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An Investigation of an Octahedral Platform Using Equiform Motions

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Abstract. In this paper, we investigate motions of the 7-parameter group of equiform transformations with the property that three points move on three circles with axes in one plane. We give an algorithm to find the corresponding one-parametric motion. It can be displayed as a curve in the space of motion parameters. As in general there seems to be no global parametrization of this curve, we give a local one up to the second order. An example demonstrates the effciency of the presented method.

Key Words: Parallel manipulator, equiform motion, flexible octahedra

MSC 2000: 53A17

1. Introduction

Stewart-Gough-platforms (SGP) are 6-leg-platforms with legs connecting points of a moving and a fixed plane. An important special case is that of the so-called "Duffy-platform" [8]. Here the telescopic legs connect two triangles in the moving and the fixed plane, respectively. If at a given position the leg lengths are fixed, this manipulator in general allows no continuous motion. The two triangles and the legs of the platform form an octahedron. Therefore these platforms are called *octahedral platforms* [13, 3]. It is well-know, that there exist snappy, shaky and even moveable models [13, 14, 15, 16] of octahedra. In [8] a rigidity-rate was assigned to the positions of an SGP. This was done by the observation, that such a polyhedron is moveable within the 7-parametric group of Euclidean similarities (equiform motions). In this paper we will investigate the equiform self-motions of such an octahedron. As the leg lengths are kept constant, the vertices of the moving plate (triangle) have to move on circular paths. These circles have axes in the edges of the fixed triangle.

An equiform displacement preserves angles, but all distances are multiplied with the so called *scaling factor* [2], denoted by ρ. The kinematics corresponding to it will be called

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equiform (similarity) kinematics. The group of equiform displacements is 7-parametric and contains the 6-parametric group of Euclidean displacements as a subgroup. In the last years special equiform Darboux-motions have for instance been used to construct overconstrained (Euclidean) mechanisms (see [10, 11, 12]), which show that the study of equiform kinematics is not a pure theoretical task. As in our case the space moves such that three points p_1, p_2, p_3, the vertices of a triangle, are compelled to remain on circles C_1, C_2, C_3 with axes in the $[xy]$-plane. In general, we will have a one-parametric equiform self-motion of this octahedral platform.

1.1. Construction of the motion

We consider three non-collinear points p_i, $i = 1, 2, 3$; each of them should move on a circle C_i with axis in the $[xy]$-plane with radius R_i, center m_i, and parametrized by

$$\vec{C}_i(u_i) = \vec{m}_i + R_i\vec{a}_i \cos u_i + R_i\vec{z} \sin u_i, \quad i = 1, 2, 3 \text{ with } u_i \in [0, 2\pi] \tag{1}$$

where

$$\vec{a}_i^2 = 1 \text{ and } \vec{a}_i \cdot \vec{z} = 0 \text{ with } \vec{z} = (0, 0, 1).$$

This guarantees, that the axis of the circle C_i is part of the $[xy]$-plane. The squared distance between two points A_i and A_j on two different circles C_i and C_j is given by

$$\begin{aligned} \overrightarrow{A_iA_j}^2 = {} & (\vec{m}_j - \vec{m}_i)^2 + R_j^2 + R_i^2 - 2R_iR_j \sin u_i \sin u_j + 2R_j\, \vec{a}_j \cdot (\vec{m}_j - \vec{m}_i) \cos u_j - \\ & -2R_i\, \vec{a}_i \cdot (\vec{m}_j - \vec{m}_i) \cos u_i - 2R_iR_j(\vec{a}_i \cdot \vec{a}_j) \cos u_j \cos u_i \,. \end{aligned} \tag{2}$$

