Journal for Geometry and Graphics
Volume 7 (2003), No. 1, 53–63.

Analagmatic Curves under the Isogonal Transformation

Ana Sliepčević

Faculty of Civil Engineering, University Zagreb
Av. V. Holjevca 15, HR 10010 Zagreb
email: anasli@gradnz.grad.hr

Abstract. This paper treats plane curves of 3^{rd} and 4^{th} order generated under an isogonal transformation. It is shown by synthetic and constructive methods that there exists a variety of analagmatic curves, i.e., curves that are invariant under the isogonal transformation. Circular analagmatic curves of 3^{rd} and 4^{th} order are constructed as well. It is proved that there are pencils of such curves. Furthermore, for these pencils the curves of singular foci are determined.

Key words: isogonal transformation, analagmatic curve, circular curve, singular focus

MSC 2000: 51N35

1. Introduction

According to [2] in the real projective plane $\mathbb{P}^2$ the involutive quadratic transformations with three different fundamental points can be defined by two involutive pencils of lines (A) and (B) in the following way: Each point $T \in \mathbb{P}^2$ lies on rays $t_A \in (A)$ and $t_B \in (B)$. Let $t'_A \in (A)$ and $t'_B \in (B)$ denote the rays corresponding to t_A, t_B, respectively, in the given involutions. Then the image T' of T is defined as $T' = t'_A \cap t'_B$. Depending on the types of involutive pencils (A) and (B), various quadratic transformations are obtained in this way. In this paper we consider a special Euclidean type.

2. Isogonal transformation

Let the two involutive pencils of lines (A) and (B) be given by their double rays d_1, d_2 and f_1, f_2, resp., which are supposed to be mutually perpendicular (Fig. 1). Then the pairs $(t_A, t'_A) \in (A)$ and $(t_B, t'_B) \in (B)$ satisfy

$$\angle(t_A, d_1) = \angle(t'_A, d_1); \quad \angle(t_B, f_1) = \angle(t'_B, f_1),$$

ISSN 1433-8157/\$ 2.50

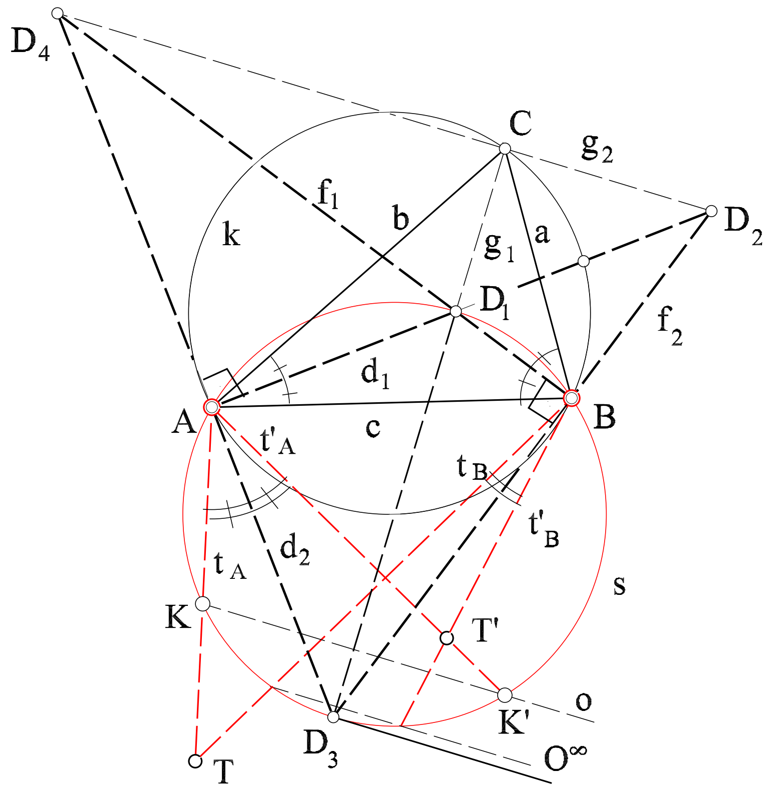

Figure 1: The isogonal transformation $I\colon\ T = t_A \cap t_B \ \mapsto\ I(T) = T' = t'_A \cap t'_B$

i.e., the angles between corresponding rays and the double rays in the pencils (A) and (B) are congruent [3].

Let C be the intersection of the rays $b \in (A)$ and $a \in (B)$ which correspond respectively to the join $c := AB$. Let g_1, g_2 denote the bisectors of the angle $\angle(a,b)$. If an involution in the pencil (C) is defined by the double rays g_1, g_2, then the pairs of involutive pencils (A) and (C) as well as (B) and (C) define the same transformation in the plane $\mathbb{P}^2$ as the pencils (A) and (B) (cf. [5]). This transformation (see Fig. 1)

$$I\colon\ \mathbb{P}^2 \to \mathbb{P}^2 \text{ with } \ T = t_A \cap t_B \ \mapsto\ I(T) = T' = t'_A \cap t'_B$$

is called the *isogonal transformation* with respect to the *base triangle* ABC.

