Journal for Geometry and Graphics
Volume 7 (2003), No. 1, 1–21.

Homology and Orthology with Triangles for Central Points of Variable Flanks

Zvonko Čerin

Kopernikova 7, HR 10010 Zagreb, Croatia
email: cerin@math.hr

Abstract. Here we continue from the paper [1] our study of the following geometric configuration. Let BR_1R_2C, CR_3R_4A, AR_5R_6B be rectangles build on sides of a triangle ABC such that oriented distances $|BR_1|$, $|CR_3|$, $|AR_5|$ are $\lambda\,|BC|$, $\lambda\,|CA|$, $\lambda\,|AB|$ for some real number λ. We explore the homology and orthology relation of the triangle on central points of triangles AR_4R_5, BR_6R_1, CR_2R_3 (like centroids, circumcenters, and orthocenters) and several natural triangles associated to ABC (as its orthic, anticomplementary, and complementary triangle). In some cases we can identify which curves trace their homology and orthology centers and which curves envelope their homology axis.

Key Words: triangle, extriangle, flanks, central points, Kiepert parabola, Kiepert hyperbola, homologic, orthologic, envelope, anticomplementary, complementary, Brocard, orthic, tangential, Euler, Torricelli, Napoleon

MSC 2000: 51N20

1. Introduction

This paper is the continuation of the author's preprint [1] where an improvement of three recent papers [3], [6], and [7] by L. Hoehn, F. van Lamoen, and C.R. Pranesachar and B.J. Venkatachala was presented. These articles considered independently the classical geometric configuration with squares BS_1S_2C, CS_3S_4A, and AS_5S_6B erected on sides of a triangle ABC and studied relationships among central points (see [4]) of the base triangle $\tau = ABC$ and of three interesting triangles $\tau_A = AS_4S_5$, $\tau_B = BS_6S_1$, $\tau_C = CS_2S_3$ (called *flanks* in [6] and *extriangles* in [3]). In order to describe their main results, recall that triangles ABC and XYZ are *homologic* provided lines AX, BY, and CZ are concurrent. The point P in which they concur is their homology *center* and the line ℓ containing intersections of pairs of lines (BC, YZ), (CA, ZX), and (AB, XY) is their homology *axis*. In this situation we use the notation $ABC \underset{\ell}{\overset{P}{\bowtie}} XYZ$ where ℓ or both ℓ and P can be omitted. Let $X_i = \underline{X}_i(\tau)$, $X_i^j = \underline{X}_i(\tau_j)$ (for $j = A, B, C$), and $\sigma_i = X_i^A X_i^B X_i^C$, where $\underline{X}_i$ (for $i = 1, \ldots$) is any of the triangle central point functions from Kimberling's lists [4] or [5].

ISSN 1433-8157/\$ 2.50 © 2003 Heldermann Verlag

Instead of homologic, homology center, and homology axis many authors use the terms *perspective*, *perspector*, and *perspectrix*. Also, it is customary to use letters I, G, O, H, F, K, and L instead of $X_1, X_2, X_3, X_4, X_5, X_6$, and X_{20} to denote the incenter, the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the symmedian (or Grebe-Lemoine) point, and the de Longchamps point (the reflection of H about O, respectively.

In [3] HOEHN proved $\tau \bowtie \sigma_3$ and $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i, j) = (1, 1), (2, 4), (4, 2)$. In [7] C.R. PRANESACHAR and B.J. VENKATACHALA add some new results because they show that $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i, j) = (1, 1), (2, 4), (4, 2), (3, 6), (6, 3)$. Moreover, they observe that if $\tau \overset{X}{\bowtie} X_A X_B X_C$ and Y, Y_A, Y_B, and Y_C are isogonal conjugates of points X, X_A, X_B, and X_C with respect to triangles τ, τ_A, τ_B, and τ_C, respectively, then $\tau \overset{Y}{\bowtie} Y_A Y_B Y_C$. Finally, they also answer in negative the question by Prakash MULABAGAL of Pune if $\tau \bowtie XYZ$, where X, Y, and Z are points where incircles of triangles τ_A, τ_B, and τ_C touch the sides opposite to A, B, and C, respectively.

In [6] VAN LAMOEN says that X_i *befriends* X_j when $\tau \overset{X_j}{\bowtie} \sigma_i$ and shows first that $\tau \overset{X_j}{\bowtie} \sigma_i$ implies $\tau \overset{X_n}{\bowtie} \sigma_m$ where X_m and X_n are isogonal conjugates of X_i and X_j. Also, he proves that $\tau \overset{X_j}{\bowtie} \sigma_i$ is equivalent to $\tau \overset{X_i}{\bowtie} \sigma_j$ and that $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i, j) = (1, 1), (2, 4), (3, 6), (4, 2), (6, 3)$. Then he notes that $\tau \overset{K(\frac{\pi}{2}-\phi)}{\bowtie} K(\phi)$, where $K(\phi)$ denotes the homology center of τ and the Kiepert triangle formed by apexes of similar isosceles triangles with the base angle ϕ erected on the sides of ABC. This result implies that $\tau \overset{X_i}{\bowtie} \sigma_i$ for $i = 485, 486$ (Vecten points — for $\phi = \pm\frac{\pi}{4}$) and $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i, j) = (13, 17), (14, 18)$ (isogonic or Fermat points X_{13} and X_{14} — for $\phi = \pm\frac{\pi}{3}$ and Napoleon points X_{17} and X_{18} — for $\phi = \pm\frac{\pi}{6}$). Finally, VAN LAMOEN observed that the Kiepert hyperbola (the locus of $K(\phi)$) befriends itself; so does its isogonal transform, the Brocard axis OK.

The idea of our generalization in [1] was in replacing squares with rectangles whose ratio of nonparallel sides is constant (see Fig. 1). More precisely, let BR_1R_2C, CR_3R_4A, AR_5R_6B be rectangles build on sides of a triangle ABC such that the oriented distances $|BR_1|$, $|CR_3|$, $|AR_5|$ are $\lambda\,|BC|$, $\lambda\,|CA|$, $\lambda\,|AB|$ for some real number λ. Let $\tau_A^\lambda = AR_4R_5$, $\tau_B^\lambda = BR_6R_1$, and $\tau_C^\lambda = CR_2R_3$ and let $X_i^j(\lambda)$ (for $j = A, B, C$) and σ_i^λ have obvious meaning. The most important central points have their traditional notations so that we shall often use these because they might be easier to follow. For example, $H^A(\lambda)$ is the orthocenter of the flank τ_A^λ and σ_G^λ is the triangle $G^A(\lambda)G^B(\lambda)G^C(\lambda)$ on the centroids of flanks.

Since triangles AS_4S_5 and AR_4R_5 are homothetic and the vertex A is the center of this homothety (and similarly for pairs BS_6S_1, BR_6R_1 and CS_2S_3, CR_2R_3) we conclude that $\{A,\, X_i^A,\, X_i^A(\lambda)\}$, $\{B,\, X_i^B,\, X_i^B(\lambda)\}$, and $\{C,\, X_i^C,\, X_i^C(\lambda)\}$ are sets of collinear points so that most statements from [3], [7], and [6] concerning triangles σ_i are also true for triangles σ_i^λ.

But, since instead of a single square on each side we have a family of rectangles it is possible to get additional information. The results in [1] explored cases when the base triangle τ is either homologic or orthologic with the triangles σ_i^λ.

The purpose of this paper is to investigate the relations of both homology and orthology for triangles σ_i^λ with some important triangles associated to τ like its anticomplementary triangle τ_a, the first Brocard triangle τ_b, the Euler triangle τ_E, the complementary triangle τ_g, the orthic triangle τ_h, the tangential triangle τ_t, the Torricelli triangles τ_u and τ_v, and the Napoleon triangles τ_x and τ_y.

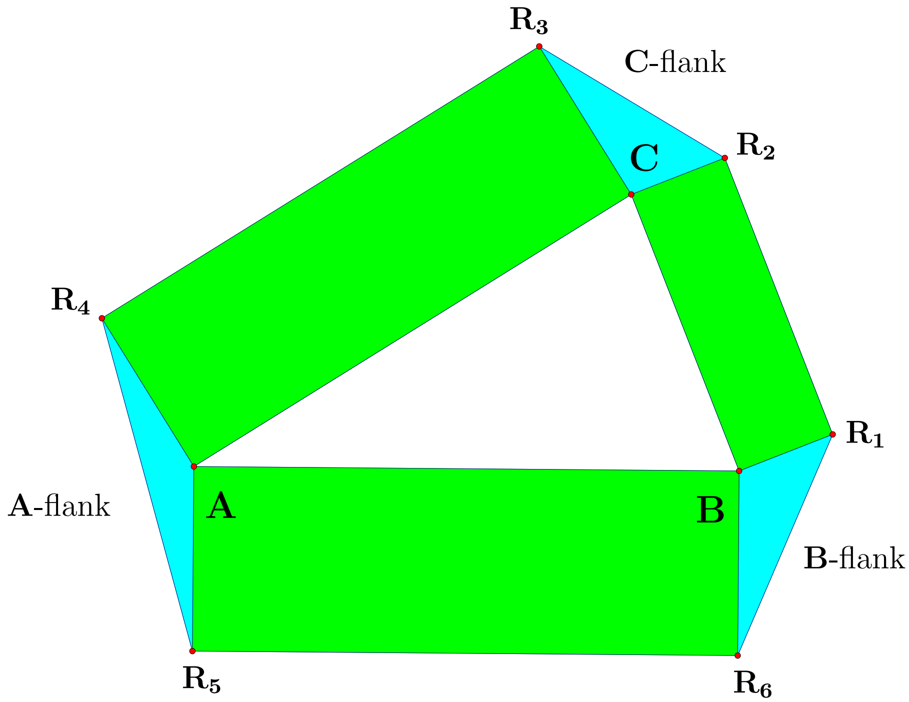

Figure 1: The triangle ABC with three rectangles and three flanks

2. The anticomplementary triangle τ_a

Let τ_a denote the anticomplementary triangle $A_a B_a C_a$ of ABC whose vertices are intersections of parallels to sidelines through opposite vertices.

Theorem 1 *For every $\lambda \in \mathbb{R}$ the triangles τ_a and σ_G^λ are homologic and their homology centers trace the Kiepert hyperbola of τ_a (see Fig. 2).*

Proof: In our proofs we shall use trilinear coordinates. Recall that the *actual trilinear coordinates* of a point P with respect to the triangle ABC are signed distances f, g, h of P from the lines BC, CA, and AB. We shall regard P as lying on the positive side of BC if P lies on the same side of BC as A. Similarly, we shall regard P as lying on the positive side of CA if it lies on the same side of CA as B, and similarly with regard to the side AB. Ordered triples $x : y : z$ of real numbers proportional to $(f,\, g,\, h)$ (that is such that $x = mf$, $y = mg$, and $z = mh$, for some real number m different from zero) are called *trilinear coordinates* of P. The advantage of their use is that a high degree of symmetry is present so that it usually suffices to describe part of the information and the rest is self evident. For example, when we write $X_1(1)$ or $I(1)$ or simply say I is 1 this indicates that the incenter has trilinear coordinates $1 : 1 : 1$. We gave only the first coordinate while the other two are cyclic permutations of the first. Similarly, $X_2\left(\frac{1}{a}\right)$ or $G\left(\frac{1}{a}\right)$ say that the centroid has trilinears $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, where a, b, c are the lengths of sides of ABC.

