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Examples of Bézier-Surfaces of Revolution

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Abstract. This paper gives examples of polynomially parametrizable surfaces of revolution. The first section presents examples which have the additional property of being ruled surfaces. The second section presents a simple and direct approach for the construction of a class of polynomial surfaces of revolution, based on the mapping $f : \mathbb{C} \to \mathbb{C}$, $w \mapsto w^m$ for $m \in \mathbb{N}$. The property of being polynomially parametrizable makes this type of surfaces useful in mathematics not only for studying the solution sets of certain diophantine equations but also in the field of CAGD, especially when Bézier-surfaces of revolution are needed.

Key Words: Surfaces: ruled, of revolution, Bézier-, polynomial

MSC 2000: 53A05, 68U05

1. Introduction

In this paper I choose and give a few examples of surfaces of revolution which admit a polynomial parametrization. Well-known examples are the paraboloid, hyperboloid and cone of revolution.

A method of constructing a polynomially parametrizable surface of revolution is presented in Section 2 and has some potential of investigation by interested readers, see Example 1 (page 4).

The problem of finding other — or even all — classes of surfaces of revolution which admit a polynomial parametrization as Bézier-surfaces is — in my opinion — non-trivial.

1.1. Cone of revolution as Bézier-surface of degree (2, 1)

The implicit representation of a circular right cone (also known as cone of revolution) is

$$x_1^2 + x_2^2 - x_3^2 = 0. \tag{1}$$

This cone is intersected with the plane $x_3 = x_1 + c$ (where $c \neq 0$ denotes a constant) by substituting into the equation (1) of the cone, which leads to

$$\begin{aligned} x_1 &= \frac{1}{2c}(x_2^2 - c^2), \\ x_3 = x_1 + c &= \frac{1}{2c}(x_2^2 - c^2 + 2c^2) = \frac{1}{2c}(x_2^2 + c^2), \end{aligned}$$

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and finally to a polynomial parametric representation of the cone of revolution, which can be written as

$$\mathbf{x}(u,v) = \frac{v}{2c}\begin{pmatrix} u^2-c^2 \\ 2cu \\ u^2+c^2 \end{pmatrix} = \left((1-u)^2,\ 2u(1-u),\ u^2\right)\begin{pmatrix} \begin{pmatrix} 0\\0\\0 \end{pmatrix} & \begin{pmatrix} -c/2\\0\\c/2 \end{pmatrix} \\ \begin{pmatrix} 0\\0\\0 \end{pmatrix} & \begin{pmatrix} -c/2\\1/2\\c/2 \end{pmatrix} \\ \begin{pmatrix} 0\\0\\0 \end{pmatrix} & \begin{pmatrix} \frac{1-c^2}{2c}\\1\\\frac{1+c^2}{2c} \end{pmatrix} \end{pmatrix}\begin{pmatrix} 1-v \\ v \end{pmatrix}.$$

We note here that the parameter lines consist of straight lines and parabolas lying in parallel planes and being similar to each other (see also Fig. 1).

