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Aspects of Geometry and Art[1]

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Dedicated to Prof. Dr. Gerhard GEISE on the occasion of his 75th birthday

Abstract. The presentation is like a mathematician's rummage through pieces of art such as sculptures, drawings, and paintings. It dwells on findings that show conspicuous traces of applied geometry or depict mathematical aspects. The findings are presented and their geometrical background is illuminated.

Key Words: Geometry, art, drawing, painting

MSC 2000: 51N05

1. Perspectives

The study of the famous copperplate engraving *"Melencholia"* by Albrecht DÜRER shows that the German master was able to handle difficult problems of perspective drawings in his time, the Renaissance.

At the first glance the angel attracts the attention. She is symbolising the artist himself and his efforts to raise his state of scientific knowledge. This interpretation is given by the opened book on the lap of the allegorical figure, the compass in her hand, and the thoughtful position of her head. Furthermore, a sphere, a ruler, a magic square of figures, and a polyhedron form a scientific ambience. The polyhedron turns out to be a truncated rhombohedron. The eye distance d_1 or d_2 of DÜRER's drawing can be reconstructed by exploiting geometric properties of the rhombohedron and the cylindrical mill-stone, respectively [5]. DÜRER was outstanding in using exact geometrical rules for art at that time.

To suggest space effect, the surrealist Salvador DALÍ also used perspective. His painting of Mae West is a portrait of the famous and effusive Hollywood actress at that time. On the other hand, a corridor is depicted leading to a living room with a red wall. This wall can be assumed in parallel position to the drawing plane because two rectangular pictures hang there. Are Mae's eyes looking through the picture frames? Are there two pictures showing eyes?

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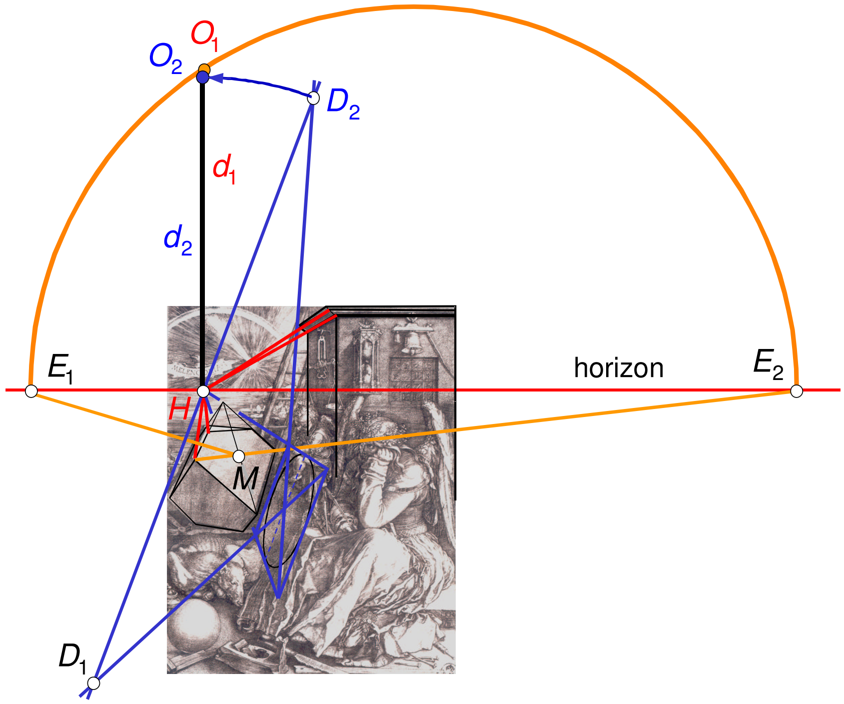


Figure 1: A. Dürer, *Melencholia,* 1514

The lips form a sofa (couch) at the same time. The nose functions as fireplace. All splices of the floor run to the center of the mantel clock which is the main vanishing point of the drawing. Ceiling lines accurately pass through this point if they originally lie perpendicular to the red wall in space. Supposing that the corridor tiles are level and quadratically shaped,

