. There are two cases: (I) R moves the faces in one 4-cycle; (II) R moves the faces in two 2-cycles.

(I) R moves the faces in one 4-cycle. Then $\mathcal{L} = \mathcal{M}/\langle R^4 \rangle$ is a hypermap with 5 faces of size 4, and so its group is of order 40. The action on the faces is the same so R in this map must also move faces in a 4-cycle. The stabilizer of a face is a 2-Sylow subgroup of $G(\mathcal{L})$, and so the action on the 2-Sylows is the same as the action on the faces. But we see in the proof of Theorem 10 that this action is a dihedral subgroup D_5 of S_5, and so contains no 4-cycles. Thus, case (I) is impossible.

(II) R moves the faces in two 2-cycles. Then $\mathcal{L} = \mathcal{M}/\langle R^2 \rangle$ is a hypermap with 5 faces of size 2, and so its group is of order 20. Theorem 10 implies that e or q is divisible by 5, and so the corresponding E or V is 1 or 2. From Theorems 3 and 4, $\mathcal{L}$ must be $D\delta_5$. Since $\mathcal{M}$ covers $\mathcal{L}$, this proves the theorem. □

Theorem 20 *If $\mathcal{M}$ is a non-orientable reflexible hypermap having exactly 5 faces then $\mathcal{M}$ or $D_{01}(\mathcal{M})$ is a map.*

Proof: Let $\mathcal{M}$ be a non-orientable reflexible hypermap with 5 faces and group G. Up to a duality, by Lemma 19, $\mathcal{M}$ covers the map $D\delta_5$, which has data line $[-1, \{2, 2, 10\}, \{5, 5, 1\}, 20]$. The symmetry group $H = G(D\delta_5)$, then, is the dihedral D_{10} generated by α_0, α_1, α_2, satisfying the relations:

$$1 = \alpha_0^2 = \alpha_1^2 = \alpha_2^2 = (\alpha_0\alpha_2)^2 = (\alpha_0\alpha_1)^2 = (\alpha_1\alpha_2)^{10} = \alpha_0(\alpha_1\alpha_2)^5\,, \tag{1}$$

Thus $\mathcal{M}$ must be of type $\{2b, 2c, 10a\}$ for some a, b, c. To show that $\mathcal{M}$ or $D_{01}(\mathcal{M})$ is a map, we need to show that $b = 1$ or $c = 1$.

Let β_0, β_1, β_2 be the corresponding generators in $G = G(\mathcal{M})$, and let R stand for $\beta_0\beta_1$. The hypermap $\mathcal{M}$ is a covering of $D\delta_5$, and so $\alpha_i \mapsto \beta_i$ defines a homomorphism of G onto H. Because both $\mathcal{M}$ and $D\delta_5$ have the same number of faces, the kernel N must be the cyclic group $\langle R^2 \rangle$.

The last relation in (1) implies that for some k in $\{1, 2, \ldots, c\}$,

$$\beta_0(\beta_1\beta_2)^5 = R^{2k}$$

This implies that $(\beta_1\beta_2)^5 = \beta_0 R^{2k} = \beta_0(\beta_0\beta_1)^{2k}$ is an involution, so $a = 1$. Moreover, $(\beta_1\beta_2)(\beta_1\beta_2)^4 = (\beta_1\beta_2)^5 = \beta_0 R^{2k}$, so $\beta_2(\beta_1\beta_2)^4 = \beta_1\beta_0 R^{2k} = R^{2k-1}$. Thus, R^{2k-1} is an involution, and so $2k - 1 \equiv c(\bmod 2c)$. Then c must be odd, and

$$\beta_2(\beta_1\beta_2)^4 = R^c = R^{-c} \tag{2}$$

It must also be true that for some j in $\{1, 2, \ldots, c\}$,

$$(\beta_0\beta_2)^2 = R^{2j}$$

This implies that b divides c, and that

$$\beta_2\beta_0\beta_2 = \beta_0 R^{2j} \tag{3}$$

Now, from (2), we see that

$$\beta_2(\beta_1\beta_2)^4\beta_0 = R^c\beta_0 = \beta_0 R^{-c} = \beta_0 R^c = \beta_0\beta_2(\beta_1\beta_2)^4$$

and so

$$\beta_0 \text{ commutes with } \beta_2(\beta_1\beta_2)^4 \tag{4}$$

From (2) and (3), we see that

$$(\beta_2\beta_0\beta_2)(\beta_2(\beta_1\beta_2)^4) = (\beta_0 R^{2j})(R^c)$$

i.e., that

$$\beta_2\beta_0(\beta_1\beta_2)^4 = \beta_0 R^{2j+c}$$

which shows us that $\beta_2\beta_0(\beta_1\beta_2)^4$ is an involution and, thus, so is $\beta_0(\beta_1\beta_2)^4\beta_2 = \beta_0(\beta_1\beta_2)^3\beta_1$. So $\beta_0(\beta_1\beta_2)^3\beta_1 = (\beta_0(\beta_1\beta_2)^3\beta_1)^{-1} = \beta_1(\beta_1\beta_2)^{-3}\beta_0 = \beta_1(\beta_2\beta_1)^3\beta_0 = (\beta_1\beta_2)^3\beta_1\beta_0$. In short,

$$\beta_0 \text{ commutes with } (\beta_1\beta_2)^3\beta_1 \tag{5}$$

Combining (4) and (5), β_0 commutes with $(\beta_1\beta_2)^3\beta_1\beta_2(\beta_1\beta_2)^4 = (\beta_1\beta_2)^8 = (\beta_1\beta_2)^{-2}$ and so

$$\beta_0 \text{ commutes with } (\beta_1\beta_2)^2 .$$

It follows from (4) that $\beta_2(\beta_1\beta_2)^4 = \beta_0\beta_2(\beta_1\beta_2)^4\beta_0 = \beta_0\beta_2\beta_0(\beta_1\beta_2)^4$, and so that $\beta_2 = \beta_0\beta_2\beta_0$, or, $(\beta_0\beta_2)^2 = I$. Since $e = 2$, the hypermap must be a map. □

Combining this theorem with Theorem 16 we get

Corollary 21 *The reflexible non-orientable hypermaps with 5 faces are the maps* opp$B^*(5, 2k)$ *for odd* k.

7. Final Note

This paper support and establish results which are important for the paper "Surfaces having no regular hypermaps" [5] which classifies the non-orientable surfaces from negative Euler characteristic 2 up to 50 which do not support any reflexible map, or hypermap. The negative Euler characteristic N is given by $N = \frac{|G|}{2}(1-(\frac{1}{e}+\frac{1}{p}+\frac{1}{q}))$ and so the order of the automorphism group G of a reflexible hypermap (or map) on a surface of negative Euler characteristic $N > 0$ is given by

$$|G| = \frac{2N}{1-(\frac{1}{e}+\frac{1}{p}+\frac{1}{q})} \leq \frac{2N}{1-(\frac{1}{2}+\frac{1}{3}+\frac{1}{7})} = 84N$$

This bound implies that the number of reflexible maps and hypermaps on an orientable or non-orientable surface is finite, allowing us to list all possible data lines for a given $N > 0$. The classification in that paper was carried out by eliminating all feasible data lines for $N = 16$, 22, 25, 37 and 46 — the only non-orientable surfaces with $N > 1$ up to 50 for which no examples of reflexible maps were known.

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