1.2. One-sheet hyperboloid of revolution as Bézier-surface of degree (2, 1)

The implicit representation of a one-sheet hyperboloid of revolution is

$$x_1^2 + x_2^2 - x_3^2 - c^2 = 0 \tag{2}$$

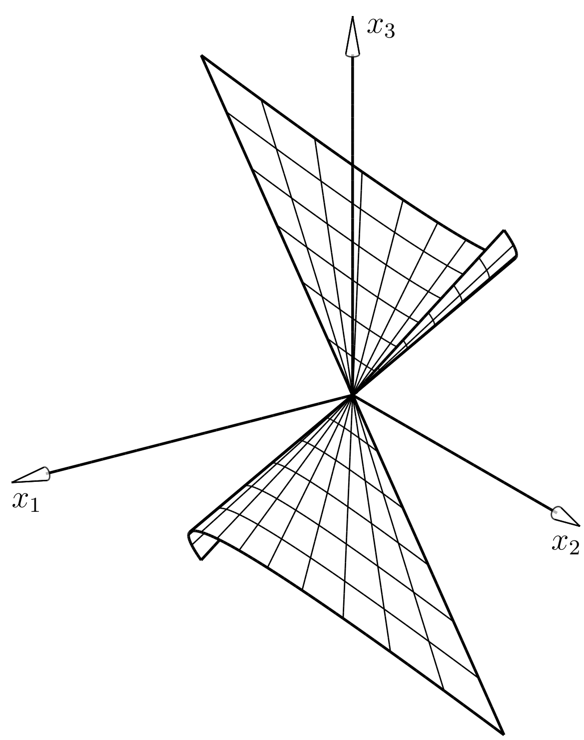

Figure 1: Cone of revolution: $x_1^2 + x_2^2 - x_3^2 = 0$

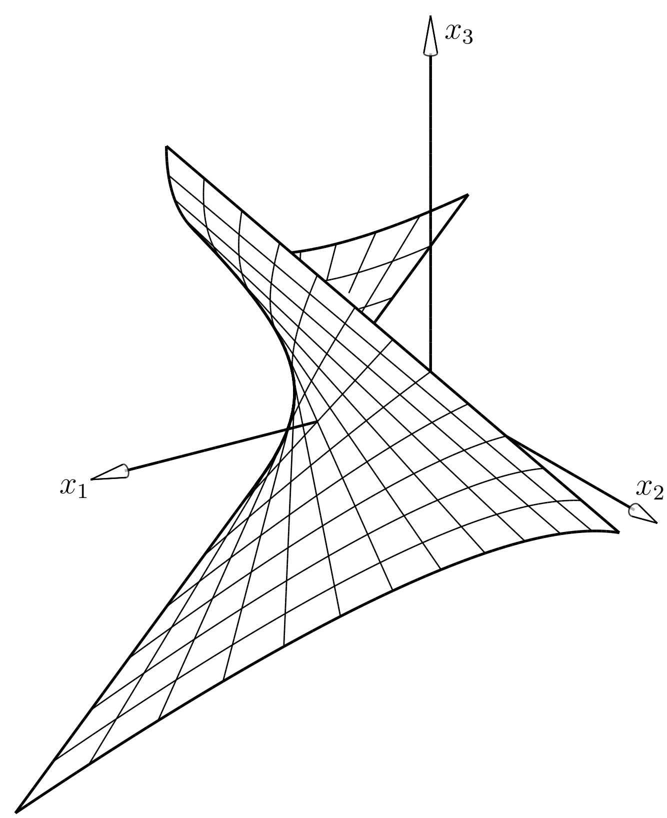

Figure 2: One-sheet hyperboloid of revolution: $x_1^2 + x_2^2 - x_3^2 - c^2 = 0$

where $c \neq 0$ denotes a constant. A *generating line* of this hyperboloid is given by the parametrization

$$\mathbf{y}(u) = \begin{pmatrix} y_1(u) \\ y_2(u) \\ y_3(u) \end{pmatrix} = \begin{pmatrix} c \\ u \\ u \end{pmatrix}.$$

By using this parametrization we can write

$$\mathbf{x}(u,v) = \begin{pmatrix} x_1(u,v) \\ x_2(u,v) \\ x_3(u,v) \end{pmatrix} = \begin{pmatrix} c \\ u \\ u \end{pmatrix} + v \begin{pmatrix} f(u) \\ g(u) \\ h(u) \end{pmatrix} \tag{3}$$

as a parametric representation of the hyperboloid, with polynoms f, g, h in u. This parametric representation is substituted into the implicit equation (2)

$$\begin{aligned} (c+vf)^2 + (u+vg)^2 - (u+vh)^2 - c^2 &= 0, \\ 2v(cf + u(g-h)) + v^2(f^2+g^2-h^2) &= 0, \\ 2(cf + u(g-h)) + v(f^2+g^2-h^2) &= 0. \end{aligned} \tag{4}$$