The expressions in terms of sides a, b, c can be shortened using the following notation.

$$d_a = b - c, \quad d_b = c - a, \quad d_c = a - b, \quad z_a = b + c, \quad z_b = c + a, \quad z_c = a + b,$$
$$t = a + b + c, \quad t_a = b + c - a, \quad t_b = c + a - b, \quad t_c = a + b - c,$$
$$m = abc, \quad m_a = bc, \quad m_b = ca, \quad m_c = ab, \quad T = \sqrt{t t_a t_b t_c}.$$

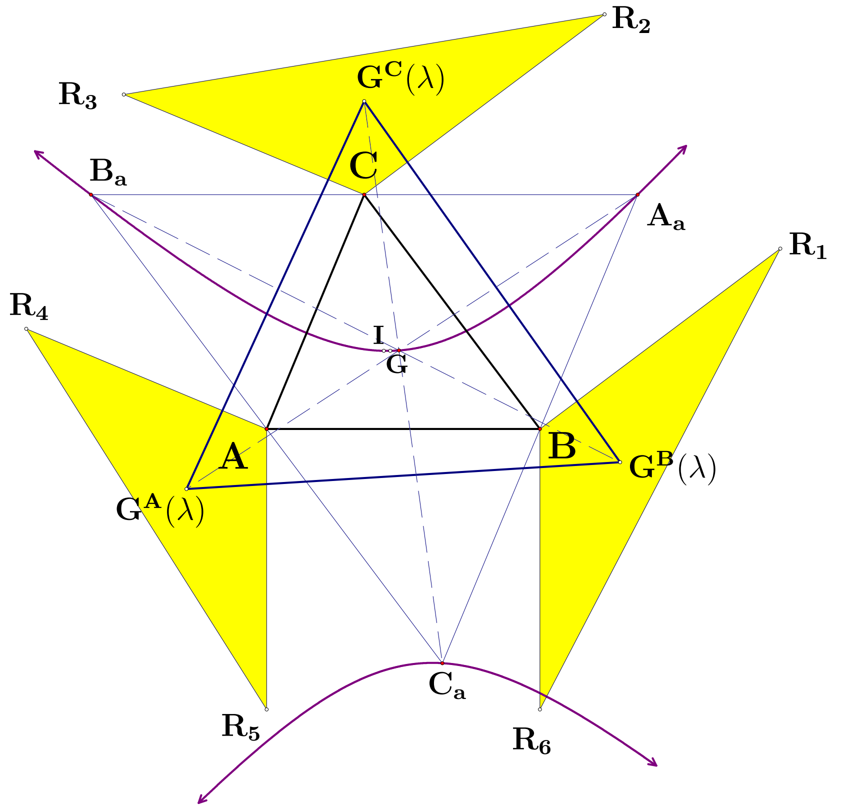

Figure 2: The homology centers of triangles τ_a and σ_G^λ trace the Kiepert hyperbola of τ_a (Theorem 1)

For an integer n, let $t_n = a^n + b^n + c^n$ and $d_{na} = b^n - c^n$ and similarly for other cases. Instead of t_2, t_{2a}, t_{2b}, and t_{2c} we write k, k_a, k_b, and k_c. Let ω denote the Brocard angle of ABC.

In order to achieve even greater economy in our presentation, we shall describe coordinates or equations of only one object from triples of related objects and use cyclic permutations φ and ψ to obtain the rest. For example, the first vertex A_a of the anticomplementary triangle $A_aB_aC_a$ of ABC has trilinears $-\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$. Then the trilinears of B_a and C_a need not be described because they are easily figured out and memorized by relations $B_a = \varphi(A_a)$ and $C_a = \psi(A_a)$. One must remember always that transformations φ and ψ are not only permutations of letters but also of positions, i.e., $\varphi(a,\, b,\, c,\, 1,\, 2,\, 3 \to b,\, c,\, a,\, 2,\, 3,\, 1)$ and $\psi(a,\, b,\, c,\, 1,\, 2,\, 3 \to c,\, a,\, b,\, 3,\, 1,\, 2)$. Therefore, the trilinears of B_a and C_a are $\frac{1}{a} : -\frac{1}{b} : \frac{1}{c}$ and $\frac{1}{a} : \frac{1}{b} : -\frac{1}{c}$.

The trilinears of the points R_1 and R_2 are equal to

$$-2\lambda m : c(T + \lambda k_c) : \lambda b k_b \quad \text{and} \quad -2\lambda m : \lambda c k_c : b(T + \lambda k_b)$$

(while $R_3 = \varphi(R_1)$, $R_4 = \varphi(R_2)$, $R_5 = \psi(R_1)$, and $R_6 = \psi(R_2)$). It follows that the centroid $X_2^A(\lambda)$ or $G^A(\lambda)$ of the triangle AR_4R_5 is $\dfrac{3T + 2a^2\lambda}{-a} : \dfrac{k_c\lambda}{b} : \dfrac{k_b\lambda}{c}$.

Hence, the line $A_aG^A(\lambda)$ is

$$\frac{3T + 6d_az_a}{a}\, x + \frac{T\lambda + 3(k_a + 2b^2)}{-b}\, y + \frac{T\lambda + 3(k_a + 2c^2)}{c}\, z = 0.$$

It follows that the homology center of τ_a and σ_G^λ is $\frac{(T^2+2k_bk_c)\lambda^2-6Tk_a\lambda-9T^2}{a}$. This point traces the conic with the equation

$$a^2(b^2-c^2)x^2+b^2(c^2-a^2)y^2+c^2(a^2-b^2)z^2=0 \text{ or (in shorter notation) } \sum a^2 d_a z_a x^2=0.$$

Since the vertices of τ_a, the common centroid G of τ and τ_a, and the orthocenter of the anticomplementary triangle $X_{20}\left(\frac{T^2+2k_bk_c}{a}\right)$ (known also as the de Longchamps point L of ABC) are on this curve, we conclude that it is the Kiepert hyperbola of the anticomplementary triangle. □

Theorem 2 *The homology axis of τ_a and σ_G^λ envelope the Kiepert parabola of τ_a.*

Proof: The line B_aC_a has the equation $by+cz=0$ while the line $G^B(\lambda)G^C(\lambda)$ is

$$a\left(T\lambda^2+6(b^2+c^2)\lambda+9T\right)x+b\lambda(T\lambda+3k_c)y+c\lambda(T\lambda+3k_b)z=0.$$

It follows that their intersection is $\frac{6d_az_a}{a} : -\frac{T\lambda^2+6(b^2+c^2)\lambda+9T}{b} : \frac{T\lambda^2+6(b^2+c^2)\lambda+9T}{c}$. Hence, the homology axis of τ_a and σ_G^λ has the following equation

$$\sum a[T^2(81-\lambda^4)-6T\lambda(k+a^2)(\lambda^2+9)+18\lambda^2(T^2-4a^2(b^2+c^2))]\,x=0.$$

It envelopes $\sum[a^2(k_a-a^2)^2\,x^2+2bc(T^2+b^2k_c+c^2k_b+m_a^2)\,y\,z]=0$. In order to see that this is the Kiepert parabola of τ_a it suffices to check that lines B_aC_a, C_aA_a, A_aB_a, the line at infinity, and the Lemoine line of τ_a (the homology axis of τ_a and its tangential triangle) are its tangents (see [2]).

Indeed, $by+cz=0$, $cz+ax=0$, $ax+by=0$, $\sum ax=0$, and $\sum a^3(b^2+c^2)\,x=0$ are their equations. By solving in one variable any of them and substituting into the left hand side of the equation of the above conic we get remaining variables in a complete square which means that these lines have a point of tangency with the conic and our proof is accomplished. □

Recall that triangles ABC and XYZ are *orthologic* provided the perpendiculars at vertices of ABC onto sides YZ, ZX, and XY of XYZ are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC,\,XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto sides BC, CA, and AB of ABC are concurrent at the point $[XYZ,\,ABC]$.

Since G befriends H it is clear that triangles τ_a and σ_G^λ are orthologic and $[\sigma_G^\lambda,\,\tau_a]=H$ (the orthocenter). Our next result shows that points $[\tau_a,\,\sigma_G^\lambda]$ trace the Kiepert hyperbola of τ_a.

Theorem 3 *The locus of the orthology centers $[\tau_a,\,\sigma_G^\lambda]$ of τ_a and σ_G^λ is the Kiepert hyperbola of $A_aB_aC_a$ (see Fig. 3).*

Proof: The perpendicular from A_a onto the line $G^B(\lambda)G^C(\lambda)$ has the equation

$$6ad_{2a}x+b(T\lambda+3k_c)y-c(T\lambda+3k_b)z=0.$$

Therefore, the orthology center $[\tau_a,\,\sigma_G^\lambda]$ is $\frac{T^2\lambda^2+6T\lambda k_a+9(T^2-2k_bk_c)}{a}$. This point traces the conic with the equation $\sum a^2d_{2a}x^2=0$ that was recognized as the Kiepert hyperbola of τ_a in the proof of Theorem 1. □

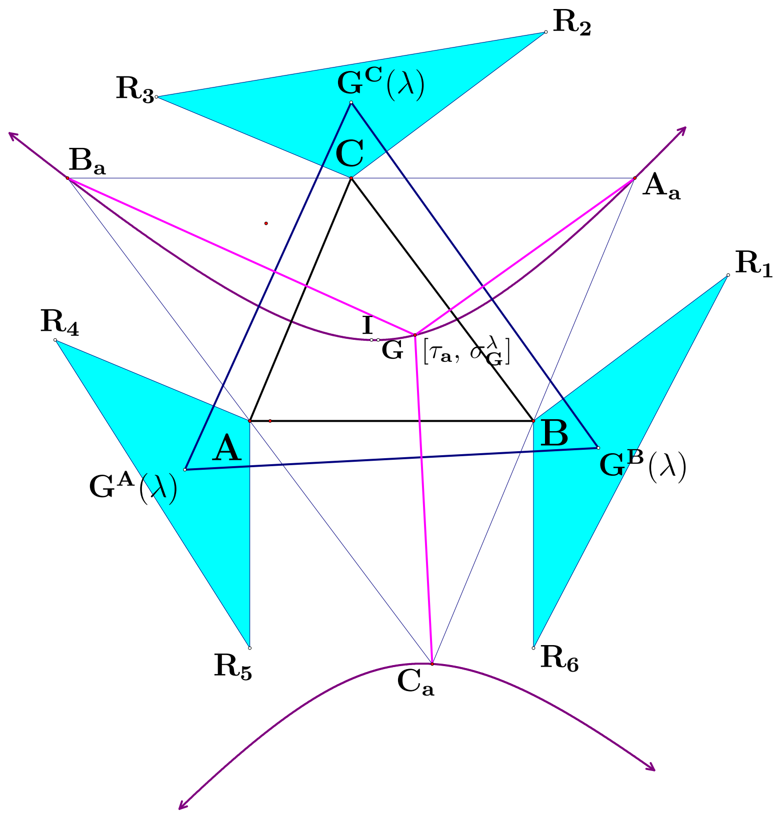

Figure 3: The orthology center $[\tau_a,\, \sigma_G^\lambda]$ of triangles τ_a and σ_G^λ traces the Kiepert hyperbola of τ_a (Theorem 3)