But for our equiform motions, the distance between any two points of the moving space is constant up to the scaling factor, thus we have

$$\overrightarrow{A_iA_j}^2 = d_{ij}^2\rho^2, \quad (i, j) \in \{(1, 2), (2, 3), (3, 1)\}, \tag{3}$$

where d_{ij} is constant. This implies the following three equations

$$\begin{aligned} d_{12}^2\rho^2 = {} & (\vec{m}_2 - \vec{m}_1)^2 + R_2^2 + R_1^2 - 2R_1R_2 \sin u_1 \sin u_2 + 2R_2\, \vec{a}_2 \cdot (\vec{m}_2 - \vec{m}_1) \cos u_2 - \\ & -2R_1\, \vec{a}_1 \cdot (\vec{m}_2 - \vec{m}_1) \cos u_1 - 2R_1R_2(\vec{a}_1 \cdot \vec{a}_2) \cos u_2 \cos u_1 \,, \end{aligned} \tag{4}$$

$$\begin{aligned} d_{23}^2\rho^2 = {} & (\vec{m}_3 - \vec{m}_2)^2 + R_3^2 + R_2^2 - 2R_2R_3 \sin u_2 \sin u_3 + 2R_3\, \vec{a}_3 \cdot (\vec{m}_3 - \vec{m}_2) \cos u_3 - \\ & -2R_2\, \vec{a}_2 \cdot (\vec{m}_3 - \vec{m}_2) \cos u_2 - 2R_2R_3(\vec{a}_2 \cdot \vec{a}_3) \cos u_3 \cos u_2 \,, \end{aligned} \tag{5}$$

$$\begin{aligned} d_{31}^2\rho^2 = {} & (\vec{m}_1 - \vec{m}_3)^2 + R_1^2 + R_3^2 - 2R_3R_1 \sin u_3 \sin u_1 + 2R_1\, \vec{a}_1 \cdot (\vec{m}_1 - \vec{m}_3) \cos u_1 - \\ & -2R_3\, \vec{a}_3 \cdot (\vec{m}_1 - \vec{m}_3) \cos u_3 - 2R_3R_1(\vec{a}_3 \cdot \vec{a}_1) \cos u_1 \cos u_3 \,. \end{aligned} \tag{6}$$

The equations (4)–(6) represent three surfaces in R^4, the space of the variables ρ, u_1, u_2, and u_3. Their intersection in general will be a curve α, which we want to discuss now:

2. Representation of the intersection curve α

The two equations (5) and (6) are linear in $\sin u_3$ and $\cos u_3$, thus we can determine $\sin u_3$ and $\cos u_3$:

$$\begin{aligned} \sin u_3 &= \frac{1}{\triangle}\big[K_1\big(R_1(\vec{a}_1 \cdot \vec{a}_3) \cos u_1 - \vec{a}_3 \cdot (\vec{m}_3 - \vec{m}_2)\big) - K_2\big(R_2(\vec{a}_2 \cdot \vec{a}_3) \cos u_2 - \vec{a}_3 \cdot (\vec{m}_3 - \vec{m}_2)\big)\big] \\ \cos u_3 &= \frac{1}{\triangle}\big[K_2R_2 \sin u_2 - K_1R_1 \sin u_1\big] \end{aligned} \tag{7}$$

where

$$\begin{aligned}\triangle &= R_2 \sin u_2 \big[\vec{a}_3\cdot(\vec{m}_1-\vec{m}_3)+R_1(\vec{a}_3\cdot\vec{a}_1)\cos u_1\big]-\\ &\quad -R_1\sin u_1\big[\vec{a}_3\cdot(\vec{m}_2-\vec{m}_3)+R_2(\vec{a}_3\cdot\vec{a}_2)\cos u_2\big],\\ K_1 &= \frac{-1}{2R_3}\big[d_{23}^2\rho^2-(\vec{m}_3-\vec{m}_2)^2-R_3^2-R_2^2+2R_2\vec{a}_2\cdot(\vec{m}_3-\vec{m}_2)\cos u_2\big],\\ K_2 &= \frac{-1}{2R_3}\big[d_{31}^2\rho^2-(\vec{m}_1-\vec{m}_3)^2-R_1^2-R_3^2+2R_1\vec{a}_1\cdot(\vec{m}_3-\vec{m}_1)\cos u_1\big].\end{aligned}$$