This equation (4) has to be fulfilled for all values of v, especially for $v = 0$ and $v := \frac{1}{t}$ with $t \to 0$. Therefore the coefficients of this polynomial v have to be zero and we get

$$cf + u(g-h) = 0, \quad f^2 + g^2 - h^2 = 0. \tag{5}$$

By assuming $c \neq 0$ we get from (5)

$$f = \frac{u}{c}(h-g). \tag{6}$$

Substituting into (5) leads to

$$g(c^2+u^2) + h(c^2-u^2) = 0.$$

One solution of this equation is obtained, when we set

$$g := u^2 - c^2, \quad h := u^2 + c^2. \tag{7}$$

From the above calculations we get equations (6), (7) which are substituted into (3) to get the following polynomial parametrization of the one-sheet hyperboloid of revolution

$$\begin{aligned} \mathbf{x}(u,v) &= \begin{pmatrix} c \\ u \\ u \end{pmatrix} + v \begin{pmatrix} 2cu \\ u^2-c^2 \\ u^2+c^2 \end{pmatrix} = \\ &= \left((1-u)^2,\ 2u(1-u),\ u^2\right) \begin{pmatrix} \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} c \\ -c^2 \\ c^2 \end{pmatrix} \\ \begin{pmatrix} c \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} & \begin{pmatrix} 2c \\ \frac{1}{2}-c^2 \\ \frac{1}{2}+c^2 \end{pmatrix} \\ \begin{pmatrix} c \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 3c \\ 2-c^2 \\ 2+c^2 \end{pmatrix} \end{pmatrix} \begin{pmatrix} 1-v \\ v \end{pmatrix}. \end{aligned}$$

We note here that the parameter curves are straight lines and parabolas lying in parallel planes being similar to each other (see also Fig. 2).

2. Bézier-surfaces of revolution constructed from an elementary conformal mapping

As a starting point let us have a look at the mapping[1] $f : \mathbb{C} \to \mathbb{C}$, $w \mapsto w^m$ for $m \in \mathbb{N}$, and note that concentric circles around the origin are transformed into circles of the same family. To see this, we apply f to the points $re^{i\varphi}$ of the circle $w\overline{w} = r^2$. Their images $r^m e^{im\varphi}$ are contained in the circle $w\overline{w} = r^{2m}$. Thus the mapping by f preserves the property "circle centered at 0". This invariant makes the analytic function f a well-suited candidate for the construction of a polynomial surface of revolution $\mathbf{F}\colon \mathbb{C} \to \mathbb{R}^3$:

$$\mathbf{F}(w) = \begin{pmatrix} \mathrm{Re}(w^m) \\ \mathrm{Im}(w^m) \\ \mathbf{Q}(w\overline{w}) \end{pmatrix} = \begin{pmatrix} \frac{w^m + \overline{w}^m}{2} \\ \frac{w^m - \overline{w}^m}{2i} \\ \mathbf{Q}(w\overline{w}) \end{pmatrix} \tag{8}$$

where $m \in \mathbb{N}$ and a polynomial $\mathbf{Q}(t) \in \mathbb{R}[t]$.

Example 1 Polynomial embeddings $\mathbf{x}\colon \mathbb{R}^2 \to \mathbb{R}^3$ of the surface of revolution $\mathbf{F}(w)$ can be obtained by setting $w := p_1(u,v) + ip_2(u,v)$ with polynomials $p_1(u,v),\ p_2(u,v) \in \mathbb{R}[u,v]$. Then we can write

$$\mathbf{x}(u,v) = \begin{pmatrix} x_1(u,v) \\ x_2(u,v) \\ x_3(u,v) \end{pmatrix} = \begin{pmatrix} \mathrm{Re}((p_1(u,v) + ip_2(u,v))^m) \\ \mathrm{Im}((p_1(u,v) + ip_2(u,v))^m) \\ \mathbf{Q}((p_1(u,v))^2 + (p_2(u,v))^2) \end{pmatrix}.$$