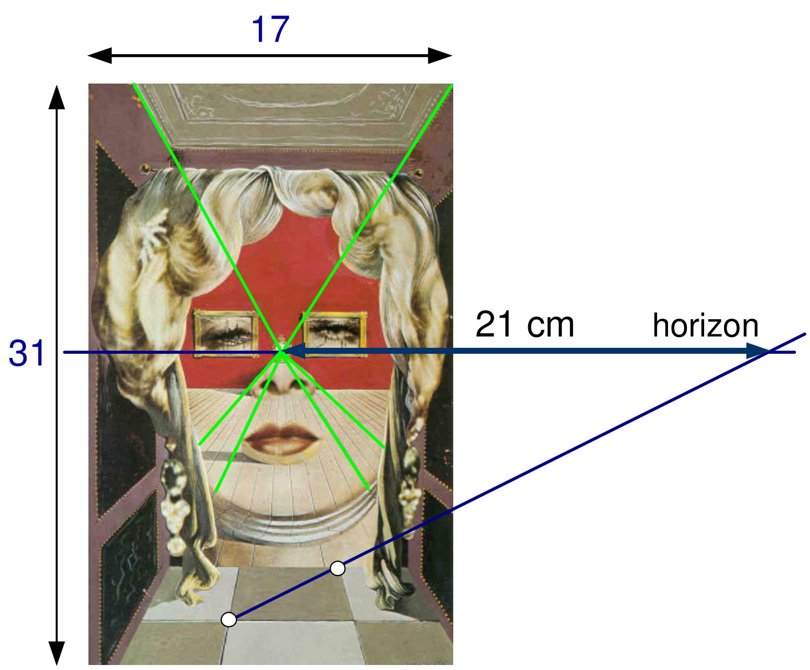


Figure 2: S. Dalí, *Face of Mae West* (may be used as surrealistic apartment), 1934-35

a diagonal line cuts the horizon in a distance point. From there to the main vanishing point we measure 21 cm (in original dimensions). It is the proper eye distance used by Dalí.

2. Concrete or paradox

The perspective projection maps the three-dimensional space onto a two-dimensional image plane. All points of a projection line have the same image point. Therefore, a loss of information is inevitable. Nevertheless, when looking at a perspective drawing people always get the idea to reconstruct the depicted objects in mind.
On the right hand side in Fig. 3, the graph of nine line segments can be the image of a concave room corner if the inner vertical line is conceived behind the other ones. With the reverse assumption the same graph appears to be the image of a convex corner (on the left hand side). If the room corner is based on an orientated Cartesian coordinate system and is formed by three pairwise rectangular plates then Fig. 3 shows the same object projected by different viewpoints.

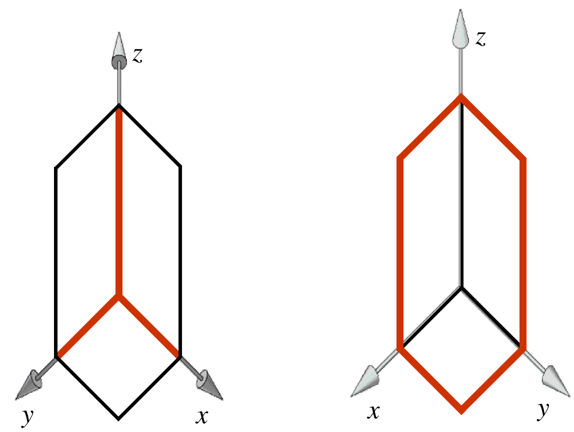


Figure 3: Interpretations of a spatial corner

Figure 4: V. Vasarely, *Shape-Rugó*, 1978 (see K. A. Schröder [6])

In his Op Art, Victor Vasarely resourcefully plays with this interpretation effect for a convex or concave room corner. For example, looking at Fig. 4 we experience the different exciting interpretations. Having accepted one interpretation at a corner of *"Shape-Rugó"* and then wandering along an edge to another corner, sometimes we confirm our interpretations and sometimes we do not.
We may experience an interpretation change. Even though the object appears clearly and geometrically basic, the overall shape can not be discovered. It is part of Vasarely's art that a secret remains for us to reveal.

Meanwhile many artist pose the question whether a depicted object is the image of a three-dimensional object in space. Oscar Reutersvärd ignited the question with his impossible *"Tribar"* in 1934. The proof that such a spatial object can not exist is very simply done by indirect conclusion. Assuming that the object exists, the sum of the angles between the bars is three times 90° that is 270°. Because three vertices of a triangle always lie in a plane, the vertices form a triangle in which the sum of the enclosed angles is always 180 degrees in contradiction to the assumption.