By using $\sin u_3^2+\cos u_3^2=1$, we gain

$$\begin{aligned}&\big[K_1\big(R_1(\vec{a}_1\cdot\vec{a}_3)\cos u_1-\vec{a}_3\cdot(\vec{m}_3-\vec{m}_2)\big)-K_2\big(R_2(\vec{a}_2\cdot\vec{a}_3)\cos u_2-\vec{a}_3\cdot(\vec{m}_3-\vec{m}_2)\big)\big]^2+\\ &\qquad +\big[K_2R_2\sin u_2-K_1R_1\sin u_1\big]^2-\triangle^2=0\end{aligned} \tag{8}$$

Eq. (8) is quadratic in ρ^2. Now we have two remaining equations (8) and (4) in ρ, u_1 and u_2. In the 3-dimensional space of (ρ, u_1, u_2) they represent two surfaces, which intersect in a curve. Its projection into the $[u_1u_2]$-plane is obtained by elimination of ρ.

From (4), one can find

$$\begin{aligned}\rho^2 &= \big((\vec{m}_2-\vec{m}_1)^2+R_2^2+R_1^2-2R_1R_2\sin u_1\sin u_2+2R_2\vec{a}_2\cdot(\vec{m}_2-\vec{m}_1)\cos u_2-\\ &\quad -2R_1\vec{a}_1\cdot(\vec{m}_2-\vec{m}_1)\cos u_1-2R_1R_2(\vec{a}_1\cdot\vec{a}_2)\cos u_2\cos u_1\big)/d_{12}^2\,.\end{aligned}$$

Using the above equation and substituting in (8), we find the equation of this case. Making use of a computer algebra system like Mathematica we can display the projection of this curve in the $[u_1u_2]$-plane.

Theorem 1. *The equiform motion of the moving triangle with respect to the fixed triangle of an octahedral platform is determined by the three equations (4)–(6). They describe the relations between the scaling factor ρ and the three angles u_1, u_2 and u_3, which define the positions of the vertices of the moving triangle on their circular paths.*
In the 4-dimensional space of coordinates $\{\rho, u_1, u_2, u_3\}$ these three equations determine three hypersurfaces, which generally intersect in a curve α. It can be seen as an image curve of the equiform self-motion of the octahedral platform.

The equations of this intersection curve demonstrate, that this curve will not have an explicit parametrization except in special cases. But according to the implicit function theorem we are able to give local parametrizations of this curve. As we are interested in local properties of the corresponding equiform motions (obtained by the curve) we will give an algorithm to generate a power series parametrization in the neighbourhood of any starting position.

3. Local parametrization of the intersection curve

In this section we present a local study of our intersection curve. We use Taylor's expansion to get a power series representation in the parameter t for the parameters u_i, $i=1,2,3$, and ρ at $t=0$. We set

$$\begin{aligned}u_i &= u_{i0}+u_{i1}t+\tfrac{1}{2}u_{i2}t^2+\dots, \qquad i=1,2,3,\\ \rho &= 1+\rho_1 t+\tfrac{1}{2}\rho_2 t^2+\dots\end{aligned} \tag{9}$$

where u_{i0} is the intial value of u_i and $u_{ik} = \left(\frac{d^k u_i}{dt^k}\right)_{t=0}$, $k = 1, 2$. Thus we have

$$\begin{aligned}\sin u_i &= \sin u_{i0} + u_{i1}\cos u_{i0} t + \tfrac{1}{2}\big[u_{i2}\cos u_{i0} - u_{i1}^2 \sin u_{i0}\big]t^2 + \dots \\ \cos u_i &= \cos u_{i0} - u_{i1}\sin u_{i0} t - \tfrac{1}{2}\big[u_{i2}\sin u_{i0} + u_{i1}^2 \cos u_{i0}\big]t^2 + \dots\end{aligned} \tag{10}$$