Example 2 We can set $w := u + iv$ to get the particular embedding

$$\mathbf{x}(u,v) = \begin{pmatrix} \mathrm{Re}((u+iv)^m) \\ \mathrm{Im}((u+iv)^m) \\ \mathbf{Q}(u^2+v^2) \end{pmatrix},$$

where $x_3(u,v) \in \mathbb{R}[u^2+v^2] \subset \mathbb{R}[u,v]$. To obtain explicit representation formulas for $x_1(u,v)$, $x_2(u,v) \in \mathbb{R}[u,v]$ one can use the binomial formula to develop $(u+iv)^m$ and remember that $i^k \in \mathbb{R}$ for even k and $i^k \in i\mathbb{R}$ for odd k, then group the terms for the real respectively the imaginary part of $(u+iv)^m$ together.

Example 3 A further specialization is the choice of $\mathbf{Q}(t) = t^n$, which gives

$$\mathbf{x}(u,v) = \begin{pmatrix} \mathrm{Re}((u+iv)^m) \\ \mathrm{Im}((u+iv)^m) \\ (u^2+v^2)^n \end{pmatrix}. \tag{9}$$

These coordinate functions satisfy the implicit equation $(x_1^2 + x_2^2)^n = x_3^m$, $x_3 > 0$.

Example 4 For $n = 1$ in eq. (9) we have $\mathbf{Q}(t) = t$ which gives

$$\mathbf{x}(u,v) = \begin{pmatrix} \mathrm{Re}((u+iv)^m) \\ \mathrm{Im}((u+iv)^m) \\ u^2+v^2 \end{pmatrix}. \tag{10}$$

[1] for more information on this elementary *conformal mapping* — because the real and imaginary parts satisfy the *Cauchy-Riemann differential equations* $g_{11} = g_{22}$, $g_{12} = 0$ — see [1, p. 90f]).

These coordinate functions satisfy the implicit equation $x_1^2 + x_2^2 = x_3^m$, $x_3 > 0$, which is the example given in the book [5, p. 143]. By taking now concrete values, we get for $m = 1$ a paraboloid of revolution and for $m = 2$ a right circular cone, which are presented in the next two sections of this paper. For values $m > 2$ the implicit surface has a higher order singularity at the origin.

Example 5 For $m = 1$ in eq. (9) we have $\mathbf{Q}(t) = t^n$ and

$$\mathbf{x}(u,v) = \begin{pmatrix} u \\ v \\ (u^2+v^2)^n \end{pmatrix} \tag{11}$$

satisfying the implicit equation $(x_1^2 + x_2^2)^n = x_3$, $x_3 > 0$.

Example 6 Another special case of eq. (9) is $m = n$. Then $\mathbf{Q}(t) = t^m$, and

$$\mathbf{x}(u,v) = \begin{pmatrix} \mathrm{Re}((u+iv)^m) \\ \mathrm{Im}((u+iv)^m) \\ (u^2+v^2)^m \end{pmatrix} \tag{12}$$

with coordinate functions satisfying the equation $x_1^2 + x_2^2 = x_3$, $x_3 > 0$. Therefore this surface is included in a right circular cone of revolution.

2.1. Paraboloid of revolution as Bézier-surface of degree (2, 2)

For $m = 1$ we get from equation (10) the following polynomial parametric representation of a paraboloid of revolution, which we write as

$$\begin{aligned} \mathbf{x}(u,v) &= \begin{pmatrix} u \\ v \\ u^2+v^2 \end{pmatrix} = \\ &= \left((1-u)^2,\ 2u(1-u),\ u^2\right) \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \end{pmatrix} \begin{pmatrix} (1-v)^2 \\ 2v(1-v) \\ v^2 \end{pmatrix}. \end{aligned}$$

We note here an interesting property of this parametrization:

1. The parameter lines form a *Tschebyscheff-net* on the surface; they satisfy the condition $\dfrac{\partial g_{11}}{\partial v} = \dfrac{\partial g_{22}}{\partial u} = 0$, where g_{11}, g_{22} are coefficients of the first fundamental form of the surface (see [4]).