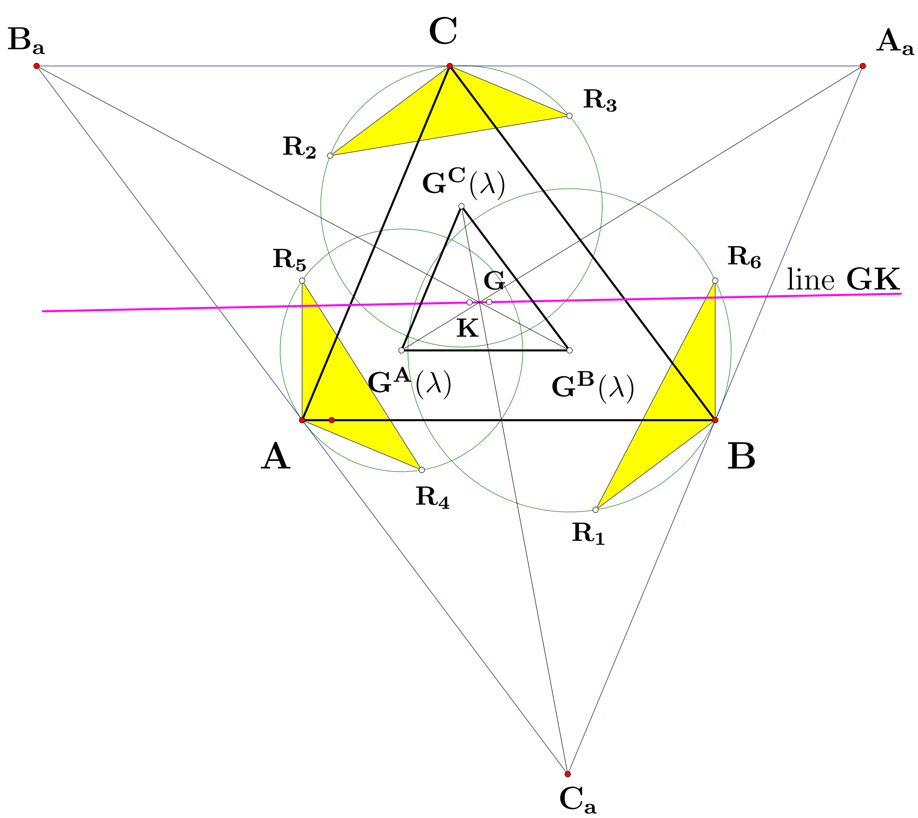

Figure 4: The homology centers of τ_a and σ_O^λ trace the line GK (Theorem 4)

Theorem 4 *For every $\lambda \in \mathbb{R}$ the triangles τ_a and σ_O^λ are homologic and their homology centers trace the line GK (see Fig. 4).*

Proof: The point $\frac{T + z_{2a}\lambda}{-m} : \frac{\lambda}{c} : \frac{\lambda}{b}$ is the circumcenter $O^A(\lambda)$ of the flank AR_4R_5. The line $A_aO^A(\lambda)$ is $a\lambda d_{2a}x + b(T+\lambda b^2)y - c(T+\lambda c^2)z = 0$. Hence, $\frac{\lambda k_a + T}{a}$ is the homology center of τ_a and σ_O^λ. It traces the line $\sum a d_{2a} x = 0$ that goes through points $G\left(\frac{1}{a}\right)$ (the centroid) and $K(a)$ (the symmedian or Grebe-Lemoine point). □

Theorem 5 *For every $\lambda \in \mathbb{R} \setminus \{-\cot\omega\}$, the triangles τ_a and σ_O^λ are orthologic. The orthology center $[\,\tau_a,\, \sigma_O^\lambda\,]$ is the de Longchamps point L or X_{20} of τ (or the orthocenter of τ_a) while the orthology centers $[\,\sigma_O^\lambda,\, \tau_a\,]$ trace the line HK (see Fig. 5).*

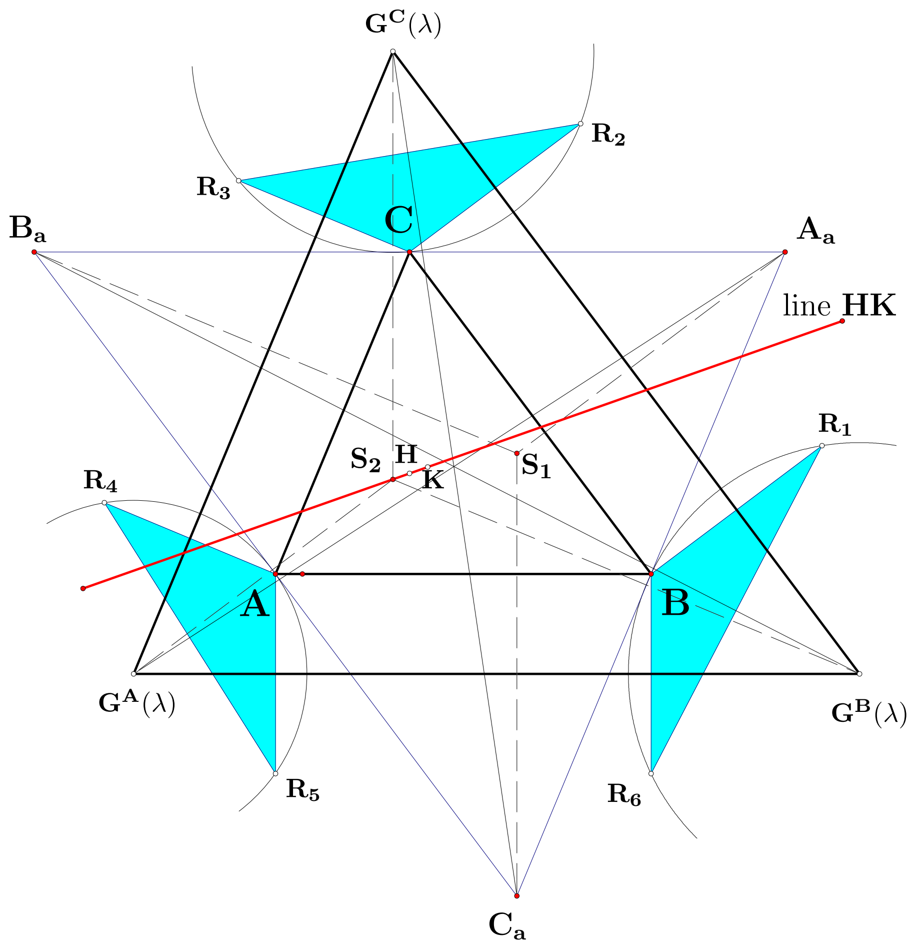

Figure 5: The orthology center $S_1 = [\tau_a,\, \sigma_O^\lambda]$ is X_{20} while the orthology centers $S_2 = [\sigma_O^\lambda,\, \tau_a]$ trace the line HK (Theorem 5)

Proof: The triangle σ_O^λ degenerates to a point if and only if $\lambda = -\cot\omega$. Since the triangles τ and σ_O^λ are homothetic and their center of similitude is the symmedian point K, the triangles τ_a and σ_O^λ have parallel corresponding sides. It follows that τ_a and σ_O^λ are orthologic and that $[\,\tau_a,\, \sigma_O^\lambda\,] = X_{20}$. The perpendicular from $O^A(\lambda)$ onto the line B_aC_a is

$$a\lambda d_{2a} k_a x + b(\lambda d_{2a} k_a - T k_b) y + c(\lambda d_{2a} k_a + T k_c) z = 0.$$

Hence, $[\sigma_O^\lambda, \tau_a]$ has coordinates $\frac{(k k_b k_c - a^2 T)\lambda + T k_b k_c}{a}$. We infer that this orthology center traces the line HK because we get its equation $\sum a d_{2a} k_a^2 x = 0$ by eliminating the parameter λ. □

Since H befriends G and the line $AH^A(\lambda)$ is the median AG that goes through the point A_a, it is clear that triangles τ_a and σ_H^λ are homologic and that their homology center is G (the centroid). The axis of these homologies envelope a complicated quartic.

Theorem 6 *The locus of the orthology centers $[\tau_a, \sigma_H^\lambda]$ of τ_a and σ_H^λ is the Kiepert hyperbola of τ_a. The locus of the orthology centers $[\sigma_H^\lambda, \tau_a]$ of σ_H^λ and τ_a is the line HK (see Fig. 6).*

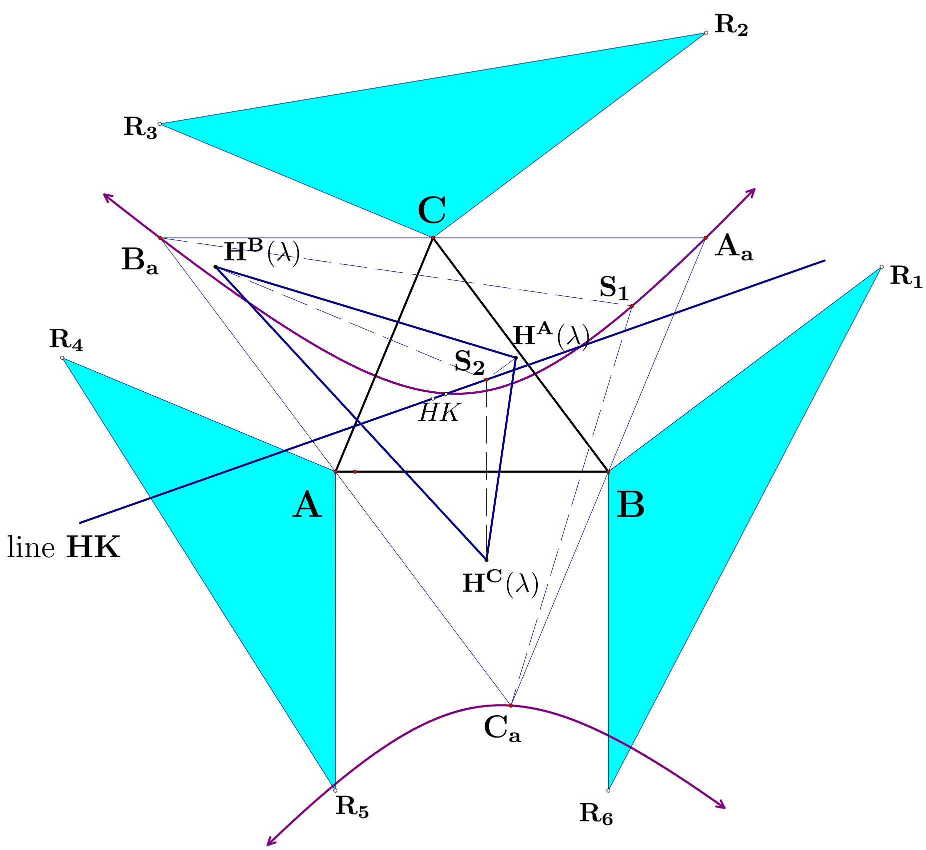

Figure 6: The orthology centers $S_1 = [\tau_a, \sigma_H^\lambda]$ and $S_2 = [\sigma_H^\lambda, \tau_a]$ trace the Kiepert hyperbola of τ_a and the line HK (Theorem 6)

Proof: The point $\frac{T - 2\lambda k_a}{a k_a} : \frac{\lambda}{b} : \frac{\lambda}{c}$ is the orthocenter $H^A(\lambda)$ of the flank AR_4R_5. The line $H^B(\lambda)H^C(\lambda)$ is

$$\frac{3k_bk_c\lambda^2 - 4a^2T\lambda + T^2}{bc}\,x + \frac{\lambda k_b(3k_c\lambda - T)}{ca}\,y + \frac{\lambda k_c(3k_b\lambda - T)}{ab}\,z = 0.$$

The perpendiculars from A_a and $H^A(\lambda)$ onto lines $H^B(\lambda)H^C(\lambda)$ and B_aC_a have equations

$$\frac{2d_az_a(T - 2k\lambda)}{bc}\,x + \frac{Tk_c + (T^2 - 2kk_c)\lambda}{ca}\,y - \frac{Tk_b + (T^2 - 2kk_b)\lambda}{ab}\,z = 0$$

and

$$\frac{2d_az_ak_a\lambda}{bc}\,x + \frac{Tk_b + 2d_az_ak_a\lambda}{ca}\,y + \frac{Tk_c - 2d_az_ak_a\lambda}{ab}\,z = 0.$$

The orthology center $[\,\tau_a,\,\sigma_H^\lambda\,]$ is

$$bc\,[(8a^2kT^2 - 16a^2k^2k_a + 8kk_aT^2 - T^4)\lambda^2 - 2T((6a^2-k)T^2 + \\ +4k_a(k_bk_c - 4a^4))\lambda + T^2(2k_bk_c - T^2)]$$

while the orthology center $[\,\sigma_H^\lambda,\,\tau_a\,]$ has coordinates $Tk_bk_c + 2(a^2T^2 - kk_bk_c)\,\lambda$. In order to see what curves trace these orthology centers we must eliminate the parameter λ. For $[\,\tau_a,\,\sigma_H^\lambda\,]$ we get the equation for the Kiepert hyperbola of τ_a as in Theorem 1 and for $[\,\sigma_H^\lambda,\,\tau_a\,]$ the equation for the line HK as in Theorem 5. □

3. The first Brocard triangle τ_b

Let $\tau_b = A_bB_bC_b$ denote the first Brocard triangle of ABC. Its vertices are projections of the symmedian point K onto perpendicular bisectors of sides.