Under quadratic transformations a generic curve of n-th order is mapped onto a curve of $2n$-th order having n-fold points at the vertices of the base triangle [2]. The genus of the curve is preserved, provided there is no degeneration. We use the transformation I to generate circular curves, among which the analagmatic curves are of special interest. A curve c is called *analagmatic* with respect to the transformation I, if c is invariant under I, i.e., transformed into itself.

Particular elements of this mapping I:

1. The vertices A, B, C of the base triangle are transformed into their opposite sides a, b, c, and vice versa. Each other point of the line a (b or c) is transformed to the point A (B or C, resp.).
2. The lines d_1, d_2; f_1, f_2; g_1, g_2 are transformed into themselves. These are the only *analagmatic lines* of the transformation I.

3. The points $D_1 = d_1 \cap f_1 \cap g_1$, $D_2 = d_1 \cap f_2 \cap g_2$, $D_3 = d_2 \cap f_2 \cap g_1$, $D_4 = d_2 \cap f_1 \cap g_2$ are the only *fixed points* of the transformation.

Steiner's circle s through the points A, B, D_1, D_3 is useful for completing the given involutive pencils (A) and (B) (see Fig. 1). This *circle is analagmatic* with respect to I. To see this, let us notice that all given involutions induce the same involution on s. Center of this involution is the point at infinity O^∞ perpendicular to the join D_1D_3. Every line through O^∞ intersects the circle s at points corresponding under I. The line at infinity, which is also a ray of the pencil (O^∞), intersects the circle s in the imaginary absolute points. As these are again corresponding points of I, the given transformation *preserves the circularity* of any curve.

Theorem 1 *There exist infinitely many analagmatic conics; they constitute six pencils.*

Proof: Let T be a generic point. Then T and the points A, B, D_1, D_3 determine a conic uniquely. This conic is incident with the assigned point T' as well, because the pencils (A), (B) are projectively related. Hence, this conic is mapped isogonally onto itself. We can therefore conclude that all conics of the pencil determined by the points A, B, D_1, D_3 are analagmatic. The analogous conclusion holds for the pencils of conics determined by the following quadruples of base elements: (A, B, D_2, D_4), (A, C, D_1, D_4), (A, C, D_2, D_3), (B, C, D_1, D_3) and (B, C, D_2, D_4). □

A generic line l is mapped onto a curve $I(l)$ of second order. The points of intersection $l \cap I(l)$ are two isogonally assigned points. This is the only pair of corresponding points located on l.

Lemma 1 *On each line $l \neq d_1, d_2, f_1, f_2, g_1, g_2$ there exists exactly one pair of real or complex conjugate points which are isogonally mapped into each other.*

Lemma 2 *The pairs of isogonally assigned points on the lines of a certain pencil (T), $T \neq A, B, C$, lie on an analagmatic cubic.*

Proof: The pencil (T) of lines l is isogonally mapped onto a pencil of conics $I(l)$ passing through A, B, C, T'. Each conic intersects its corresponding line in a pair of isogonally assigned points. The pencil of lines is projectively related to the mentioned pencil of conics. The set of intersection points of such two pencils is a curve of third order [3]. This cubic is obviously analagmatic. □

Lemma 1 and Lemma 2 directly imply:

Theorem 2 *There is a continuum of analagmatic cubics passing through $A, B, C, D_1, \ldots, D_4$.*

The line at infinity is mapped onto a circle k circumscribed to the base triangle ABC [5]. We obtain

Corollary 1 *The pairs of isogonally assigned points on parallel lines lie on an analagmatic circular cubic.*

Theorem 3 *Any pencil of conics, which is the image of a pencil of lines under the isogonal transformation I, includes an equilateral hyperbola.*