2.2. Cone of revolution as Bézier-surface of degree (2, 2)

For $m = 2$ we get from equation (10) the polynomial parametric representation of a right circular cone which can be expressed as

$$\mathbf{x}(u,v) = \begin{pmatrix} u^2-v^2 \\ 2uv \\ u^2+v^2 \end{pmatrix} =$$

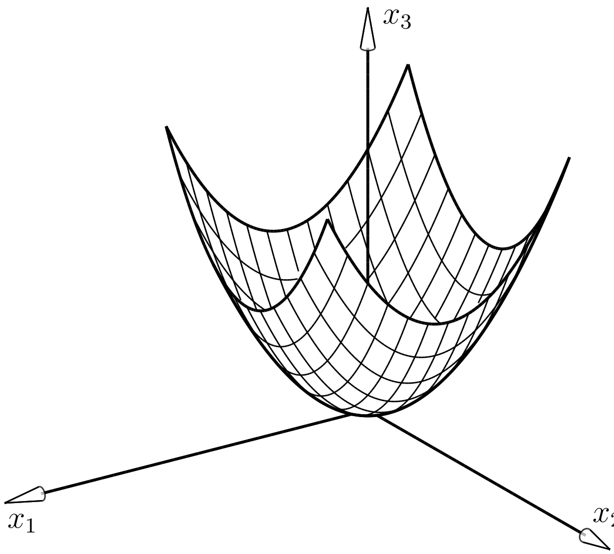

Figure 3: Paraboloid of revolution: $x_1^2 + x_2^2 - x_3 = 0$

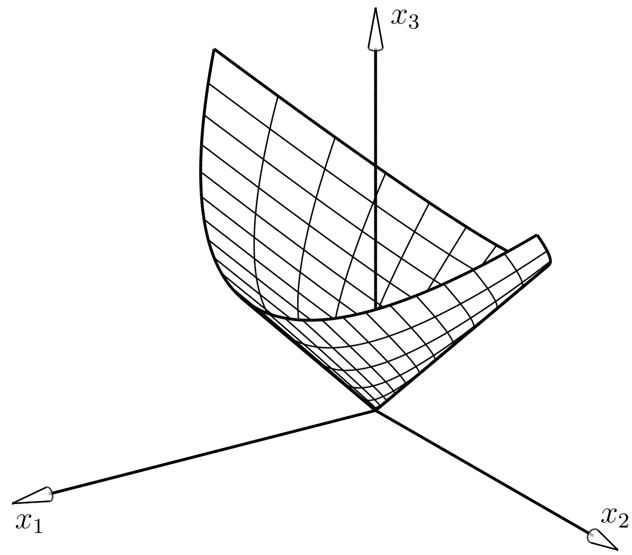

Figure 4: Cone of revolution: $x_1^2+x_2^2-x_3^2 = 0$

$$= \quad \big((1-u)^2,\ 2u(1-u),\ u^2\big) \left(\begin{array}{ccc} \begin{pmatrix} 0\\0\\0 \end{pmatrix} & \begin{pmatrix} 0\\0\\0 \end{pmatrix} & \begin{pmatrix} -1\\0\\1 \end{pmatrix} \\ \begin{pmatrix} 0\\0\\0 \end{pmatrix} & \begin{pmatrix} 0\\ \frac{1}{2}\\0 \end{pmatrix} & \begin{pmatrix} -1\\1\\1 \end{pmatrix} \\ \begin{pmatrix} 1\\0\\1 \end{pmatrix} & \begin{pmatrix} 1\\1\\1 \end{pmatrix} & \begin{pmatrix} 0\\2\\2 \end{pmatrix} \end{array} \right) \begin{pmatrix} (1-v)^2 \\ 2v(1-v) \\ v^2 \end{pmatrix}.$$