Theorem 7 *For every $\lambda \in \mathbb{R}$ the triangles τ_b and σ_G^λ are homologic and their homology centers trace the Kiepert hyperbola of τ_b (see Fig. 7).*

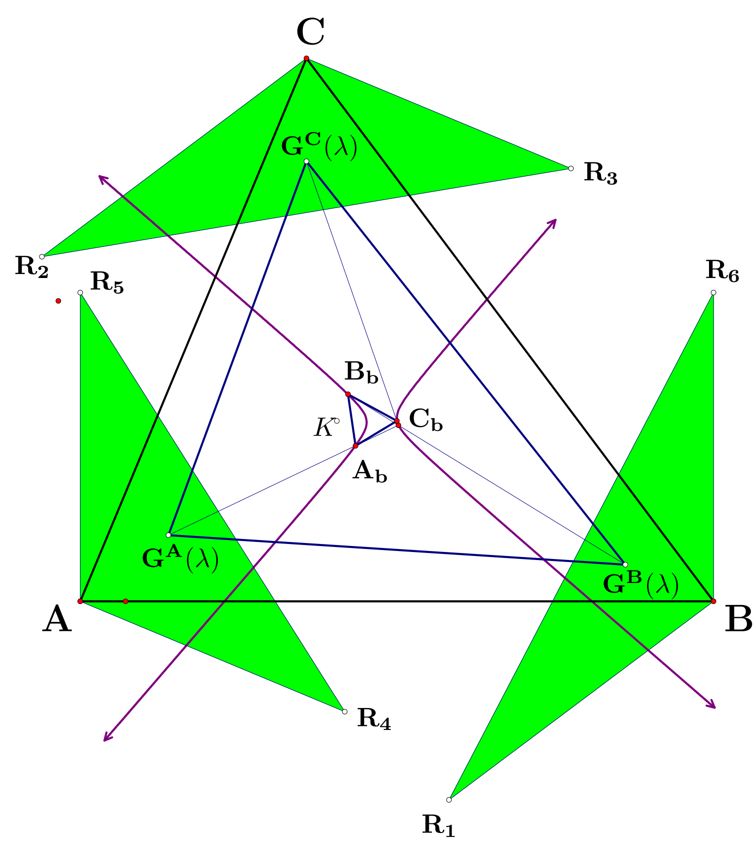

Figure 7: The homology centers of τ_b and σ_G^λ trace the Kiepert hyperbola of τ_b (Theorem 7)

Proof: The line $A_bG^A(\lambda)$ is

$$\frac{kd_az_a\lambda}{bc}\,x + \frac{a^2k\lambda + 3b^2T}{ca}\,y - \frac{a^2k\lambda + 3c^2T}{ab}\,z = 0$$

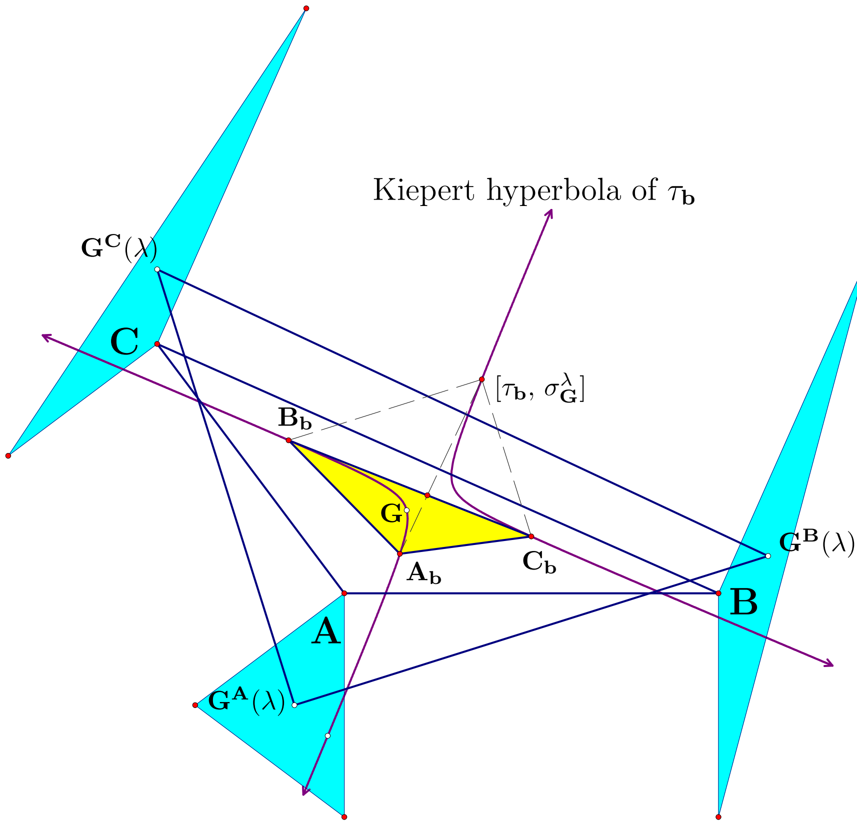

Figure 8: The points $[\tau_b,\, \sigma^{\lambda}_{G}]$ trace the Kiepert hyperbola of τ_b (Theorem 8)

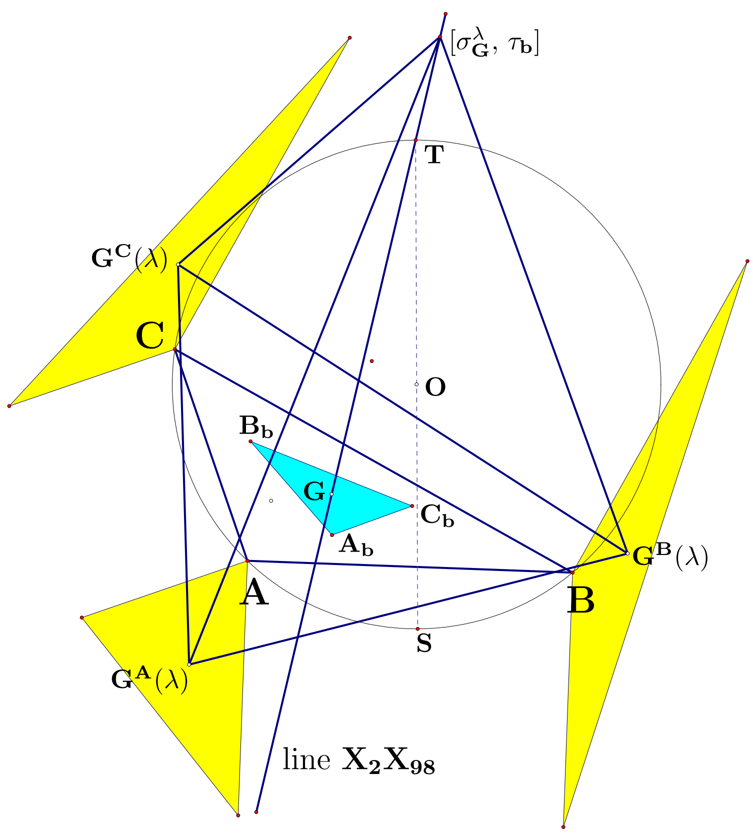

Figure 9: The orthology centers $[\sigma^{\lambda}_{G},\, \tau_b]$ trace the line X_2X_{98} (Theorem 8)

since A_b is $abc : c^3 : b^3$. Hence, $\frac{a^2 k_a k^2 \lambda^2 + 3kT(b^4+c^4)\lambda + 9m_a^2 T^2}{a}$ is the homology center of τ_b and σ_G^λ. This center will trace the curve $\sum d_a z_a (a^4 x^2 + m_a(b^2+c^2)yz) = 0$ while the parameter λ changes. Since the vertices of τ_b, the common centroid $G\left(\frac{1}{a}\right)$ of τ and τ_b, and the orthocenter $2a(T^2 - a^2 k_a - 2m_a^2) - \frac{k k_b k_c}{a}$ of the first Brocard triangle τ_b are on this curve, we conclude that it is the Kiepert hyperbola of τ_b. Notice that the circumcenter $O(a\,k_a)$ and the 3rd Brocard point $X_{76}\left(\frac{1}{a^3}\right)$ are also on this hyperbola. □

Theorem 8 *For every $\lambda \in \mathbb{R}$ the triangles τ_b and σ_G^λ are orthologic. The locus of the orthology centers $[\tau_b,\, \sigma_G^\lambda]$ and $[\sigma_G^\lambda,\, \tau_b]$ is the Kiepert hyperbola of τ_b and the line X_2X_{98} joining the centroid with the Tarry point of ABC, respectively (see Figs. 8 and 9).*

Proof: The perpendicular from A_b onto the line $G^B(\lambda)G^C(\lambda)$ has the equation

$$a d_{2a}(T\lambda + 3k)\,x + b(T z_{2c}\lambda + 3a^2 k)\,y - c(T z_{2c}\lambda + 3a^2 k)\,z = 0.$$

Therefore, the orthology center $[\tau_b,\, \sigma_G^\lambda]$ is $\frac{T^2(2\,k - a^2)\lambda^2 + 6T\lambda(a^2 k_a + z_{4a}) + 9a^2 k k_a}{a}$. This point traces the conic with the equation $\sum d_{2a}(a^4 x^2 + m_a z_{2a} yz) = 0$ that was recognized as the Kiepert hyperbola of τ_b in the proof of Theorem 7.

The perpendicular from $G^A(\lambda)$ onto the line B_bC_b has the equation

$$a d_{2a} T x + b\left(d_{2a} T\lambda - 3(c^4 - a^2\,k_a)\right) y + c(d_{2a} T\lambda + 3(b^4 - a^2\,k_a))z = 0.$$

So, the orthology center $[\,\sigma_G^\lambda,\, \tau_b\,]$ is $\frac{T(k k_b k_c + a^2(a^2 k_a - 2T^2) + 2m^2)\lambda + 3(b^4 - a^2 k_a)(c^4 - a^2 k_a)}{a}$. This point traces the line with the equation $\sum d_{2a}(k_b k_c + a^2 k_a - 2m_a^2)x = 0$. The points $X_2\left(\frac{1}{a}\right)$ and $X_{98}\left(\frac{1}{a(z_{4a} - a^2 z_{2a})}\right)$ are on it. Note that the points $X_{110}\left(\frac{a}{d_{2a}}\right)$ (the focus of the Kiepert parabola), $X_{114}\left(\frac{(a^2\,k - T^2)(z_{4a} - a^2 z_{2a})}{a}\right)$ (the Kiepert antipode), and $X_{125}\left(\frac{d_{2a}^2 k_a}{a}\right)$ (the center of the Jerabek hyperbola) also belong to this line. □

Since the vertices of τ_b are on perpendicular bisectors of sides of τ and triangles τ and σ_O^λ are homothetic it follows that τ_b and σ_O^λ are orthologic and $[\,\tau_b,\, \sigma_O^\lambda\,] = O$.