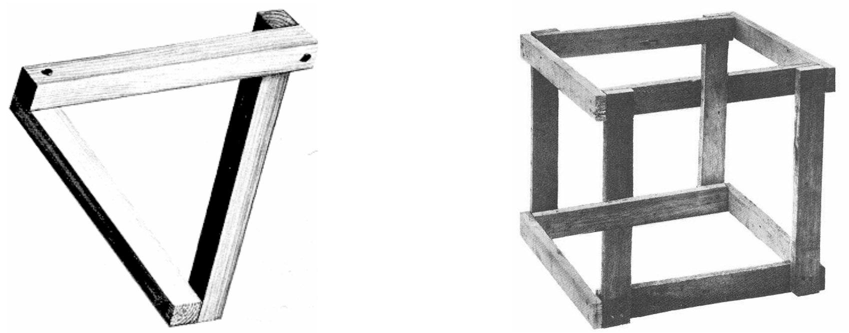


Figure 5: *Tribar* and *Crazy Crate*

Until the present time, a lot of impossible objects had been found and published, for example the *"Crazy Crate"* also shown in Fig. 5. Nowadays, we are used to encountering such pictures. Obviously knowing that the artist F. Kracht opens another intelligent play. He paints objects that seem to be impossible objects but some are not. Therefore, on the front page of an exhibition catalog [3] (Fig. 6, left) he puts the question *"Concrete or Paradox"*. For the two objects in Fig. 6, for instance, the answer is "concrete". In the case of the fourfold bent beam, the proof is given on the left hand side in Fig. 7. On the right hand side the orthographic top view of the *"Space Knot"* is added supposing that the artist gave us the front view of the knot.

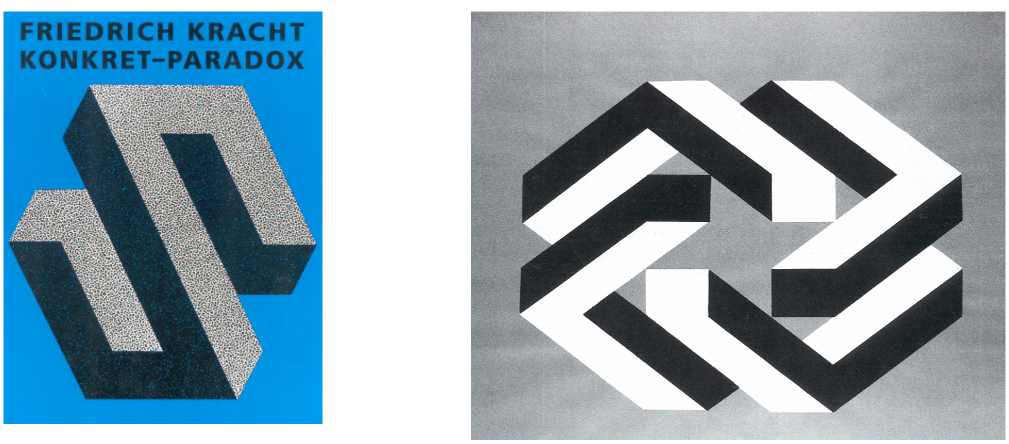


Figure 6: F. Kracht, *Concrete or Paradox*, 2000, and *Space Knot 5*

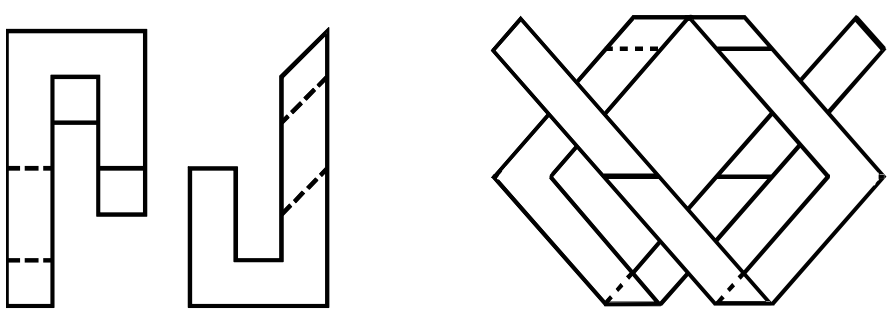


Figure 7: Orthographic views of the solution objects

3. Polyhedra

In former times polyhedra were often represented in art. The solids themselves and their images reflect admirable relations of evenness, order, rotational or reflection symmetry. The world of crystals is strongly related to them. The beauty of light refraction at crystal facets attracts everybody. Here, I restrict myself to presenting artistic by-products that appear in the process of constructing, unwinding, and folding polyhedra. The sculpture by H. Glöckner is an example (Fig. 8).