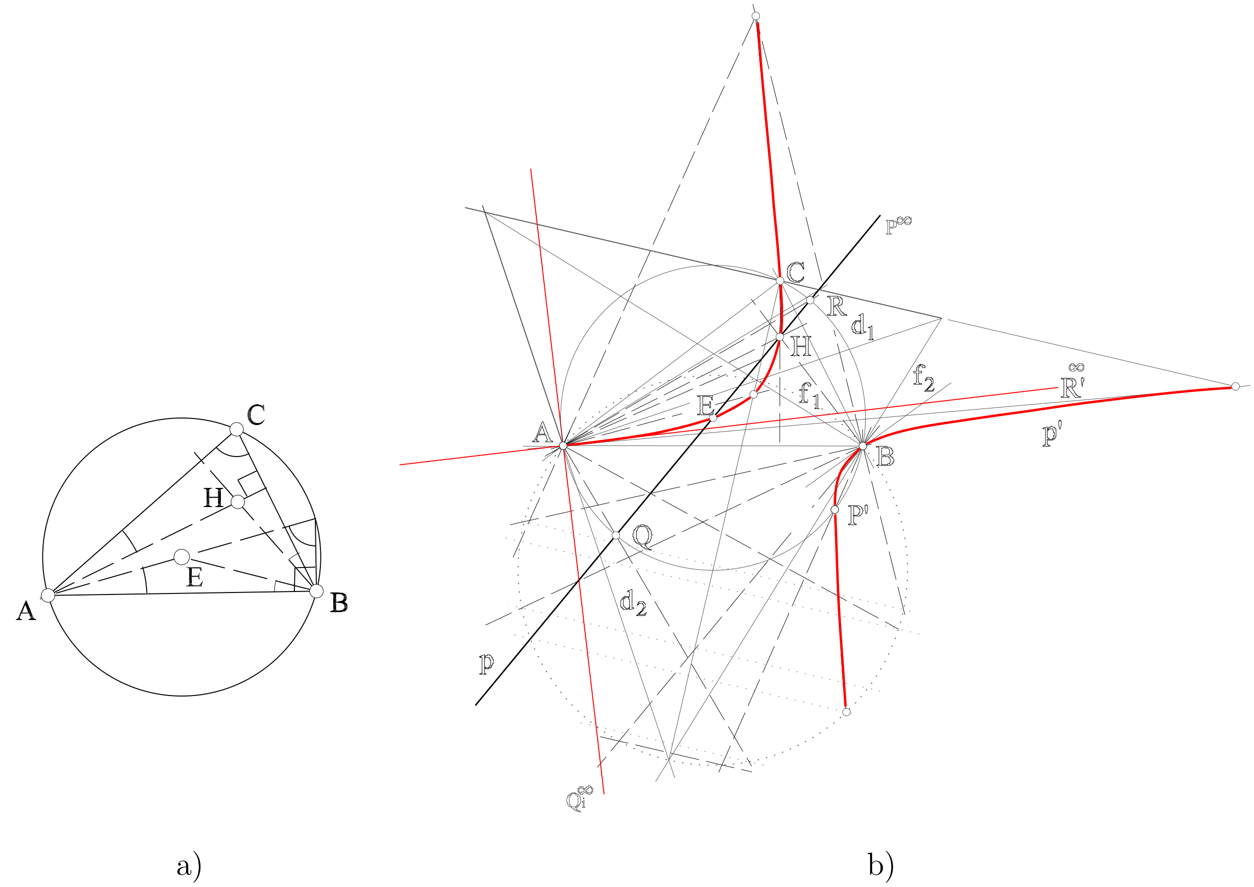

Figure 2: The isogonal image of the line $p = EH$

Proof: Because of the similarity of triangles in Fig. 2a, the center E of the circle circumscribed to the base triangle ABC is mapped onto its orthocenter H. Therefore, every line through E is mapped onto a conic which passes through the vertices of the triangle and its orthocenter H, hence to an equilateral hyperbola. In particular, the pencil of lines with vertex E is mapped to a pencil of equilateral hyperbolas passing through A, B, C, H. In Fig. 2b an equilateral hyperbola of this pencil is constructed as the set of points isogonally assigned to the points of the line $p = EH$. □

3. Generation of circular curves of third and fourth order

Under the isogonal transformation I a circle k generally is mapped onto a circular curve $I(k)$ of fourth order with three singular points at the vertices of the base triangle [6]. The characters of the singular points depend on the position of k with respect to the sides of the base triangle. k can intersect a side of the base triangle in a pair of real or complex conjugate points or it can be tangent to a side, and therefore the assigned quartic has a knot, an isolated point or a cusp at the opposite vertex of the triangle.

If the circle k passes through a vertex of the base triangle, say A, then the circular curve $I(k)$ splits into the opposite side $a = BC$ and a circular cubic with a double point at A (Fig. 3).

Let a pencil of circles be defined by the base points A and T, $T \neq A, B, C$ (Fig. 4): This pencil of circles is mapped onto a pencil of circular cubics with a common double point at