We substitute eqs. (9) and (10) in (4)–(6) and compare the coefficients of t up to the second order. This results in

$$[t^0]: \quad d_{ij}^2 = (\vec{m}_j - \vec{m}_i)^2 + R_j^2 + R_i^2 - 2R_iR_j \sin u_{i0} \sin u_{j0} + 2R_j \vec{a}_j\cdot(\vec{m}_j - \vec{m}_i)\cos u_{j0} - \\ - 2R_i\vec{a}_i\cdot(\vec{m}_j - \vec{m}_i)\cos u_{i0} - 2R_iR_j(\vec{a}_i\cdot\vec{a}_j)\cos u_{j0}\cos u_{i0}, \tag{11}$$

$$[t^1]: \quad d_{ij}^2\rho_1 + F_{ij}u_{i1} + F_{ji}u_{j1} = 0, \tag{12}$$

$$[t^2]: \quad d_{ij}^2\rho_1^2 + d_{ij}^2\rho_2 + Q_{ij}u_{i1}u_{j1} + L_{ij}u_{i1}^2 + L_{ji}u_{j1}^2 + F_{ij}u_{i2} + F_{ji}u_{j2} = 0, \tag{13}$$

where F_{ij}, Q_{ij} and L_{ij} are constants and given by the initial values

$$\begin{aligned}F_{ij} &= R_i\big[-\sin u_{i0}\big(R_j(\vec{a}_j\cdot\vec{a}_i)\cos u_{j0} + (\vec{m}_j - \vec{m}_i)\cdot\vec{a}_i\big) + R_j\cos u_{i0}\sin u_{j0}\big], \\ Q_{ij} &= 2R_iR_j[\cos u_{i0}\cos u_{j0} + (\vec{a}_j\cdot\vec{a}_i)\sin u_{i0}\sin u_{j0}], \\ L_{ij} &= R_i\big[-\cos u_{i0}\big(R_j(\vec{a}_j\cdot\vec{a}_i)\cos u_{j0} + (\vec{m}_j - \vec{m}_i)\cdot\vec{a}_i\big) - R_j\sin u_{i0}\sin u_{j0}\big].\end{aligned}$$

The eqs. (11) are satisfied from the initial conditions. For the first order t^1 we gain the three linear and homogenous equations (12) for the unknowns $\rho_1, u_{11}, u_{21}, u_{31}$. It is straightforword to get

$$\begin{aligned}\rho_1 &= -(F_{12}F_{23}F_{31} + F_{21}F_{32}F_{13})v, \\ u_{21} &= (d_{12}^2F_{32}F_{13} - d_{31}^2F_{12}F_{32} + d_{23}^2F_{12}F_{31})v \\ u_{31} &= (d_{31}^2F_{23}F_{12} + d_{23}^2F_{21}F_{13} - d_{12}^2F_{23}F_{13})v,\end{aligned} \tag{14}$$

where

$$v = \frac{u_{11}}{d_{12}^2F_{31}F_{23} + d_{31}^2F_{21}F_{32} - d_{23}^2F_{21}F_{31}}$$

with arbitrary u_{11}.

In an anologous way we use the equations (13) from the quadratic terms in t and get

$$\begin{aligned}\rho_2 &= \frac{(T_{12}F_{31}F_{23} + T_{31}F_{21}F_{32} - T_{23}F_{21}F_{31})v + \rho_1u_{12}}{u_{11}}, \\ u_{22} &= \frac{(T_{12}(d_{31}^2F_{32} - d_{23}^2F_{31}) + T_{23}d_{12}^2F_{31} - T_{31}d_{12}^2F_{32})v + u_{21}u_{12}}{u_{11}}, \\ u_{32} &= \frac{(T_{31}(d_{12}^2F_{23} - d_{23}^2F_{21}) + T_{32}d_{31}^2F_{21} - T_{12}d_{31}^2F_{23})v + u_{31}u_{12}}{u_{11}},\end{aligned} \tag{15}$$

where T_{ij} are given by

$$T_{ij} = -[d_{ij}^2\rho_1^2 + Q_{ij}u_{i1}u_{j1} + L_{ij}u_{i1}^2 + L_{ji}u_{j1}^2]$$

for arbitrarily chosen u_{12}.

Remark 1: The behaviour of the problem does not change, if we go for higher powers in t.