Theorem 9 *The locus of the orthology centers $[\sigma_O^\lambda,\, \tau_b]$ of the triangles σ_O^λ and τ_b is the line X_6X_{98} joining the symmedian point X_6 with the Tarry point X_{98} (see Fig. 10).*

Proof: The perpendicular from the point $O^A(\lambda)$ onto the line B_bC_b has the equation

$$\frac{\lambda d_{2a} k_a}{bc}\,x + \frac{\lambda d_{2a} k_a - T k_b}{ca}\,y + \frac{\lambda d_{2a} k_a + T k_c}{ab}\,z = 0.$$

Hence, $\frac{\lambda(k k_b k_c - a^2 T^2) + T k_b k_c}{a}$ is the orthology center $[\sigma_O^\lambda,\, \tau_b]$. It traces the line with the equation

$$\sum a d_{2a}(a^2\,k_a + 2\,m_a^2)(z_{4c} - a^2\,z_{2c})x = 0.$$

One can easily check that the points X_6 and X_{98} are on this line. □

Theorem 10 *For every number $\lambda \in \mathbb{R}$ and $j = 4,\, 5,\, 20$ the triangles τ_b and σ_j^λ are orthologic. The locus of the orthology centers $[\,\tau_b,\, \sigma_j^\lambda\,]$ of the triangles σ_j^λ and τ_b is the Kiepert hyperbola of τ_b. The orthology centers $[\,\sigma_j^\lambda,\, \tau_b\,]$ trace the line X_4X_{98} for $j = 4$, the line through X_{98} parallel to the line X_3X_{66} for $j = 5$, and a line through X_{98} for $j = 20$.*

Proof: We leave proofs of the statements of this theorem to the reader as an exercise because they are similar to the above proofs. □

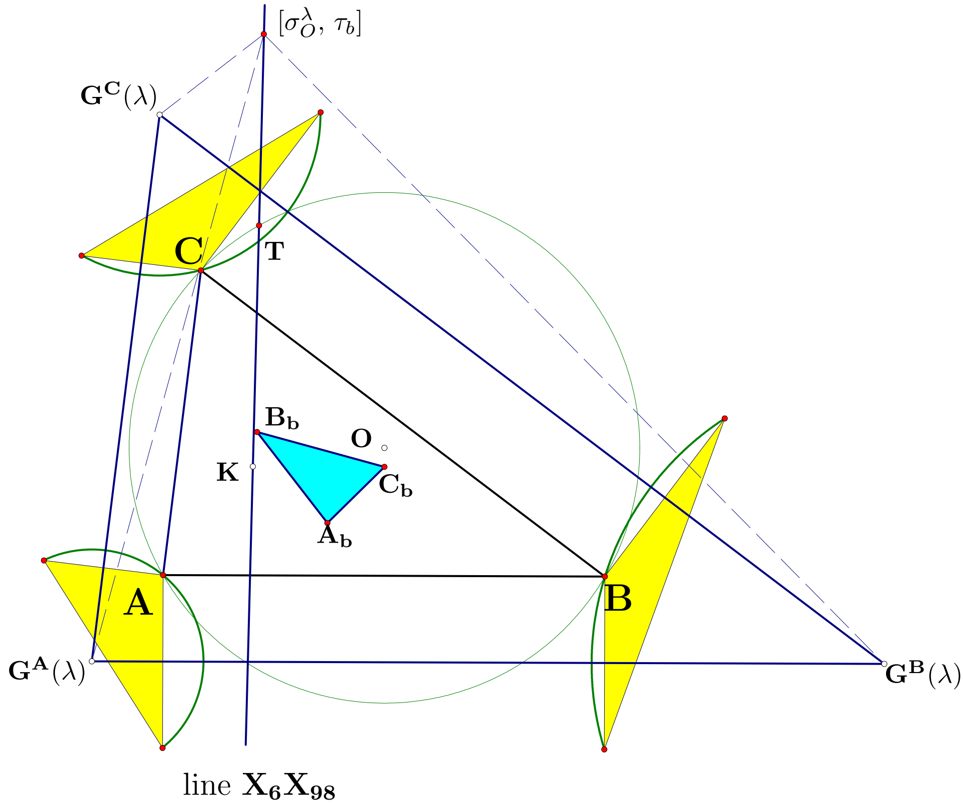

Figure 10: The orthology centers $[\sigma_O^\lambda, \tau_b]$ trace the line X_6X_{98} (Theorem 9)

4. The Euler triangle

The Euler triangle $\tau_E = A_E B_E C_E$ has the midpoints A_E, B_E, C_E of segments AH, BH, CH joining vertices with the orthocenter H as vertices.

Since the lines $AG^A(\lambda)$, $BG^B(\lambda)$, $CG^C(\lambda)$ are altitude lines of ABC it is obvious that for every $\lambda \in \mathbb{R}$ the triangles τ_E and σ_G^λ are homologic and their homology center is the orthocenter H of τ.

Theorem 11 *For every $\lambda \in \mathbb{R}$ the triangles τ_E and σ_O^λ are homologic and their homology centers trace the line HK (see Fig. 11).*

Proof: Since $\frac{T}{a\,k_a} - a : \frac{k_c}{2b} : \frac{k_b}{2c}$ are trilinears of A_E and from the proof of Theorem 4 we know that $\frac{T + (b^2+c^2)\lambda}{-m} : \frac{\lambda}{c} : \frac{\lambda}{b}$ are trilinears of $O^A(\lambda)$, we infer that $A_E O^A(\lambda)$ is

$$\lambda a d_{2a} k_a^2 x + b\,[\lambda M_+ + T k_a k_b]\,y + c[\lambda M_- + T k_a k_c] z = 0,$$

where $M_\pm = z_{2a} a^4 - 2 z_{2a}^2 a^2 + d_{2a}(d_{2a} z_{2a} \pm 4 m_{2a})$. Note that the lines $A_E O^A(\lambda)$, $B_E O^B(\lambda)$, and $C_E O^C(\lambda)$ are concurrent at the point $\frac{k_b k_c T + (3 z_{2a} a^4 - 2 d_{2a}^2 a^2 - z_{2a} d_{2a}^2)\lambda}{a(2k\lambda + T)}$. This point traces the line HK whose equation is $\sum a d_{2a} k_a^2 x = 0$. □

Theorem 12 *For every $\lambda \in \mathbb{R}$ the triangles τ_E and σ_H^λ are homologic and their homology centers trace the Kiepert hyperbola of τ_E (see Fig. 12).*

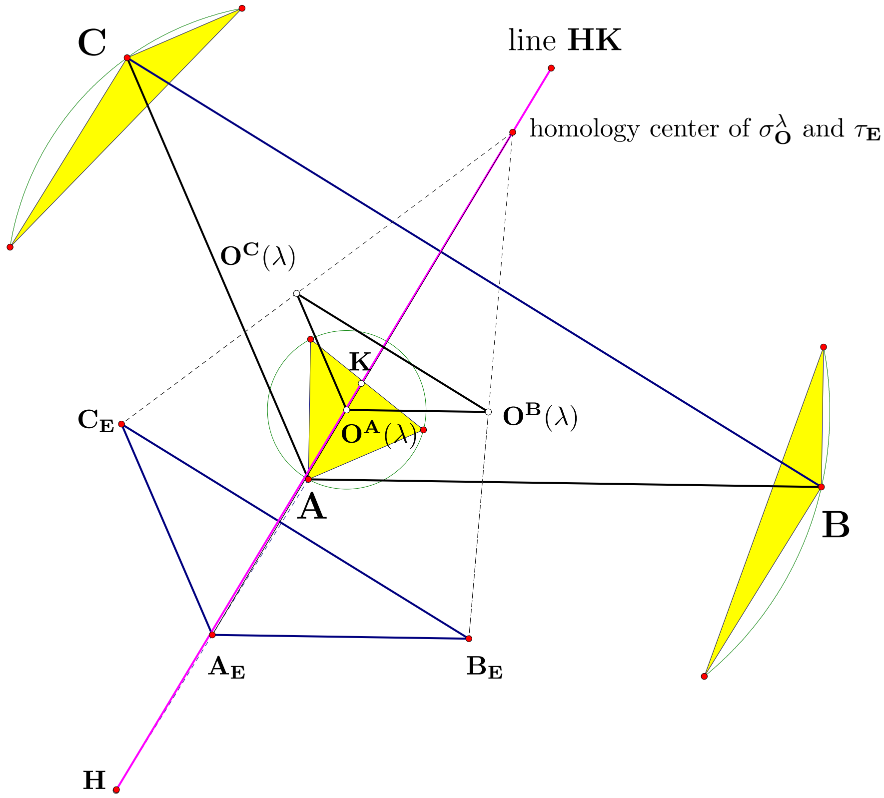

Figure 11: The homology centers of σ_O^λ and τ_E trace the line HK (Theorem 11)

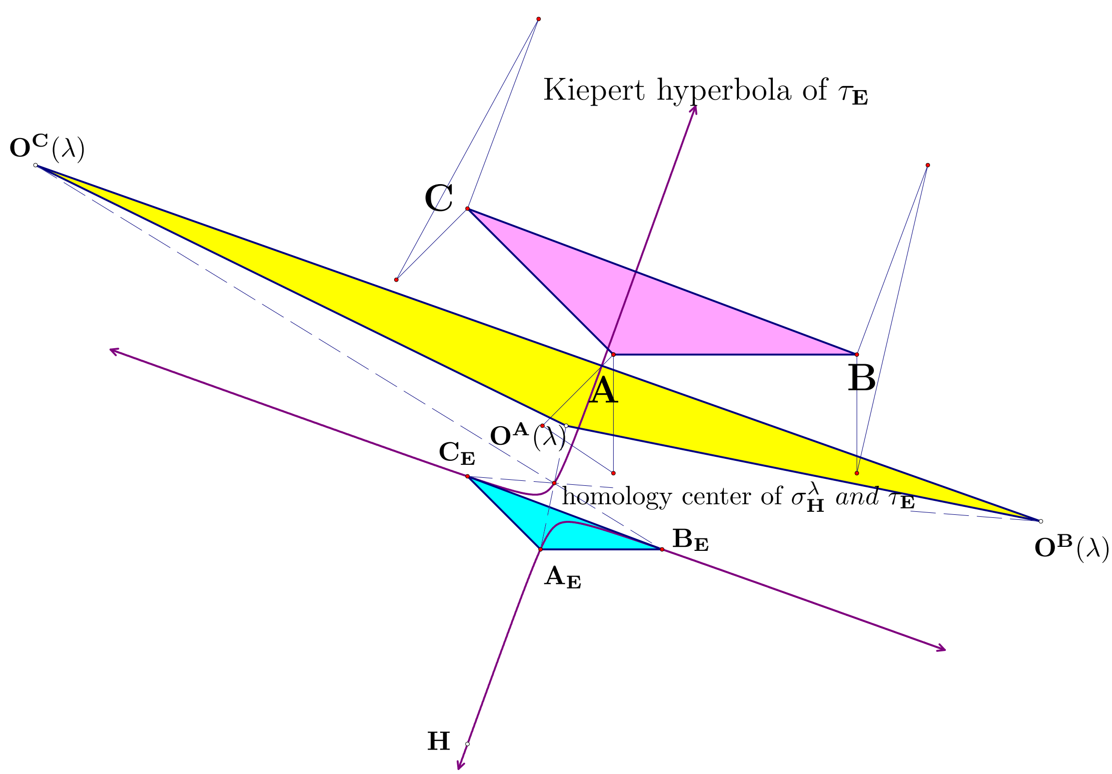

Figure 12: The homology centers of σ_H^λ and τ_E trace the Kiepert hyperbola of τ_E (Theorem 12)