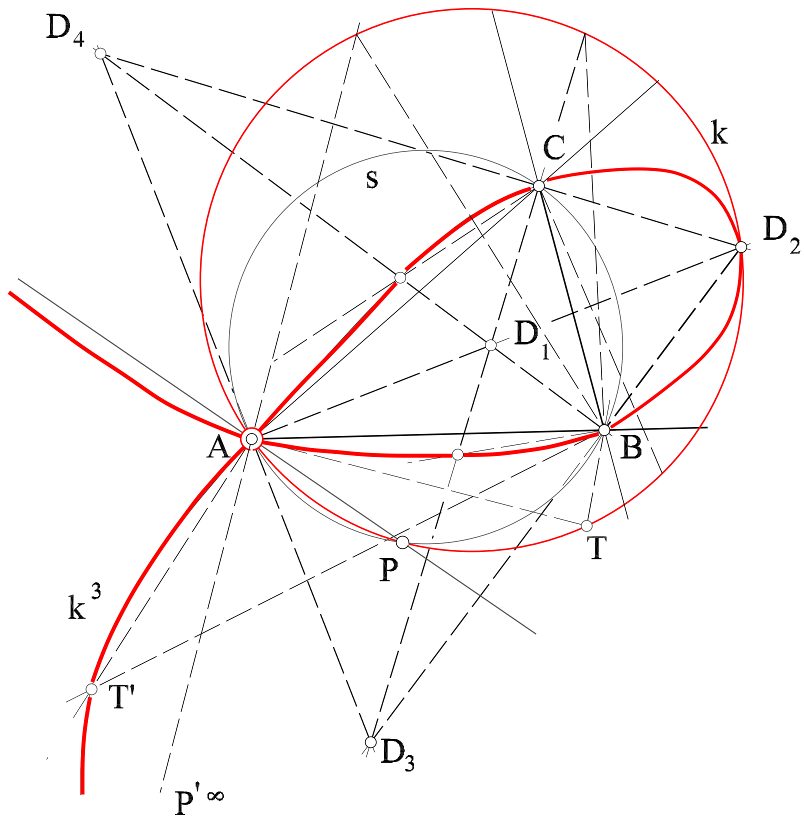

Figure 3: The image $I(k) = k^3$ of a circle k passing through A

the vertex A of the base triangle. Each cubic of this pencil has a *singular focus*, i.e., the point of intersection between the tangent lines at the absolute points. The singular foci of all cubics in this pencil lie on a certain circle (compare [4]). In order to determine this circle f of foci, it is sufficient to consider three cubics in the pencil which split into a circle and a line. These are the cubics obtained under the isogonal transformation either from those circles of the generating pencil which pass through the vertices B or C, or from the degenerated circle $AT \cup p^\infty$. The foci F_B, F_C, F_S of these degenerated cubics determine the circle f of foci uniquely (Fig. 4).

Due to this circle f it is easy to determine graphically the singular focus for each circular cubic in such a pencil: For this purpose it is sufficient to consider the projectivity between e.g. the points at infinity of the cubics in the pencil and their singular foci on the mentioned circle f.

It is interesting to notice that the pencil of circular cubics includes a *strophoid*. This is the isogonal image of the circle through A and T which has its center on the side BC of the triangle ABC. To see this, recall that the intersections of the circle with the side BC are mapped onto the double point A, and the tangents at A are obviously perpendicular. This property characterizes a strophoid.

4. Analagmatic circular cubics

Corollary 1 implies that there are infinitely many analagmatic circular cubics. These cubics pass through $A, B, C, D_1, D_2, D_3, D_4$ and through the absolute points. They form a pencil.

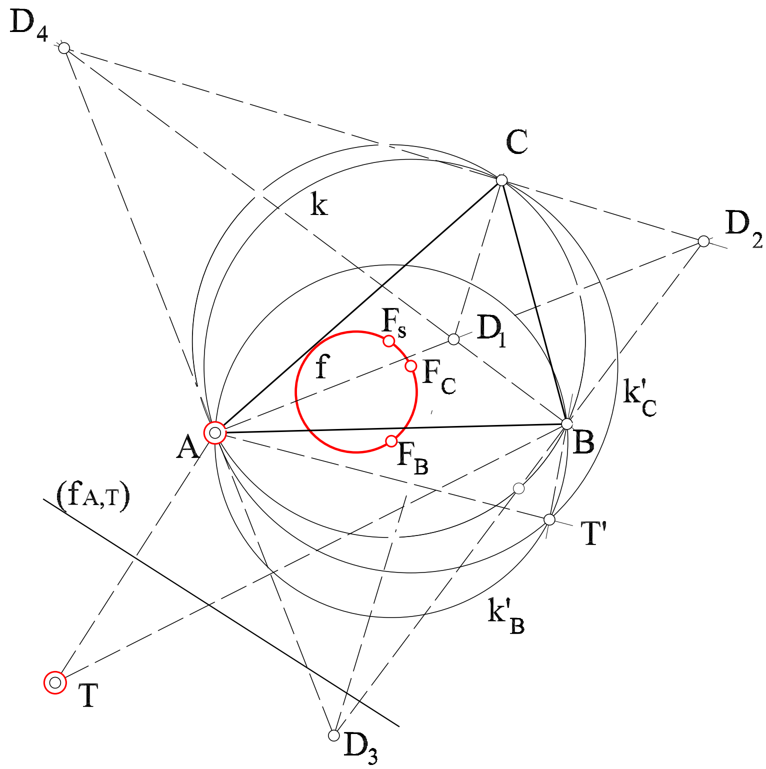

Figure 4: Focal curve f of a pencil of circular cubics

The included irreducible cubics are all of genus one. It is possible to determine constructively each of them as a *"projective product"*, i.e., as the set of intersection points, of a certain pencil of circles and a pencil of lines. In order to see this, let us notice that in the mentioned pencil of cubics there are six cubics which split into a circle and a line. A pencil of circles is spanned by two of them. Furthermore, a certain pencil of lines can be projectively assigned to it such that their product is a cubic of the required type.

An example of such a cubic k^3 is displayed in Fig. 5. The pencil of lines (C) is projectively related to the pencil of circles (A, B). We define this projectivity by the lines $p_1 = D_2D_4$, $p_2 = D_1D_3$, and an arbitrary p_3 through C and their image circles $k_1(A, B, D_2, D_4)$, $k_2(A, B, D_1, D_3)$ and $k_3 = AB \cup p_1^\infty$.