The sculpture is formed by a rectangular strip of stainless sheet metal. It is a nice exercise of spatial imagination to reconstruct the artist's idea without having the opportunity to measure the real sculpture. How to fold up a rectangular strip to build such a sculpture? To make the problem more interesting, the answer has to consider the stringent condition that the folded strip has two points of contact in equal heights as observed in the photos. A solution is given in Fig. 8. The calculated measures are rounded up and down, respectively. The points P_1 and P_2, and also the points Q_1 and Q_2 had to be identical after the folding up of the strip. The strip and folding lines are central symmetric with respect to the strip midpoint. This relation transforms to the spatial sculpture which allows a half turn about a vertical mid axis to move onto itself.

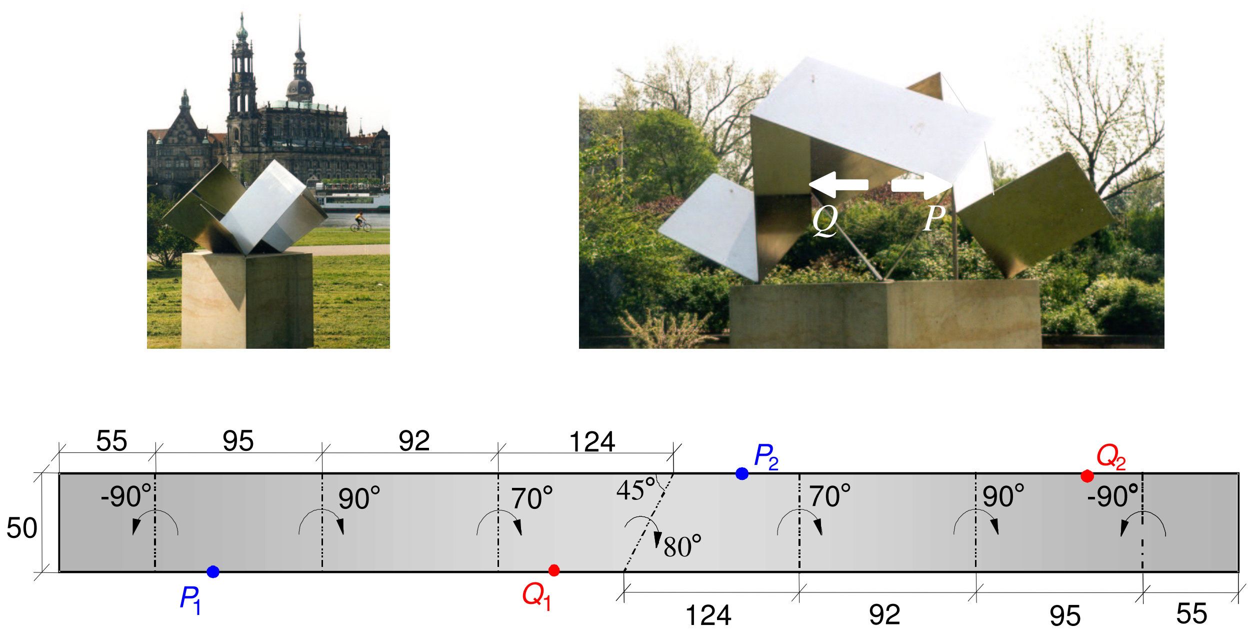


Figure 8: H. Glöckner, *Sixfold Broken Strip,* 1967

In the 90's T. Stengel created installations and monoprints by the help of *"rolled-out" cubes.* He systematically studied the various possible unrollings — each one became a so-called *module* of his installations.

Fig. 9 shows only five such modules arranged along a straight line. Practically, the modules are black wax castings manufactured by unrolling the cube in sand-filled cast-iron frames. Until now nobody answered the question, *"How many incongruent unrollings exist?"* But there is an interim result for unrollings that only allow revolutions about cube edges. In this case exact 11 modules exist, of course consisting of different arranged squares [1]. For large-scale installations (see Fig. 10) Stengel used about 40 different modules. These *"mor-*

Figure 9: T. Stengel, *Basic Line*, 1995

phemes" are combined to *"words"* and *"phrases"*, forming ornaments or patterns [7].

The left hand side of Fig. 11 shows a photo of *"Container III"*, a wall arrangement that includes the transportation cardboard box on which six cardboard wrappers are piled for the single modules. Each wrapper is the frame for a photocopy of the intended content: so aspects of the artist's working process are translated into the installation.