Proof: Since $\frac{T}{a k_a \lambda} - \frac{2}{a} : \frac{1}{b} : \frac{1}{c}$ are trilinears of $H^A(\lambda)$ and from the proof of the previous theorem we know that $\frac{T}{a k_a} - a : \frac{k_c}{2b} : \frac{k_b}{2c}$ are trilinears of A_E, we infer that $A_E H^A(\lambda)$ is

$$2\lambda a d_{2a} k_a x - M_+(b) y - M_-(c) z = 0, \text{ where } M_\pm(b) = b\left[2\lambda(d_{2a}a^2 - d_{4a} \pm T) \mp T k_b\right].$$

The lines $A_E H^A(\lambda)$, $B_E H^B(\lambda)$, and $C_E H^C(\lambda)$ are concurrent at the point

$$\frac{4T(a^4 + z_{2a}a^2 - 2d_{2a}^2)\lambda^2 - 2(3z_{2a}a^4 - 2d_{2a}^2 a^2 - z_{2a}d_{2a}^2)\lambda + T k_b k_c}{a}$$

that traces the conic whose equation is

$$\sum d_{2a}[a^2 k_a^2 x^2 + bc(a^4 + 2z_{2a}a^2 - 3d_{2a}^2)yz] = 0.$$

Since it goes through the vertices of τ_E, its centroid $\frac{a^4 + z_{2a}a^2 - 2d_{2a}^2}{a}$, and the common orthocenter H of τ and τ_E, we conclude that it is the Kiepert hyperbola of τ_E. □

The proof of the following theorem is left to the reader.

Theorem 13 *For every number $\lambda \in \mathbb{R}$ and $j = 2, 3, 4, 5, 20$ the triangles τ_E and σ_j^λ are orthologic. The orthology centers $[\tau_E, \sigma_3^\lambda]$ and $[\sigma_2^\lambda, \tau_E]$ are the orthocenter H. For $i =$ 2, 4, 5, 20 the locus of orthology centers $[\tau_E, \sigma_i^\lambda]$ is the Kiepert hyperbola of τ_E. The locus of the orthology centers $[\sigma_i^\lambda, \tau_E]$ is the line HK for $i = 3, 4, 5, 20$.*

5. The complementary triangle

The complementary triangle $\tau_g = A_g B_g C_g$ has the midpoints A_g, B_g, C_g of sides BC, CA, AB as vertices. It is also the Cevian triangle of the centroid G.

Since the lines $AH^A(\lambda)$, $BH^B(\lambda)$, $CH^C(\lambda)$ are median lines of ABC it is obvious that for every $\lambda \in \mathbb{R}$ the triangles τ_g and σ_H^λ are homologic and their homology center is the centroid G of τ.

Theorem 14 *For every $\lambda \in \mathbb{R}$ the triangles τ_g and σ_G^λ are homologic and their homology centers trace the Kiepert hyperbola of τ_g.*

Proof: Since from the proof of Theorem 1 we know that $\frac{3T + 2a^2\lambda}{-a} : \frac{k_c\lambda}{b} : \frac{k_b\lambda}{c}$ are trilinears of $G^A(\lambda)$ while the trilinears of A_g are $0 : c : b$, we infer that $A_g G^A(\lambda)$ is

$$2\lambda a d_{2a} x + M(b) y - M(c) z = 0, \text{ where } M(b) = b(2\lambda a^2 + 3T).$$

The lines $A_g G^A(\lambda)$, $B_g G^B(\lambda)$, and $C_g G^C(\lambda)$ concur at the point $\frac{4k_a a^2\lambda^2 + 6T z_{2a}\lambda + 9T^2}{a}$ that traces the conic whose equation is $\sum d_{2a}[a^2x^2 + bcyz] = 0$. Since it goes through the vertices of τ_g, the common centroid G of τ and τ_g, and the orthocenter O of τ_g, it follows that this is the Kiepert hyperbola of τ_g. □

Theorem 15 *For every $\lambda \in \mathbb{R}$ the triangles τ_g and σ_O^λ are homologic and their homology centers trace the line GK (see Fig. 13).*

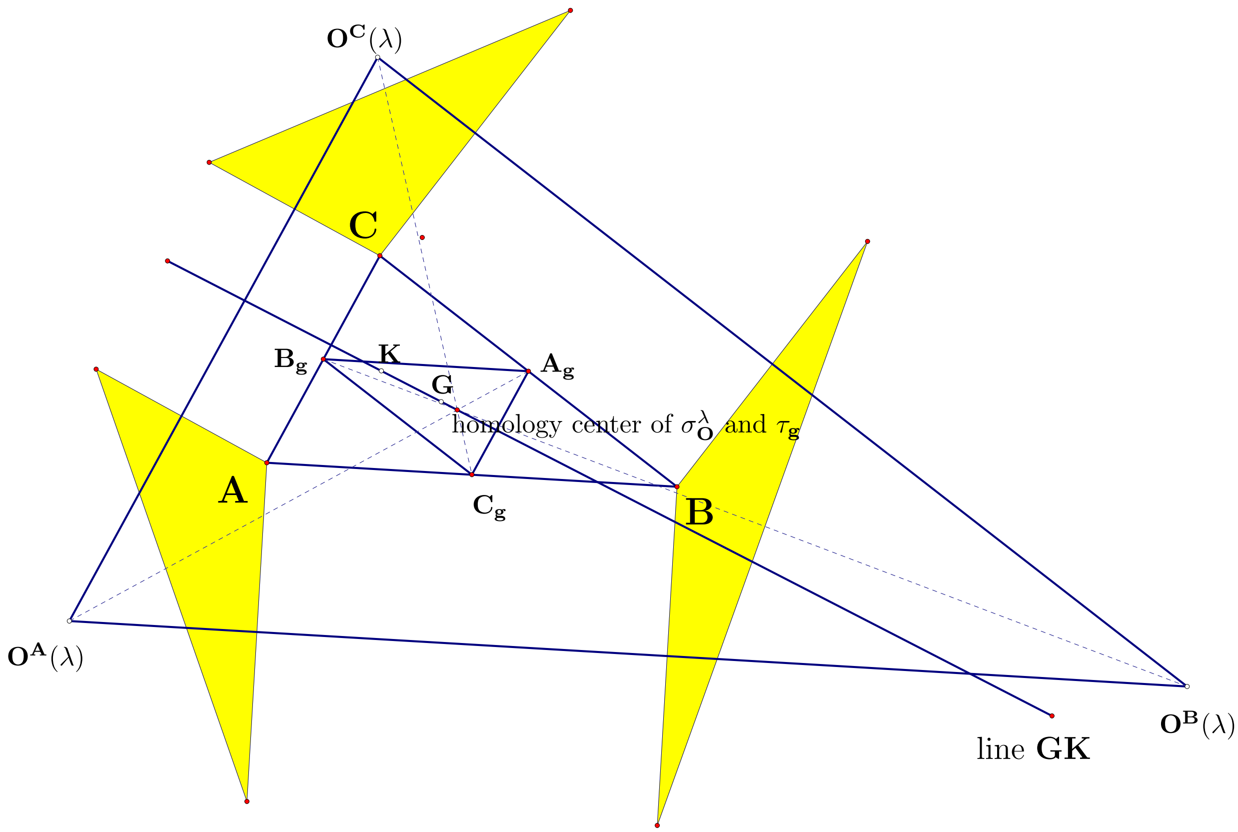

Figure 13: The homology centers of σ_O^λ and τ_g trace the line GK (Theorem 15)

Proof: Now we recall from the proof of Theorem 4 that $\frac{T + z_{2a}\lambda}{-m} : \frac{\lambda}{c} : \frac{\lambda}{b}$ are trilinears of $O^A(\lambda)$. Hence, the line $A_g O^A(\lambda)$ is $\lambda a d_{2a} x + M(b) y - M(c) z = 0$, where $M(b)$ is equal to $b(\lambda z_{2a} + T)$. The lines $A_g O^A(\lambda)$, $B_g O^B(\lambda)$, and $C_g O^C(\lambda)$ concur at the point $\frac{z_{2a}\lambda + T}{a}$ that traces the line whose equation is $\sum a d_{2a} x = 0$. One can easily check that the points $G\left(\frac{1}{a}\right)$ and $K(a)$ are on this line. □

Theorem 16 *For every number $\lambda \in \mathbb{R}$ and $j = 2, 3, 4, 5, 20$ the triangles τ_g and σ_j^λ are orthologic. The orthology center $[\tau_g, \sigma_3^\lambda]$ is the circumcenter O and $[\sigma_2^\lambda, \tau_g]$ is the orthocenter H. For $i = 2, 4, 5, 20$ the locus of orthology centers $[\tau_g, \sigma_i^\lambda]$ is the Kiepert hyperbola of τ_g. The locus of the orthology centers $[\sigma_i^\lambda, \tau_g]$ is the line HK for $i = 3, 4, 5, 20$.*

Proof for the locus of $[\tau_g, \sigma_F^\lambda]$. Since $F^A(\lambda)$ is $\frac{(a^2 - k_a)\lambda + 2T}{a} : \frac{d_{2b}\lambda}{b} : \frac{d_{2c}\lambda}{-c}$, the perpendicular $p\left(A_g, F^B(\lambda)F^C(\lambda)\right)$ from the point A_g onto the line $F^B(\lambda)F^C(\lambda)$ has the equation

$$a d_{2a}(k\lambda - 2T)x + M(b)y - M(c)z = 0, \text{ where } M(b) = b[(2a^4 - z_{2a}a^2 + d_{2a}^2)\lambda - 2a^2 T].$$

Then the perpendiculars

$$p\left(A_g, F^B(\lambda)F^C(\lambda)\right), \quad p\left(B_g, F^C(\lambda)F^A(\lambda)\right), \text{ and } p\left(C_g, F^A(\lambda)F^B(\lambda)\right)$$

concur at the point

$$\frac{(z_{4a} - z_{2a}a^2)(2a^4 - z_{2a}a^2 + d_{2a}^2)\lambda^2 + T(2a^6 - z_{2a}a^4 - z_{2a}d_{2a}^2)\lambda + 2a^2 k_a T^2}{a}$$

that traces the Kiepert hyperbola of τ_g (see the proof of Theorem 14). □

6. The orthic triangle

The orthic triangle $\tau_h = A_h B_h C_h$ has the feet A_h, B_h, C_h of altitudes of ABC as vertices. It is also the Cevian triangle of the orthocenter H.

Since the lines $AG^A(\lambda)$, $BG^B(\lambda)$, $CG^C(\lambda)$ are altitude lines of ABC it is obvious that for every $\lambda \in \mathbb{R}$ the triangles τ_h and σ_G^λ are homologic and their homology center is the orthocenter H of τ.