The line p_3 intersects the corresponding degenerated circle k_3 in a finite point and in the point P^∞ at infinity. The isogonal image of this point is the point $P' = I(P^\infty)$ on the circle f circumscribed to the base triangle (Fig. 5). k^3 is identical with the cubic containing the pairs of isogonally corresponding points on the lines through P^∞ in the sense of Lemma 2 as this analagmatic cubic shares with k^3 the nine points $A, B, C, D_1, \ldots, D_4, P^\infty, P'$ together with the absolute points. This implies that the tangent lines of k^3 at $D_1, \ldots, D_4$ pass through P^∞, and the join $P^\infty P'$ is the asymptote of k^3.

By varying the line p_3 in the pencil (C), all curves of the considered pencil of analagmatic circular cubics of genus 1 are obtained. In Fig. 6 the line p_3 is chosen parallel to AB. Therefore, it is an asymptote for the obtained cubic. The point P' coincides with C. The circle f is tangent to the cubic at the point C.

A fancy Remark: If the given double rays d_1, d_2, f_1, f_2 are considered as the base tangents in a 'dual pencil' of conics, then point C is the focus of the only parabola included in this pencil

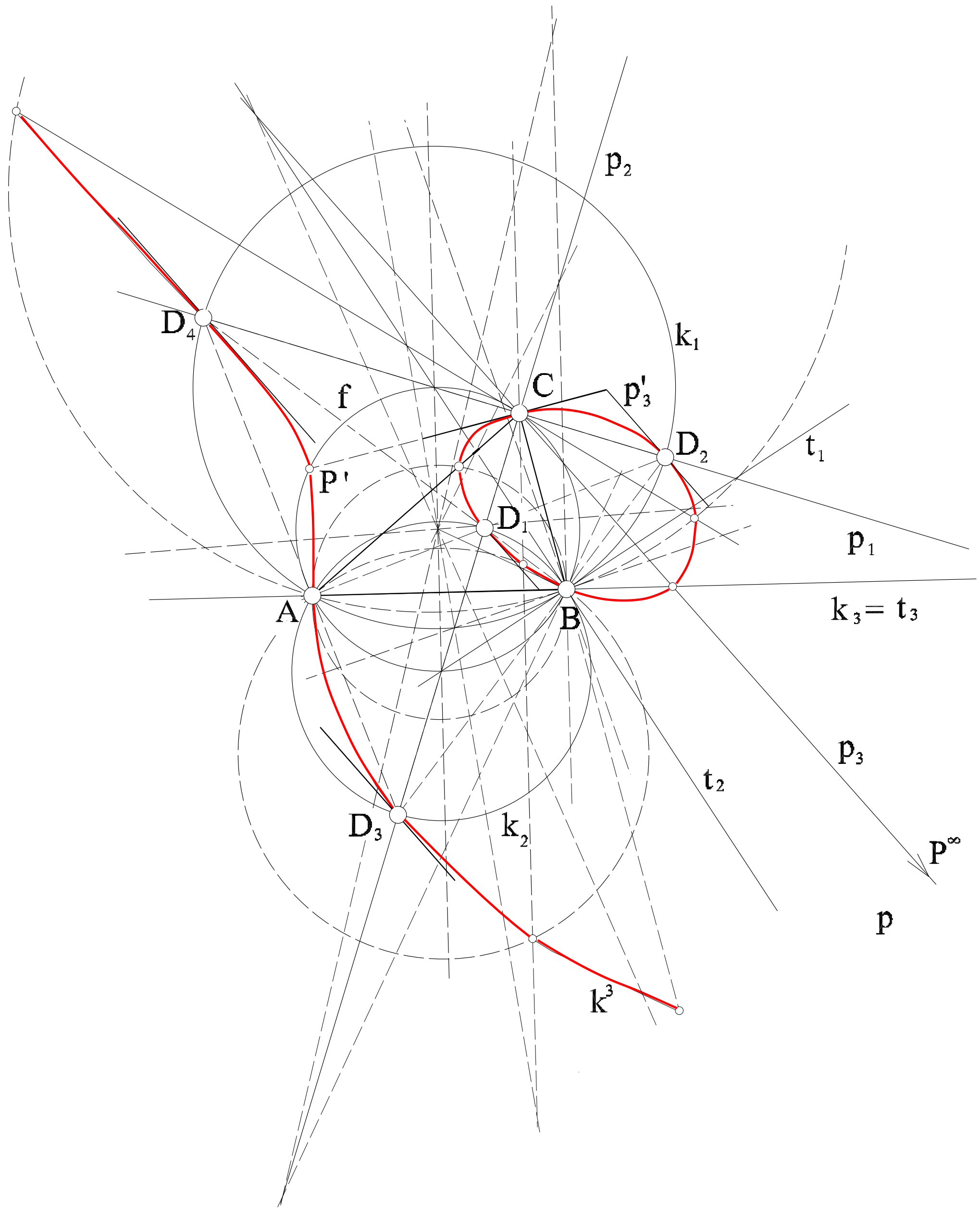

Figure 5: An analagmatic circular cubic k^3. In the sense of Lemma 2 this cubic k^3 includes all pairs of isogonally assigned points on the lines through P^∞

[5]. The point at infinity of this parabola (perpendicular to the director line AB) determines a cubic in the considered pencil of cubics. This cubic coincides with the focal curve of the mentioned pencil of conics [7].