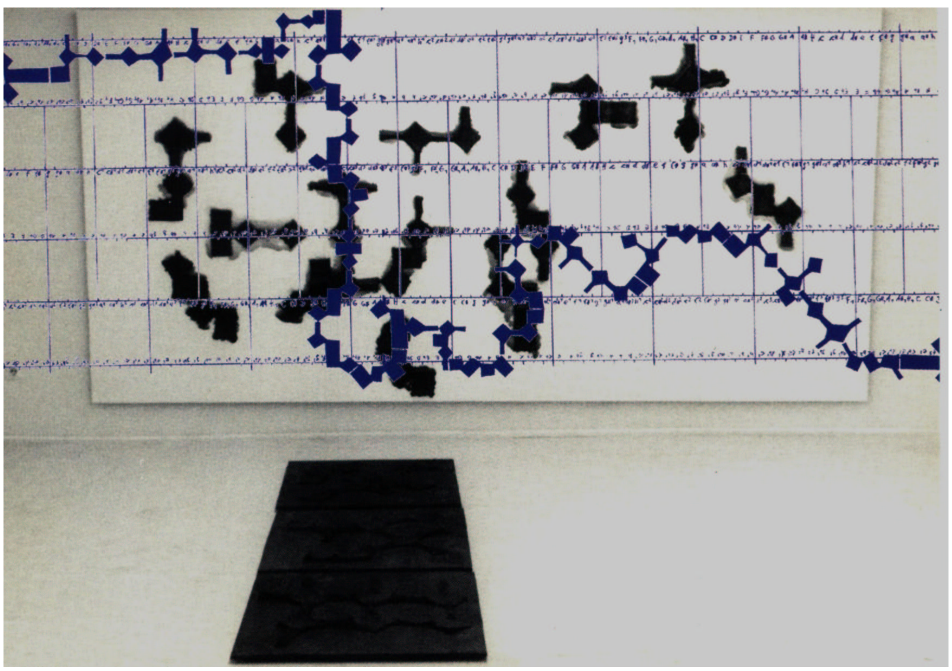


Figure 10: T. Stengel, *Rauminstallation*, 1994
(combined with a composition for 4 instruments by H. Dorschner)

The right hand side of Fig. 11 shows a pictorial montage by Stengel. The back ground consists of a measuring map of a landscape. Above that an unfolded net of a crystal, the mica quartz, is drawn. An equilateral triangle composed by letters of the crystal's German nickname *"Katzengold"*, and seal vertices are added. It is a graphically attractive and sophisticated example of visualized mathematics by bringing together aspects of our micro- and macro-world.

4. Visualizing mathematics

The visualization of mathematics can be carried out by showing symbols, patterns, drawings, models, and other physical concretizations of our mathematical imagination. A very pure and heavy example in this sense presently stands in front of the New National Gallery in Berlin, Germany. By the help of very simple geometric solids — two pyramides and a prismatic body — Barnett Newman created his famous *"Broken Obelisk"* (see Fig. 12, [2]). We can detect in