We note here two interesting properties of this parametrization:

1. The parameter lines are parabolas lying in two systems of parallel planes (see Fig. 4).
2. By inserting natural numbers u and v, this parametrization leads to the pythagorean triples (one well-known example among them, generated by $(u, v) = (2, 1)$ is (3, 4, 5), with $3^2 + 4^2 = 5^2$) provided the difference $u - v$ is odd. There are many books, mainly on algebra, which use or derive this parametrization for the pythagorean triples. I only want to mention the introductory book [2] on algebraic geometry.

2.3. Particular surface of revolution of degree (3, 3)

For $m = 3$, $n = 1$ we get from equation (9) the polynomial parametric representation

$$\mathbf{x}(u,v) = \begin{pmatrix} \mathrm{Re}((u+iv)^3) \\ \mathrm{Im}((u+iv)^3) \\ u^2+v^2 \end{pmatrix} = \begin{pmatrix} u^3 - 3uv^2 \\ 3u^2v - v^3 \\ u^2+v^2 \end{pmatrix}.$$

These coordinate functions satisfy the implicit equation $x_1^2 + x_2^2 = x_3^3$, $x_3 > 0$ (see Fig. 5).

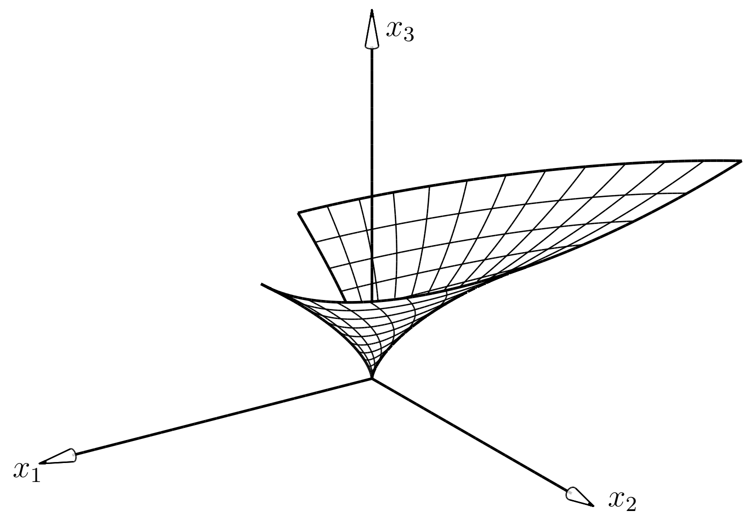

Figure 5: Surface of revolution: $x_1^2 + x_2^2 = x_3^3$.

2.4. Particular surface of revolution of degree (4, 4)

For $m = 4$, $n = 1$ we get from eq. (9) the polynomial parametric representation

$$\mathbf{x}(u, v) = \begin{pmatrix} \mathrm{Re}((u+iv)^4) \\ \mathrm{Im}((u+iv)^4) \\ u^2+v^2 \end{pmatrix} = \begin{pmatrix} u^4 - 6u^2v^2 + v^4 \\ 4uv(u^2-v^2) \\ u^2+v^2 \end{pmatrix}.$$

These coordinate functions satisfy the implicit equation $x_1^2 + x_2^2 = x_3^4$, $x_3 > 0$ (see Fig. 6).