Each cubic in this pencil of analagmatic circular cubics of genus 1 has its singular focus. All singular foci lie on a circle f circumscribed to the base triangle ABC (Fig. 7). This circle includes in particular the centers of the circular components of those cubics which split into a circle and a line. There are the following reducible cubics: $k_1 + d_1$, $k_2 + g_2$, $k_3 + f_1$, $k_4 + g_1$, $k_5 + f_2$, $k_6 + d_2$ with centers $F_1, \dots, F_6$. It should be noticed that the points A, B, C are also singular foci of particular cubics in this pencil. These are three *strophoidals* in the pencil, i.e., cubics which include their singular focus.

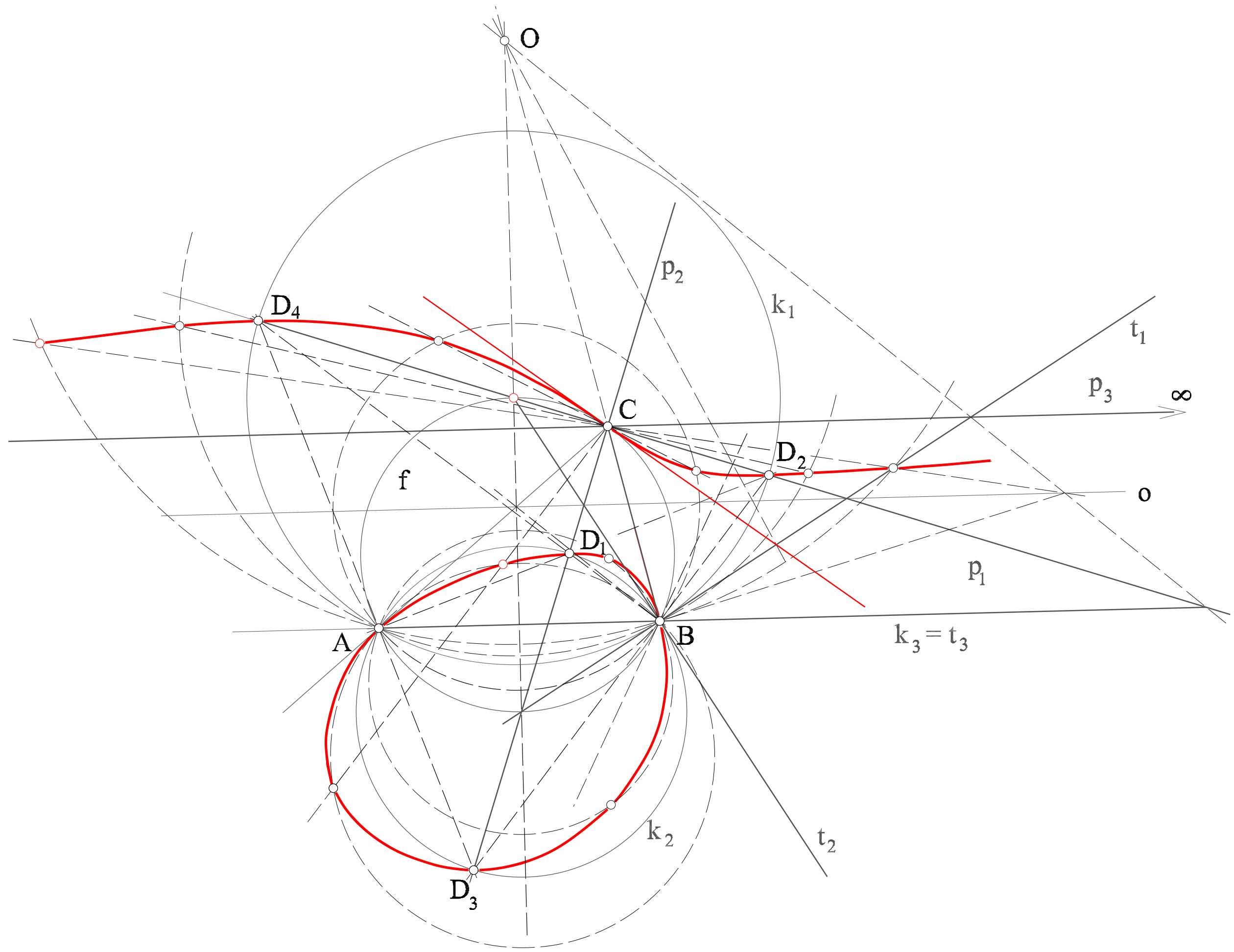

Figure 6: Another analagmatic circular cubic. The lines connecting corresponding points of this cubic are parallel to AB

Beside the mentioned pencil of analagmatic circular cubics of genus 1 there are four pencils of rational analagmatic circular cubics. One example is the pencil of circular cubics which pass through the points A, B, C and which have the double point D_3. Under the isogonal transformation this pencil is mapped onto itself. This pencil contains three degenerate cubics consisting of the join of D_3 with one base point and the circumcircle of D_3 and the two remaining base points. As each of these degenerate cubics is analagmatic, each curve of this pencil remains fixed under I.[1] Any point $S \in f$ determines a cubic of this pencil uniquely. This cubic intersects the line at infinity at the isogonal image S'^{∞} of S (Fig. 8).