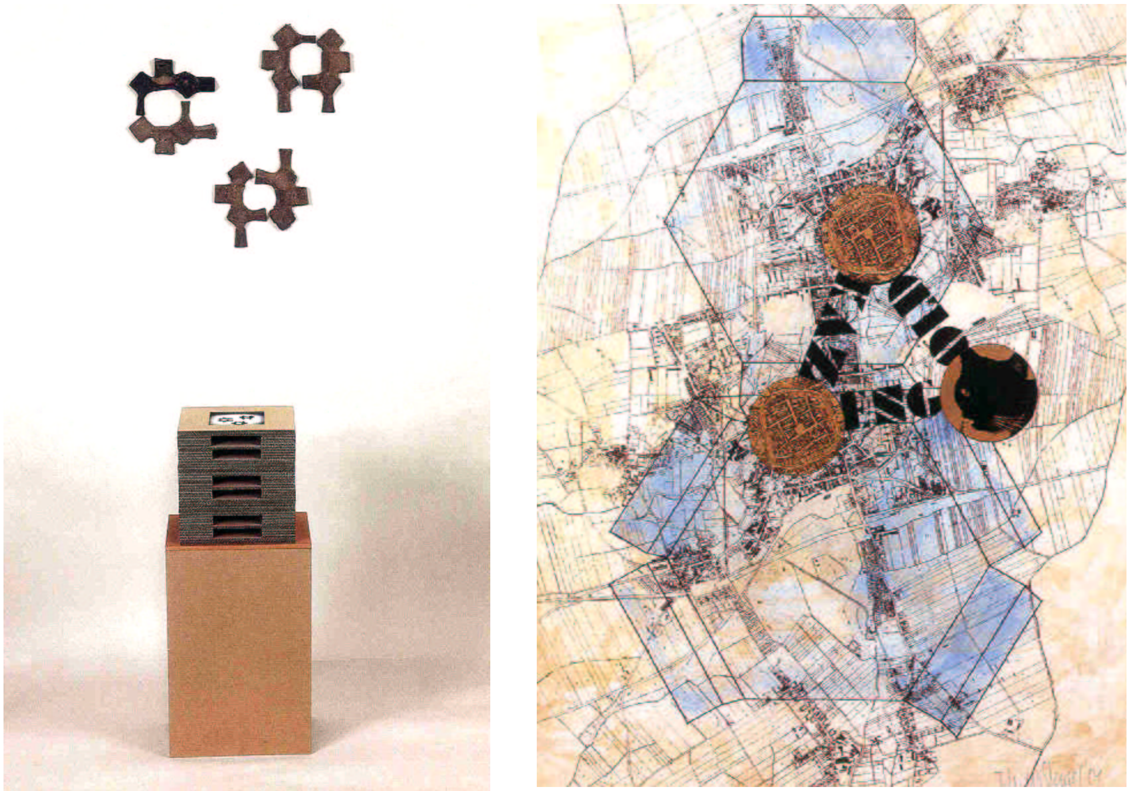


Figure 11: T. Stengel, *Container III*, 1995, and *Mica Quartz*, 2001

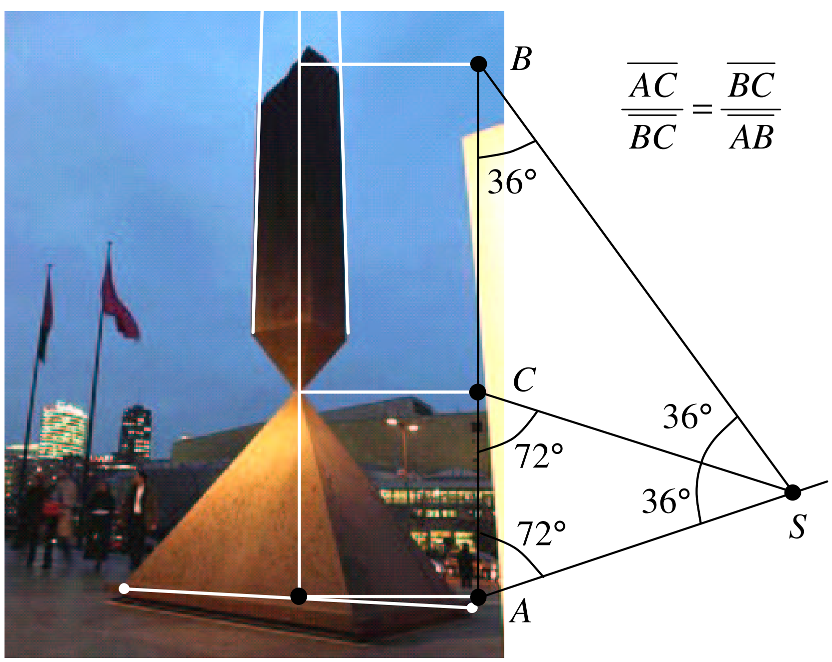
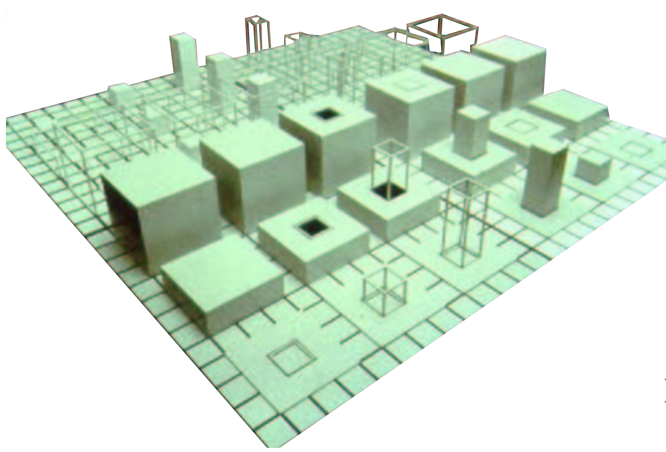


Figure 12: B. Newman, *Broken Obelisk*, 1963-69
(in front of the Neue Nationalgalerie Berlin)



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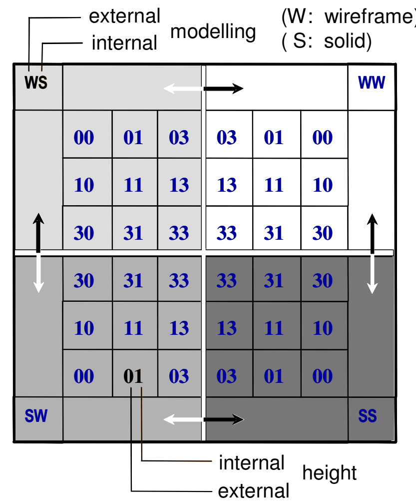

Figure 13: Sol LeWitt, *Serial Project # 1 (ABCD)*, 1966
Museum of Modern Art, New York

Fig. 12 that the artist also applied the golden ratio. Here, by the help of two special similar isosceles triangles we see that the apex of the pyramid divides the total height according to the golden ratio.