Theorem 17 *For every $\lambda \in \mathbb{R}$ the triangles τ_h and σ_O^λ are homologic and their homology centers trace the equilateral hyperbola that goes through the vertices of τ_h and the central points H (the orthocenter), K (the symmedian point), X_{52} (the orthocenter of the orthic triangle), and X_{113} (Jerabek antipode) of the triangle ABC. The homology axis trace a parabola.*

Proof: Since A_h is $0 : \frac{1}{b\,k_b} : \frac{1}{c\,k_c}$ and $O^A(\lambda)$ is $\frac{T + z_{2a}\lambda}{-m} : \frac{\lambda}{c} : \frac{\lambda}{b}$, the line $A_hO^A(\lambda)$ has the equation

$$\lambda a d_{2a} k_a x - M(b)y + M(c)z = 0, \text{ where } M(b) = bk_b(\lambda z_{2a} + T).$$

It follows that the lines $A_hO^A(\lambda)$, $B_hO^B(\lambda)$, and $C_hO^C(\lambda)$ concur at the point

$$\frac{2a^2 z_{2a}\lambda^2 + T(k + a^2)\lambda + T^2}{a\,k_a}$$

that traces an equilateral hyperbola with the equation $\sum d_{2a}(a^2x^2 + bck_bk_cyz) = 0$. One can easily check that the vertices of τ_h, the orthocenter $H\left(\frac{1}{a\,k_a}\right)$, the symmedian point $K(a)$, the orthocenter $X_{52}\,(a(a^2k_a - T^2)(2m_a^2 - T^2))$ of the orthic triangle, and the Jerabek antipode $X_{113}\left(\frac{(T^2 - 3a^2k_a)(z_{2a}a^4 - 2(z_{4a} - m_a^2)a^2 + z_{2a}d_{2a}^2)}{a}\right)$ all lie on it. □

Theorem 18 *For every real number λ and for $j = O, K$ the triangles τ_h and σ_j^λ are orthologic. The orthology center $[\tau_h, \sigma_O^\lambda]$ is the orthocenter H and $[\sigma_K^\lambda, \tau_g]$ is the circumcenter O. The locus of orthology centers $[\tau_h, \sigma_K^\lambda]$ is the rectangular hyperbola $A_hB_hC_hH$. The locus of the orthology centers $[\sigma_O^\lambda, \tau_h]$ is the line OK (see Fig. 14).*

Proof for the locus of $[\tau_h, \sigma_K^\lambda]$. Since the point $K^A(\lambda)$ is

$$\frac{2[T(3k_a - 2a^2) + (z_{2a}a^2 - d_{2a}^2)\lambda]}{-a} : M(b,c) : M(c,b)$$

where the function $M(b,c)$ is $bt_c[(k_b + 2m_b)^2 - T^2]$, the perpendicular $p\left(A_h,\, K^B(\lambda)K^C(\lambda)\right)$ from the point A_h onto the line $K^B(\lambda)K^C(\lambda)$ has the equation

$$2\lambda a d_{2a} k_a T x - M(b,c)y + M(c,b)z = 0,$$

where the function $M(b,c)$ is $bk_b[T(k + 2a^2)\lambda + (3k - 4b^2)(3k - 4c^2)]$. Then the lines

$$p\left(A_h,\, K^B(\lambda)K^C(\lambda)\right),\quad p\left(B_h,\, K^C(\lambda)K^A(\lambda)\right),\ \text{and}\ p\left(C_h,\, K^A(\lambda)K^B(\lambda)\right)$$

concur at the point

$$\frac{T^2(k + 2a^2)\lambda^2 + 2T(3a^4 + 7z_{2a}a^2 + 8m_a^2)\lambda + (3k - 4a^2)(3k - 4b^2)(3k - 4c^2)}{k_a}$$

that traces the rectangular hyperbola with the equation

$$\sum (3k - 4a^2)d_{2a}[a^2k_a^2x^2 + m_ak_bk_cyz] = 0.$$

It is easy to check that the vertices of τ_h and the orthocenter H lie on it. □

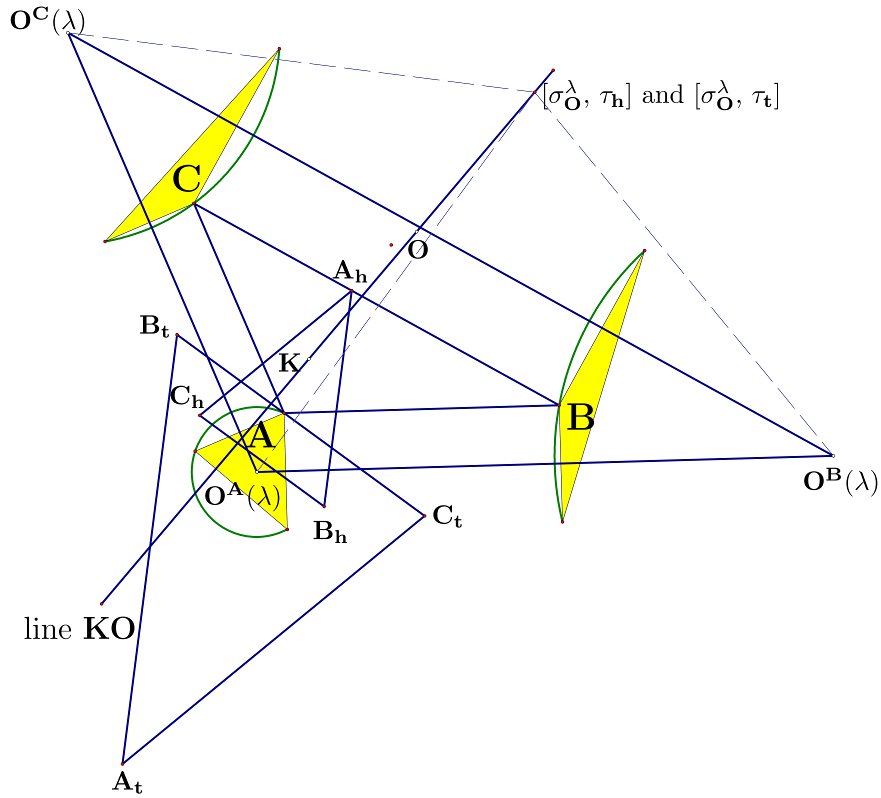

Figure 14: Orthology centers $[\sigma^{\lambda}_{O},\, \tau_h]$ and $[\sigma^{\lambda}_{O},\, \tau_t]$ trace the line OK (Theorems 18 and 19)

7. The tangential triangle

Let p_a, p_b, and p_c be perpendiculars at vertices A, B, and C to segments AO, BO, and CO joining the vertices with the circumcenter. The tangential triangle $\tau_t = A_tB_tC_t$ has the intersections $p_b \cap p_c$, $p_c \cap p_a$, and $p_a \cap p_b$ as vertices. It is also the antipedal triangle of the circumcenter O.

Since the lines $AO^A(\lambda)$, $BO^B(\lambda)$, $CO^C(\lambda)$ are the symmedians of ABC it is obvious that for every $\lambda \in \mathbb{R}$ the triangles τ_t and σ^{λ}_{O} are homologic and their homology center is the symmedian point K of τ.

Theorem 19 *For every number $\lambda \in \mathbb{R}$ and $j = O,\, K$ the triangles τ_t and σ^{λ}_{j} are orthologic. The orthology centers $[\tau_t,\, \sigma^{\lambda}_{O}]$ and $[\sigma^{\lambda}_{K},\, \tau_t]$ are the circumcenter O. The locus of orthology centers $[\tau_h,\, \sigma^{\lambda}_{K}]$ is the rectangular hyperbola $A_tB_tC_tO$. The locus of the orthology centers $[\sigma^{\lambda}_{O},\, \tau_t]$ is the line OK (see Fig. 14).*

Proof for the locus of $[\sigma^{\lambda}_{O},\, \tau_t]$. Since the point $O^A(\lambda)$ is $\frac{T + z_{2a}\lambda}{-m} : \frac{\lambda}{c} : \frac{\lambda}{b}$ and A_t has trilinears $-a : b : c$, the perpendicular $p\left(O^A(\lambda),\, B_tC_t\right)$ from the point $O^A(\lambda)$ onto the line B_tC_t has the equation

$$2\lambda abcd_{2a}x + c[2\lambda b^2 d_{2a} + k_cT]y + b[2\lambda c^2 d_{2a} - k_bT]z = 0.$$

Then the lines

$$p\left(O^A(\lambda),\, B_tC_t\right), \quad p\left(O^B(\lambda),\, C_tA_t\right), \text{ and } p\left(O^C(\lambda),\, A_tB_t\right)$$

concur at the point $a(2\lambda(z_{2a}a^2 - z_{2a}) - k_aT)$ that traces the line with the equation $\sum m_a d_{2a} x = 0$. It is easy to check that the circumcenter O and the symmedian point K lie on it. □

8. The Torricelli triangles

Let A_u, B_u, and C_u be vertices of equilateral triangles built on sides BC, CA, and ABb of ABC towards inside. When they are built towards outside then their vertices are denoted A_v, B_v, and C_v. The negative Torricelli triangle τ_u is $A_uB_uC_u$ while $A_vB_vC_v$ is the positive Torricelli triangle τ_v of ABC.

Theorem 20 *For every $\lambda \in \mathbb{R}$ the triangles τ_u and σ_G^λ are homologic and their homology centers trace the Kiepert hyperbola of τ_u that goes through the vertices of τ_u and the central points G (the centroid), O (the circumcenter), and X_{14} (the negative isogonic point) of the triangle ABC.*

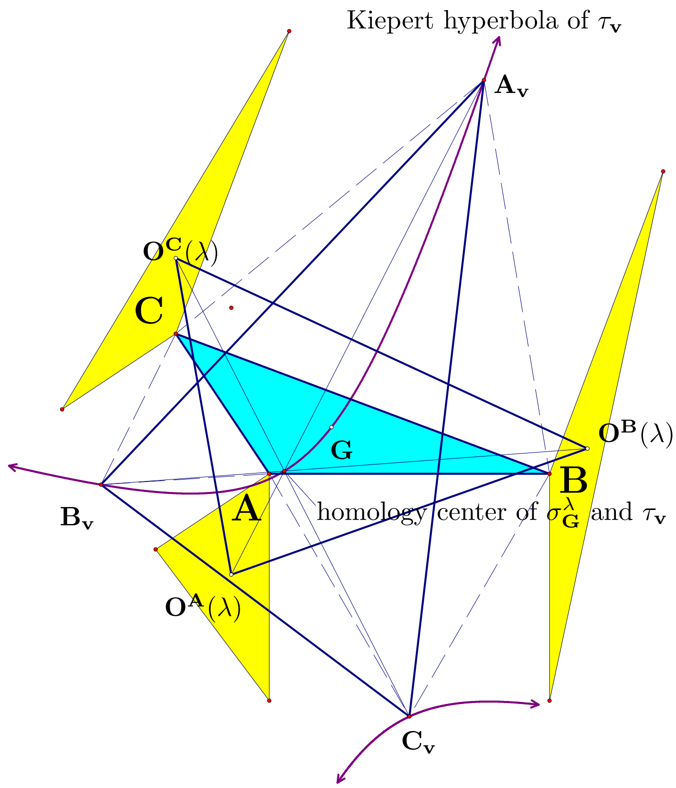

Figure 15: The homology centers of σ_G^λ and τ_v trace the Kiepert hyperbola of τ_v (analogue of Theorem 20)

Proof: Since the point A_u has trilinears $-1 : \frac{T\sqrt{3} - 3k_c}{6ab} : \frac{T\sqrt{3} - 3k_b}{6ca}$ and the point $G^A(\lambda)$ is $\frac{3T + 2\lambda a^2}{-a} : \frac{\lambda k_c}{b} : \frac{\lambda k_b}{c}$, the line $A_uG^A(\lambda)$ has the equation

$$\lambda a d_{2a}(3k_a + T\sqrt{3})x - M(b)y + M(c)z = 0,$$

where the function $M(b)$ is equal to $b[\lambda(3k_a d_{2a} - z_{2a}T\sqrt{3}) + (3k_b - T\sqrt{3})T]$. Hence, the lines $A_uG^A(\lambda)$, $B_uG^B(\lambda)$, and $C_uG^C(\lambda)$ concur at the point

$$\frac{2k_a a^2\lambda^2 + 3[z_{2a}T - (z_{2a}a^2 - d_{2a}^2)\sqrt{3}]\lambda + 9[a^4 + (z_{2a} - T\sqrt{3})a^2 - 2d_{2a}^2]}{a}$$

that traces the conic $\sum d_{2a}[a^2(3k_a - T\sqrt{3})x^2 - m_a(3k_a + T\sqrt{3})yz] = 0$. It is easy to check that this is the Kiepert hyperbola of τ_u because it goes through the vertices of τ_u, the common centroid G of τ_u and τ, and the orthocenter of τ_u with trilinears $\frac{(k_a - 2a^2)T\sqrt{3} + a^4 + 2z_{2a}a^2 - 3d_{2a}^2}{a}$. In the same way one can prove that O (the circumcenter), and X_{14} (the negative isogonic point) of the triangle ABC are also on it. □