Remark 2: The result in (14) and (15) allows to rate the instantaneous rigidity of the platform up to the 2^{nd} order. So, $\rho_1 = 0$ will characterize a singular or shaky position [13]. If additionally $\rho_2 = 0$, the position is singular of order 2.

Theorem 2. *The intersection curve α of Theorem 1 generally will have no simple representation. Beginning with a starting position ($t = 0$) the equations (14)–(15) determine the Taylor expansion of a representation of α near $t = 0$ up to the second order.*

4. Example

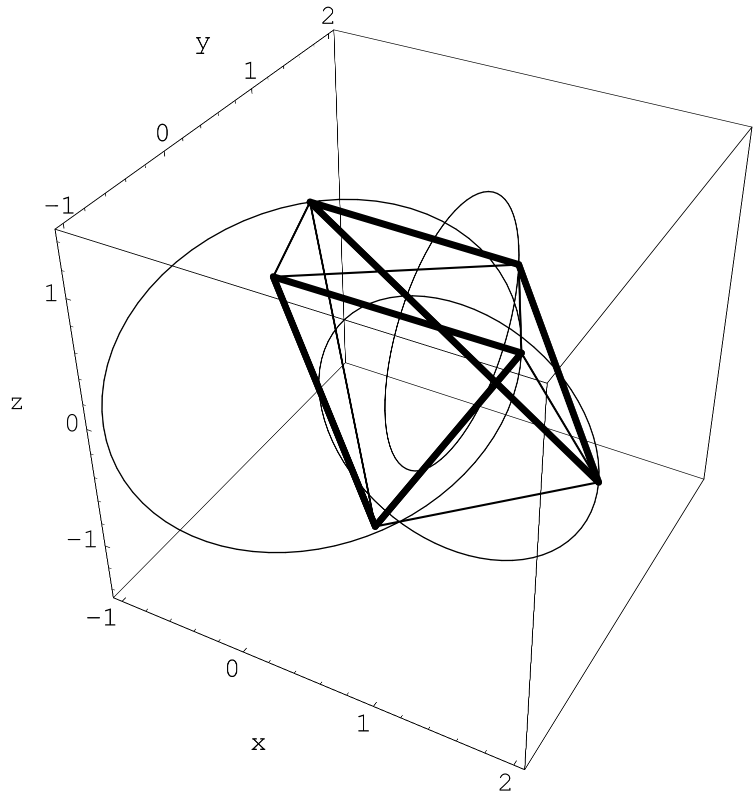

Figure 1: The platform and the three path circles at the intial position

In this section we give a numeric example to show how the theory can be applied. Consider the three points moving on three circles in the fixed space with centers $\vec{m}_1 = (0,0,0)$, $\vec{m}_2 = (1,0,0)$, $\vec{m}_3 = (\frac{1}{2},1,0)$, axes $\vec{a}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\vec{a}_2 = (1,0,0)$, $\vec{a}_3 = (0,1,0)$, radii $R_1 = \sqrt{2}$, $R_2 = 1$, $R_3 = 1$ and angles at the intial point $u_{10} = \frac{\pi}{2}$, $u_{20} = 0$, $u_{30} = 0$. Fig. 1 displays this platform.

For this platform, one can find

$$d_{12} = \sqrt{6}, \quad d_{23} = 5/2\,, \quad d_{31} = 5/2\,.$$

The equations (4)–(6) give now

$$\begin{aligned} 3\rho^2 + \cos u_1 - \cos u_2 + \cos u_1 \cos u_2 + \sqrt{2}\sin u_1 \sin u_2 - 2 &= 0, \\ 25\rho^2 - 4\cos u_2 - 8\cos u_3 + 8\sin u_2 \sin u_3 - 13 &= 0, \\ 25\rho^2 + 12\cos u_1 - 8\cos u_3 + 8\cos u_1 \cos u_3 + 8\sqrt{2}\sin u_1 \sin u_3 - 17 &= 0. \end{aligned} \tag{16}$$