The next example does not reveal the underlying mathematics at the first glance. In Fig. 13 the installation *"Serial Project No. 1"* is shown, already made 1966 by Sol LeWitt [2]. He is acclaimed as one of the most powerful artists of Minimal and Conceptual Art. We infer that he already introduced in 1966 what today is called wire frame and solid modelling in the scientific world of computer graphics.

So, we detect in this piece of art $4 * 9$ double cubes. A double cube consists of two cubes which are wireframe or solid-models. We observe three different heights of the cubes.

Distributed on a square plate the double cubes are ordered according to a square matrix. The first matrix row is supposed to correspond with the last row in the installation. The first column should correspond to the left hand side of the installation. Consequently, each matrix element can encode the properties of the corresponding double cube. In the background of the installation all external cubes are modelled by wireframe, the internal cubes by solid or wireframe. With letter W and S for wireframe and solid, respectively, the encoding is WS for the first three rows and columns etc. The three different heights may be encoded by 0, 1, and 3. The art consumer is challenged to imagine cubes which have the height zero. Furthermore, it is not always apparent what happens inside a cube. Then my suggestion is to follow the most adjacent principle of order and symmetry to complete the matrix. In the end a very symmetrical encoding for this piece of art is derived that shows clearly Sol LeWitt's conceptual idea.

Some mathematicians refused to use any concrete objects to explain a structure or a formula — others considered models very helpful for understanding and deduction. For a long time practical needs of architects and technicians developed the discipline "Descriptive Geometry" and also the manufacture of artistic drawings and sculptures. By algebraic equations and parametric representations, geometry describes curves and surfaces which are abstractions of investigated natural forms. Computer Aided Geometric Design and Computer Vision took up familiar methods and developed new principles.

So, our world today is full of visualized mathematics. Sometimes the output is considered to be art. In this case it hits our senses of excitement and astonishment and we wonder about the beauty and originality of a form, pattern or structure. Sometimes symbolic interpretations are possible.

As an example, the symbolic sculptures by J. Robinson can be named [4]. Fig. 14 shows his work *"Eternity"* (height 5 ft, polished bronze). The surface can be generated by moving a regular triangle (fibre) around a circle (base), and at the same time the triangle rotates about the tangent line of the circle which is the axis of this second rotation. After one revolution around the circle, the triangle reaches its original position if it is rotated by any integer multiple of 120°. For making *"Eternity"*, this multiple is one. In this case the triangle vertices sweep out only one edge on the surface — and each edge of the generating triangle sweeps out the same eternal (closed) surface.

Obviously, the generating triangle can be replaced by any other rotationally symmetrical curve. For the surface design of Fig. 15, I used an astroid (hypotrochoid with four cusps) as fibre. A part of the generated surface has been omitted in order to depict the cross section clearly. Such surfaces illustrate a general concept, the theory of fibre bundles, which is very important in modern mathematics.

Figure 14: J. Robinson, *Eternity*, 1980

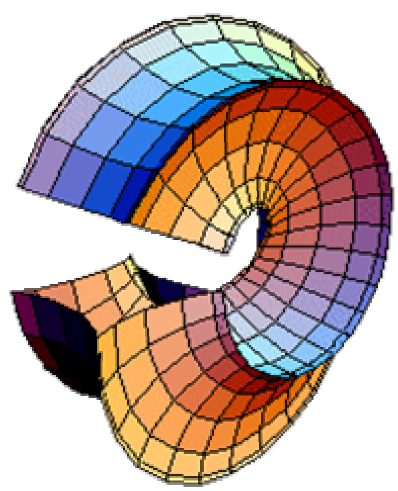

Figure 15: The fibre is taken to be an asteroid

Finally, Fig. 16 shows a piece of visualized mathematics that the author found during testing various graphics programs on his home PC. In mathematical terms, this piece *"XY-modulo11"* is simply a density plot of the function $f(x, y) = xy$ (modulo 11) of two real variables x and y over a specific definition area. The choice of the area is very important for the graphical result because the possible value of this real function is the remainder of the product xy when it is divided by 11. So, all possible values lie between 0 and 10.9999... This interval was divided into an arbitrary number of subintervals with the same number of provided colors. Points are colored identically if their function values lie in the same subinterval.