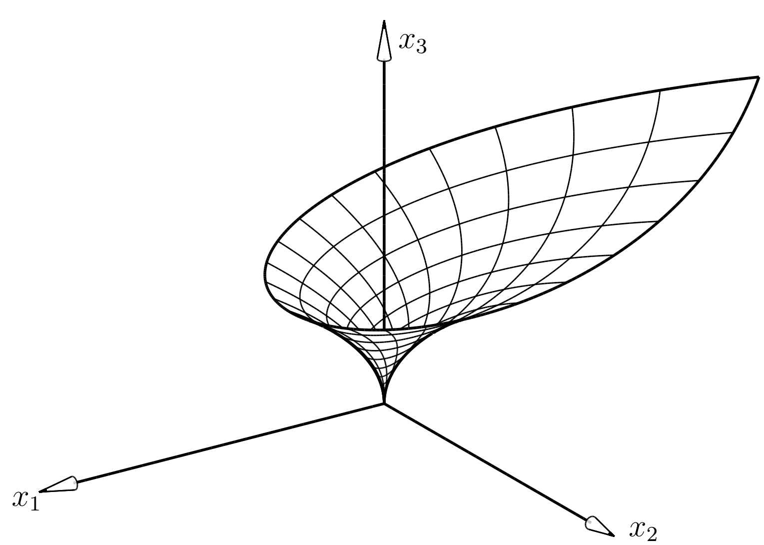

Figure 6: Surface of revolution: $x_1^2 + x_2^2 = x_3^4$

2.5. Particular surface of revolution of degree (6, 6)

For $m = 2$, $n = 3$ in (9) we obtain the polynomial parametric representation

$$\mathbf{x}(u,v) = \begin{pmatrix} \mathrm{Re}((u+iv)^2) \\ \mathrm{Im}((u+iv)^2) \\ (u^2+v^2)^3 \end{pmatrix} = \begin{pmatrix} u^2 - v^2 \\ 2uv \\ (u^2+v^2)^3 \end{pmatrix}.$$

These coordinate functions satisfy the implicit equation $(x_1^2 + x_2^2)^3 = x_3^2$, $x_3 > 0$ (see Fig. 7).

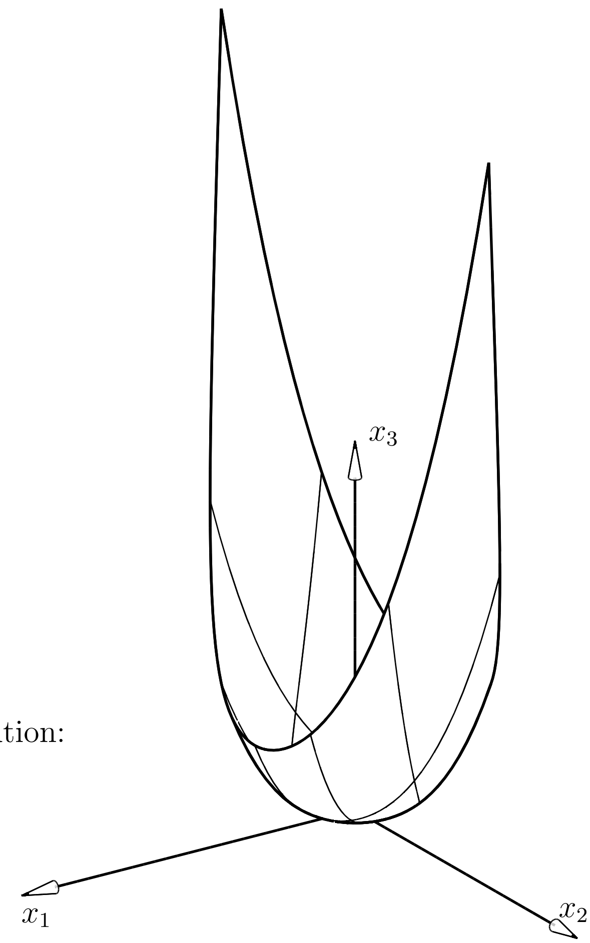

Figure 7: Surface of revolution: $(x_1^2 + x_2^2)^3 = x_3^2$

3. Conclusion

We see that although the parallel circles of a revolution surface cannot be represented as Bézier-curves, there are examples of surfaces of revolution which admit a representation as Bézier-surfaces and do not need to be treated as *rational* Bézier-surfaces or even more generalized notions.

4. Acknowledgements

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