Theorem 21 *For every $\lambda \in \mathbb{R}$ the triangles τ_u and σ_{18}^λ are homologic and their homology center is the negative isogonic point X_{14}.*

Proof: The point $X_{18}^A(\lambda)$ is $f : g(b,c) : g(c,b)$, where

$$g(b,c) = \lambda m_b t_b(3k_c + 6m_c + T\sqrt{3})(k_c + 2m_c - T\sqrt{3}),$$

and

$$f = \frac{4T^2 m_a \left[\left((k_a - 3a^2)T\sqrt{3} + 3z_{2a}a^2 - 3d_{2a}^2\right)\lambda + 3(3k_a + 2a^2)T - 5T^2\sqrt{3}\right]}{t_a(k_a + T\sqrt{3})}.$$

One can easily check that this point lies on the line A_uX_{14} and thus complete the proof. □

Theorem 22 *The triangle τ_u is orthologic to σ_i^λ for $i = 2, 3, 4, 5, 14, 20$. The orthology centers $[\tau_u, \sigma_O^\lambda]$ and $[\sigma_{14}^\lambda, \tau_u]$ are the circumcenter O and the second Napoleon point X_{18}, respectively. The locus of the orthology centers $[\tau_u, \sigma_j^\lambda]$ is the Kiepert hyperbola of $A_uB_uC_u$ for $j = 2, 4, 5, 20$. The orthology centers $[\tau_u, \sigma_{14}^\lambda]$ trace a hyperbola that goes through the vertices of τ_u and the circumcenter O. The locus of the orthology centers $[\sigma_k^\lambda, \tau_u]$ for $k = 2, 3, 4, 5, 20$ are the lines GX_{18}, KX_{18}, HX_{18}, $X_{15}X_{18}$, and a line through X_{18}, respectively.*

Proof of the case $i = 14$. The point $X_{14}^A(\lambda)$ has trilinear coordinates $f : g(b) : g(c)$, where $g(b) = \lambda m_b(3b^2k_b + T(2k_a - b^2)\sqrt{3})$ and

$$f = m_a\left[\left((3k_a - a^2)T\sqrt{3} - 3d_{2a}^2 + 3z_{2a}a^2\right)\lambda + 3(3k_a + 2a^2)T - 3T^2\sqrt{3}\right].$$

We infer easily that the perpendicular $p\left(X_{14}^A(\lambda),\, B_uC_u\right)$ from the point $X_{14}^A(\lambda)$ onto the line B_uC_u has the equation $b(k_b - T\sqrt{3})y - c(k_c - T\sqrt{3})z = 0$ and it goes through the point X_{18} with the first trilinear coordinate $\frac{a^4 + (T\sqrt{3} - 3z_{2a})a^2 + 2d_{2a}^2}{a}$. This shows that the triangles σ_{14}^λ and τ_u are orthologic and that $[\sigma_{14}^\lambda, \tau_u] = X_{18}$.

On the other hand, the perpendicular $p\left(A_u,\, X_{14}^B(\lambda)X_{14}^C(\lambda)\right)$ from the point A_u onto the line $X_{14}^B(\lambda)X_{14}^C(\lambda)$ has the equation $f_x x - g_+(b,c)y + g_-(c,b)z = 0$, with

$$\begin{aligned}
f_x &= ad_{2a}(g_1\lambda + g_2),\\
g_1 &= a^4 + 10z_{2a}a^2 - 5z_{4a} + 16m_a^2 + (2a^2 - 3k)T\sqrt{3},\\
g_2 &= (3a^4 + 6z_{2a}a^2 - 3z_{4a} + 8m_a^2)\sqrt{3} - 3(k + 2a^2)T, \text{ and}\\
g_\pm(b,c) &= b\{[9a^6 + 4(d_{2a} - b^2)a^4 + (2m_a^2 \pm 7d_{4a} - 4c^4)a^2 + 2(c^2 \mp d_{2a})d_{2a}^2 +\\
&\quad +(-a^4 + (2c^2 \mp d_{2a})a^2 \pm 2b^2d_{2a})T\sqrt{3}]\lambda + g_2a^2\}.
\end{aligned}$$

The lines

$$p\left(A_u,\, X_{14}^B(\lambda)X_{14}^C(\lambda)\right),\quad p\left(B_u,\, X_{14}^C(\lambda)X_{14}^A(\lambda)\right),\ \text{and } p\left(C_u,\, X_{14}^A(\lambda)X_{14}^B(\lambda)\right)$$

concur at the point $h_2\lambda^2 + 2h_1\lambda\sqrt{3} - 3a^2 k_a h_0$, where

$$\begin{aligned}
h_2 &= h_{20}\sqrt{3} + 3T h_{21},\\
h_{20} &= 18a^{10} - 65z_{2a}a^8 + (68z_{4a} + 49m_a^2)a^6 - z_{2a}(33z_{4a} - 40m_a^2)a^4 +\\
&\quad +(14z_{8a} - 9m_a^2 z_{4a} - 22m_{4a})a^2 - 2z_{2a}d_{2a}^2(z_{4a} - 5m_a^2),\\
h_{21} &= 12a^8 - 7z_{2a}a^6 - (z_{4a} + 7m_a^2)a^4 + z_{2a}(2z_{4a} + 3m_a^2)a^2 - 2d_{2a}^2(3z_{4a} + m_a^2),\\
h_1 &= h_{10}\sqrt{3} + 3T h_{21},\\
h_{10} &= 12a^{10} - 32z_{2a}a^8 + (12z_{4a} + m_a^2)a^6 + 3z_{2a}(3z_{4a} - m_a^2)a^4 +\\
&\quad +(2z_{4a} + 17m_a^2)d_{2a}^2 a^2 - 3z_{2a}d_{2a}^2(d_{2a}^2 - m_a^2),\\
h_0 &= h_{00}\sqrt{3} + 21T(m_a^2 + m_b^2 + m_c^2), \text{ and at last}\\
h_{00} &= 6a^6 - 9z_{2a}a^4 - (9z_{4a} + 29m_a^2)a^2 + 3z_{2a}(2d_{2a} + c^2)(d_{2a} - c^2).
\end{aligned}$$

In order to find the curve which traces this point we must eliminate the parameter λ. We obtain an equilateral hyperbola that goes through the vertices of τ_u and the circumcenter O. □

Of course, there are versions of the above three theorems for the positive Torricelli triangle τ_v of ABC (see Fig. 15). Instead of numbers 18 and 14 now the numbers 17 and 13 play important role.

9. The Napoleon triangles

Let A_x, B_x, and C_x be centers of equilateral triangles built on sides BC, CA, and AB of ABC towards inside. When they are built towards outside then their vertices are denoted A_y, B_y, and C_y. The negative Napoleon triangle τ_x is $A_xB_xC_x$ while $A_yB_yC_y$ is the positive Napoleon triangle τ_y of ABC.

Theorem 23 *For every $\lambda \in \mathbb{R}$ the triangles τ_x and σ_G^λ are homologic and their homology centers trace the hyperbola that goes through the vertices of τ_x and the central points G (the centroid), O (the circumcenter), and X_{18} (the second Napoleon point) of the triangle ABC.*

Proof: Since A_x has coordinates $-1 : \frac{k_c - T\sqrt{3}}{2m_c} : \frac{k_b - T\sqrt{3}}{2m_b}$, the line $A_xG_\lambda^A$ has the equation $2\lambda a d_{2a}x + b(2\lambda a^2 - \sqrt{3}\,k_b + 3T)y - c(2\lambda a^2 - \sqrt{3}\,k_c + 3T)z = 0$. It follows that the lines $A_xG_\lambda^A$, $B_xG_\lambda^B$, and $C_xG_\lambda^C$ concur at the point

$$\frac{2a^2k_a\lambda^2 + (3z_{2a}T - \sqrt{3}(z_{2a}a^2 - d_{2a}^2))\lambda - 3\sqrt{3}\,a^2T - 3a^4 + 9z_{2a}a^2 - 6d_{2a}^2}{a}.$$

This point traces the equilateral hyperbola with the equation

$$\sum d_{2a}[a^2(k_a - T\sqrt{3})x^2 - m_a(k_a + T\sqrt{3})yz] = 0.$$

It goes through the vertices of τ_x, the centroid G, the circumcenter O, and the second Napoleon point X_{18}. □

Theorem 24 *For every $\lambda \in \mathbb{R}$ the triangles τ_x and σ_{14}^λ are homologic and their homology center is the second Napoleon point X_{18}.*

Proof: The point $X_{14}^A(\lambda)$ whose coordinates have been described in the proof of Theorem 22 is easily seen to lie on the line A_xX_{18}. □

Theorem 25 *The triangle* τ_x *is orthologic to* σ_i^λ *for* $i = 2, 3, 4, 5, 18, 20$. *The orthology centers* $[\tau_x, \sigma_O^\lambda]$ *and* $[\sigma_{18}^\lambda, \tau_x]$ *are the circumcenter* O *and the second isogonic point* X_{14}. *The locus of orthology centers* $[\tau_x, \sigma_j^\lambda]$ *is the hyperbola that goes through the vertices of* τ_x *and points* G, O, *and* X_{18} *of the triangle* ABC *for* $j = 2, 4, 5, 20$. *The locus of the orthology centers* $[\sigma_k^\lambda, \tau_x]$ *for* $k = 2, 3, 4, 5, 20$ *are the lines* GX_{14}, KX_{14}, HX_{14}, $X_{14}X_{16}$, *and a line through* X_{14}, *respectively. The orthology centers* $[\sigma_{18}^\lambda, \tau_x]$ *trace a hyperbola that goes through the vertices of* τ_x *and* O.

Proof: The proofs of the claims in this theorem are left to the reader as an exercise (see the proof of Theorem 22). □

Of course, there are versions of the above three theorems for the positive Napoleon triangle τ_y of ABC. Instead of numbers 18 and 14 now the numbers 17 and 13 play important role.

References

[1] Z. Čerin: *Loci related to variable flanks*. Forum Geometricorum **2**, 105–112 (2002).

[2] R.H. Eddy, R. Fritsch: *The conics of Ludwig Kiepert: A comprehensive lesson in the geometry of the triangle*. Mathematics Magazine **67**, 188–205 (1994).

[3] L. Hoehn: *Extriangles and excevians*. Mathematics Magazine **74**, no. 5, 384–388 (2001).

[4] C. Kimberling: *Central points and central lines in the plane of a triangle*. Mathematics Magazine **67**, 163–187 (1994).

[5] C. Kimberling: *Encyclopedia of Triangle Centers*. 2000, http://cedar.evansville.edu /~ck6/encyclopedia/.

[6] F. van Lamoen: *Friendship Among Triangle Centers*. Forum Geometricorum **1**, 1–6 (2001).

[7] C.R. Pranesachar, B.J. Venkatachala: *On a Curious Duality in Triangles*. Samasyā **7** (2), 13–19 (2001).

Received September 23, 2002; final form June 5, 2003