Furthermore, the boundaries of neighboring value regions may be rendered by various methods — dotted, linear or smoothly interpolated etc. There are a lot of graphical options

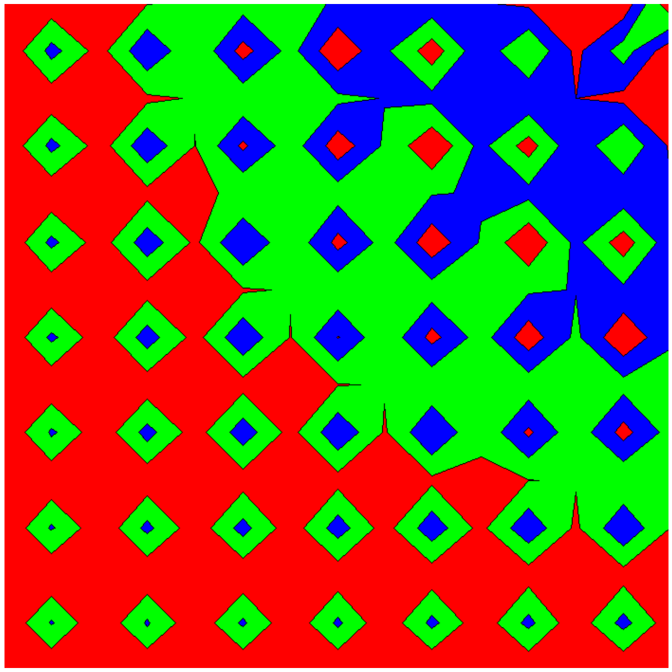

Figure 16: G. Bär, *xymodulo11col3*, 2002

for the curves and points. It depends on all these input data whether the graphics output appears to be nearly a piece of art. Fig. 16 is ornamental and clear but it is surprising for a mathematician because it hides the reality of the function which it renders. The graphical software works in this example with input data that are intentionally chosen unsuitably. Look at Fig. 17, on the left hand side a cross section of the function $f(x, y) = xy$ (modulo 11) is plotted in the plane $y = 5$. We observe 15 peaks and depths along the cross section (as it has to be). In the plane $y = 33$, the cross section oscillates 99 times. So, the function describes a surface which is extremely rich concerning peaks at 10.9999... and depths at 0. In general, graphics software doesn't worry about unsuitable data. Standard 3D-rendering omits a lot of peaks. So, be especially meticulous with computer output when you are a mathematician by profession. Otherwise be prepared and enjoy when beautiful output surprise you. Be pleased with the originality of forms hidden in a simple formula.

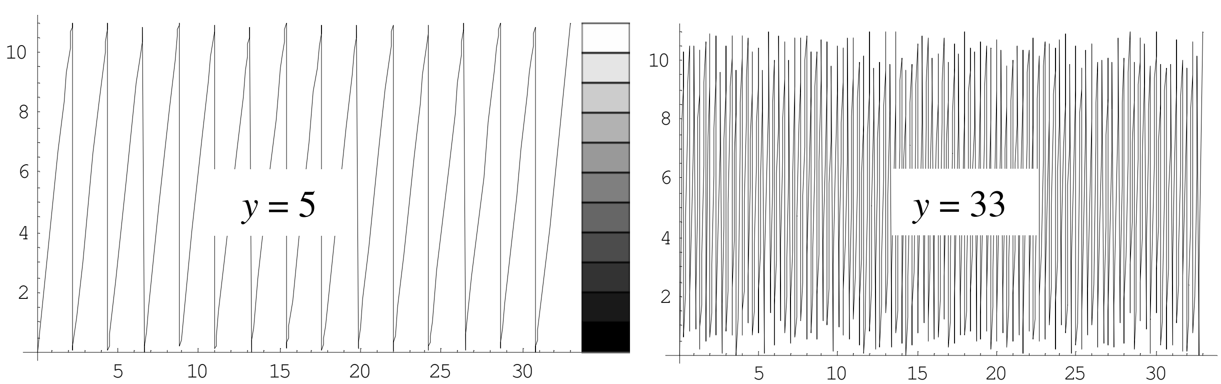

Figure 17: Cross sections of the function

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