It is not difficult to conclude that the circle f circumscribed to the base triangle ABC is again the focal circle of this pencil of cubics. How can any cubic of this pencil be constructed? Any rational circular cubic can be generated as the pedal curve of a parabola. Therefore it is sufficient to determine a negative pedal curve. The tangents of such a parabola are obtained as perpendicular lines a, b, c, s through points A, B, C, S to their joins with the double point D_3 (Fig. 8). As D_3 is located on the director line of this parabola, all cubics in this pencil

[1]That each cubic of this pencil is analagmatic follows also from the fact that all line elements with point D_i remain fixed. This results from Lemma 1 since any line l through D_i is mapped under I onto a conic $I(l)$ which touches l at D_i (note also Theorem 1 as well as in Fig. 3 the curves k and $k^3 = I(k)$ at D_2).

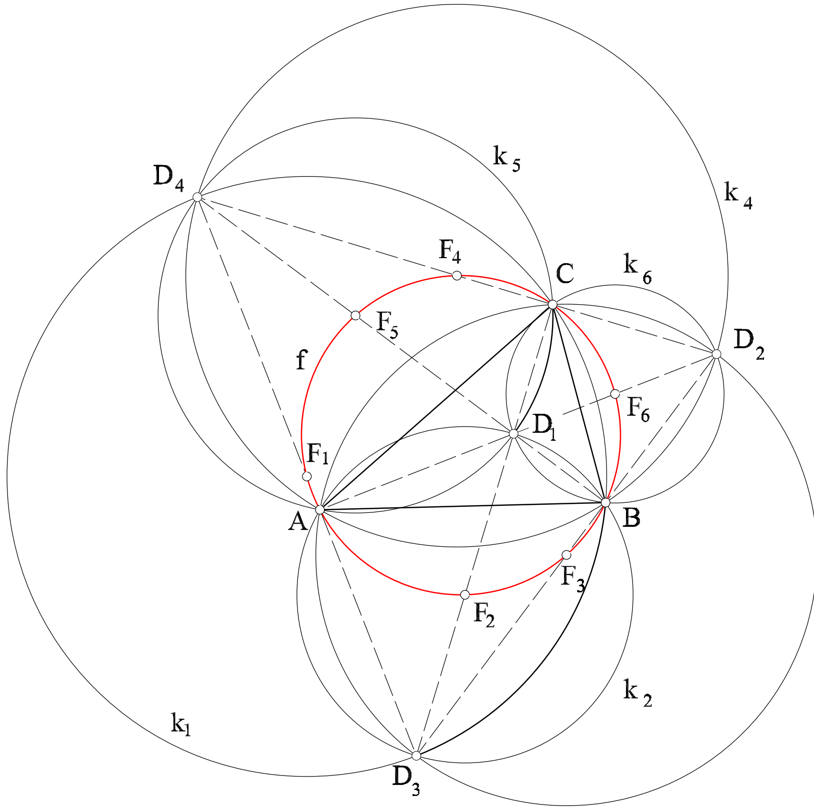

Figure 7: The circle f of singular foci of all analagmatic circular cubics of genus 1

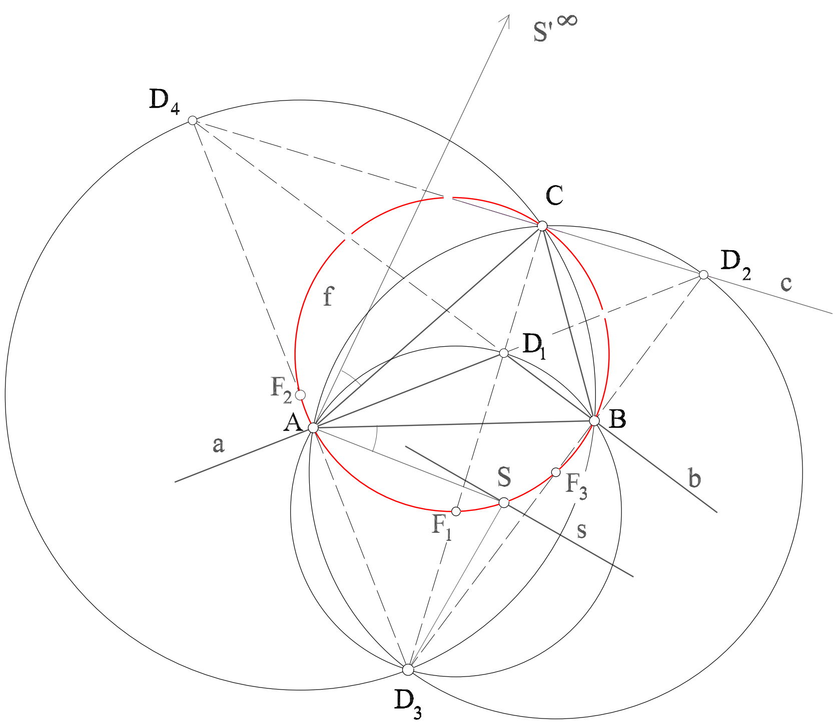

Figure 8: A rational analagmatic circular cubic passing through A, B, C, S and with the knot D_3 is defined as the pedal curve of the parabola tangent to a, b, c, s