We eliminate u_3 from the second and third equation of (16), and get

$$\begin{aligned} &\left[\sin u_2(25\sigma + 12\cos u_1 - 17) - \sqrt{2}\sin u_1(25\sigma - 4\cos u_2 - 13)\right]^2 + \\ &\left[(\cos u_1 - 1)(25\sigma - 4\cos u_2 - 13) + 25\sigma + 12\cos u_1 - 17)\right]^2 = \\ &= 64\left[(1-\cos u_1)\sin u_2 - \sqrt{2}\sin u_1\right]^2 \end{aligned} \tag{17}$$

where $\sigma = \rho^2$, and from the first equation of (16) we have

$$\sigma = (\cos u_2 - \cos u_1 - \cos u_1 \cos u_2 - \sqrt{2}\sin u_1 \sin u_2 + 2)/3\,.$$

Substituting the above relation in (17), we find a relation between u_1 and u_2. We can use the Mathematica program and find the contourplot of this relation with contour $\{0\}$ (see Fig. 2). It displays the projection of the intersection curve α onto the $[u_1u_2]$-plane.

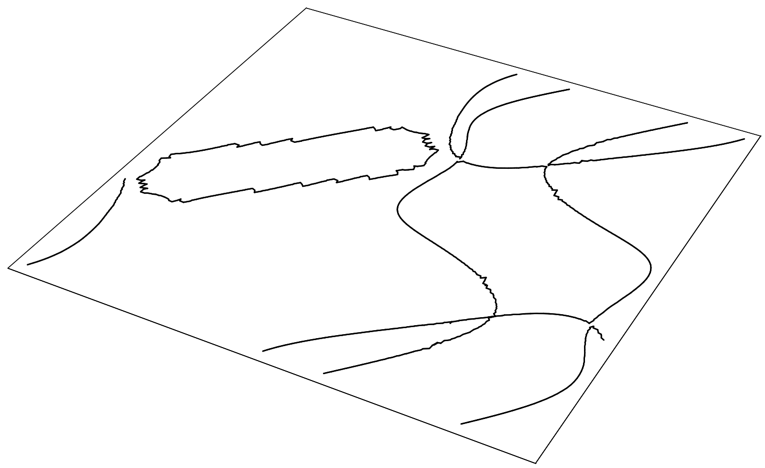

Figure 2: The projection of the curve α into the $[u_1u_2]$-plane

Now, we find the scaling factor up to the second order. In this case we have $u_{10} = \pi/2$, $u_{20} = 0$ and $u_{30} = 0$, using (9), (10) and substituting in (16), one can find that

$$-4u_{11} + 2\sqrt{2}u_{21} + 12\rho_1 = 0, \quad 25\rho_1 = 0, \quad -10u_{11} + 4\sqrt{2}\,u_{31} + 25\rho_1 = 0$$

and

$$\begin{aligned} -2u_{12} + u_{21}^2 + \sqrt{2}u_{22} + 6\rho_1^2 + 6\rho_2 &= 0, \\ 2u_{21}^2 + 8u_{21}u_{31} + 4u_{31}^2 + 25\rho_1^2 + 25\rho_2 &= 0, \\ -10u_{12} + 4u_{31}^2 + 4\sqrt{2}u_{32} + 25\rho_1^2 + 25\rho_2 &= 0. \end{aligned}$$

By solving the first three equations, we find that

$$\rho_1 = 0, \qquad u_{21} = \sqrt{2}u_{11}, \qquad u_{31} = \frac{5\sqrt{2}}{4}\,u_{11}\,,$$

and by solving the other three equations, we get

$$\rho_2 = -\frac{73}{50}\,u_{11}^2\,, \quad u_{22} = \sqrt{2}\Big(\frac{169}{50}\,u_{11}^2 + u_{12}\Big), \quad u_{32} = \sqrt{2}\Big(3u_{11}^2 + \frac{5}{4}\,u_{12}\Big).$$

Remark 3: The octahedral platform of this example is shaky of order one, but not of order two, because $\rho_2 \neq 0$ at our starting position.

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