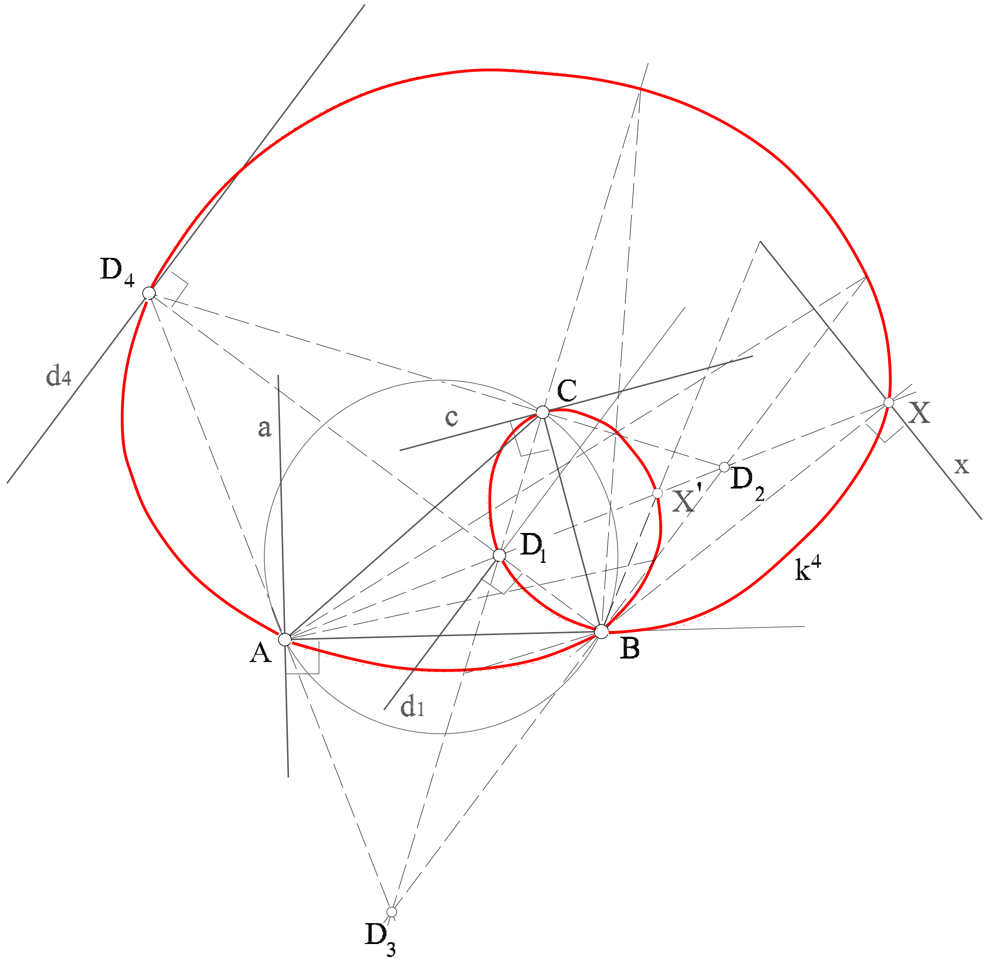

Figure 9: An analagmatic bicircular quartic constructed as the pedal curve of a conic tangent to the lines a, c, d_1, d_4, x

have perpendicular tangents at the knot D_3.

If instead of D_3 the points D_1, D_2 or D_4 are chosen, three additional pencils of rational analagmatic cubics are obtained. We summarize:

Theorem 4 *There are four pencils of rational circular cubics which are analagmatic with respect to I. All irreducible cubics in these pencils are strophoids passing through A, B, C, and with the knot at a fixed point D_i.*

One of the further logical questions is that for the existence of circular curves of fourth order which are analagmatic with respect to the isogonal transformation I. Of course, such curves do exist: All *bicircular* curves of fourth order which have the double point at a vertex of the base triangle, say at B, and which pass through points A, C, D_1, D_4, are an example. They form a pencil which remains fixed under I. Footnote 1 implies again that each curve in this pencil is analagmatic. Any point X on the line D_1D_2, which is isogonally mapped onto a point X' on the same line, determines a curve of this pencil uniquely. Obviously there exist at least three pencils of bicircular curves of fourth order.

The negative pedal curve of any rational bicircular quartic with respect to its third double point is a conic. Thus we can determine an arbitrary number of points of the considered curve. Fig. 9 shows one example of such a bicircular quartic. Its pedal conic with respect to the double point B is determined by the tangents a, c, d_1, d_4, x which are perpendicular to the joins of B with the points A, C, D_1, D_4, X, respectively.

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Received April 14, 2003; final form July 18, 2003