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Moving Central Axonometric Reference Systems

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Abstract. In this paper we give a new, synthetic condition under which a central axonometric mapping is a central projection. This condition is applied for controlling the change of unit points of a central axonometric reference system. Correction of a central axonometric system to be of central projection type is also discussed by the help of our condition.

Key Words: central projection, central axonometry

MSC 2000: 51N05

1. Introduction

Descriptive geometry and its applications widely use central axonometry and central projection to map the projective space $\mathbb{P}^3$ onto the projective plane $\mathbb{P}^2$. Given an orthonormal Cartesian basis in $\mathbb{P}^3$ with origin O, unit points of the axes E_1, E_2, E_3 and points at infinity of the axes U_1, U_2, U_3, the central axonometry is a surjective collinear transformation onto $\mathbb{P}^2$ defined by a *central axonometric reference system* $(O^c, E_1^c, E_2^c, E_3^c, U_1^c, U_2^c, U_3^c) \subset \mathbb{P}^2$. The central projection is a more specific mapping, where the Cartesian basis is projected from a given spatial center onto the plane. Central projection mappings obviously form a subset of the set of central axonometries and this fact leads to the classical problem of this field: how can one characterize central projections among central axonometries. Early results include the general synthetic condition of Kruppa [1] and an algebraic condition for a special case by Stiefel [2]. In the last decade several papers dealt with this problem. General algebraic conditions are given in [3], [4] and [5]. A specific case of the first condition is discussed in [6], while geometric interpretation of the latter ones and generalizations for higher dimensions are discussed in [7] and [8].

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Throughout this paper the Szabó-Stachel-Vogel condition [3] will frequently be referred: let us denote the distances $O^cE_i^c$ by e_i and the distances $E_i^cU_i^c$ by f_i, $(i = 1, 2, 3)$. Considering the three angles $\alpha_1 = \angle(U_3^cU_1^cU_2^c), \alpha_2 = \angle(U_1^cU_2^cU_3^c)$ and $\alpha_3 = \angle(U_2^cU_3^cU_1^c)$, the condition can be stated as follows:

$$\left(\frac{e_1}{f_1}\right)^2 : \left(\frac{e_2}{f_2}\right)^2 : \left(\frac{e_3}{f_3}\right)^2 = \tan\alpha_1 : \tan\alpha_2 : \tan\alpha_3. \tag{1}$$

In this paper, similarly to the Szabó-Stachel-Vogel condition, all points of the reference system are supposed to be finite. Moreover, Euclidean metric will be used in the computations, so the image plane can rather be considered as the projective closure of the Euclidean plane. For the sake of simplicity the notation $\mathbb{P}^2$ will be preserved for this closure as well. We will also use spatial homogenous coordinates in the form (wx, wy, wz, w).

As we have seen, given a *central axonometric reference system* $(O, E_1, E_2, E_3, U_1, U_2, U_3)$ in $\mathbb{P}^2$ we have several ways to characterize that system as a central projection (from now on we will omit the upper index c, since only the planar points will be considered). If the system fulfills these conditions we will call it *central projection reference system* (for the sake of brevity we will denote these two systems by *CA-system* and *CP-system*, respectively). It is an obvious fact, that if a general CA-system is defined, moving any of its base points while preserving the 3-tuples (O, E_i, U_i) to be collinear, the new system will also remain a CA-system. If, however, the original CA-system was a CP-system, after an arbitrary reposition the new system will generally not hold this property, i.e., the new system will only be a general CA-system. A simple example is the following: consider a system which fulfills the Szabó-Stachel-Vogel condition. Moving E_1 along the x-axis all the values remain unchanged except the first ratio, thus eq. (1) will not hold any more.

Our final purpose is to describe some geometric and/or analytical conditions under which moving one or more of its base points, a CP-system is transformed to a system of the same kind. From another point of view, if only part of the reference system is given, how one can choose the missing points in a way that the final system will be of central projection type. Beyond its theoretical interest it may have some practical sense if we could replace the computation of the movement of a spatial coordinate system and its projection by some planar conditions. Interactive change of a central projection view by drag-and-drop technique may also use this theoretical background. Here we describe only the movement of the unit points with the help of some new conditions for a CA-system to be a CP-system.

2. A simple geometric condition

At first we describe a necessary and sufficient geometric condition under which a CA-system is of central projection type. As we have mentioned, there are numerous conditions known for this problem, but, apart from Kruppa's work, all of them are analytical. This means that, having an existing CA-system, we have to measure lengths and/or angles, compute matrices etc. to decide if the system is a CP-system. Here we give a condition with the help of which one can solve this problem by a simple construction. We will use the following property and notion (cf. [9]).

Lemma 1 *If ABC is an acute-angled triangle, consider an interior point P with traces T_a, T_b, T_c. Find the point R_a on the side BC for which the signed distances satisfy*

$$\frac{R_aB}{CR_a} = \sqrt{\frac{T_aB}{CT_a}}. \tag{2}$$

If we define R_b and R_c on the sides AC and AB in a similar way, then the lines AR_a, BR_b and CR_c are concurrent in a point R_P.

Here the points R_a, R_b and R_c are supposed to be inner points of the sides of the triangle. Eq. (2) obviously yields another solution for each of these points along the lines BC, AC and AB, respectively. In this paper, however, we always consider the solutions for which the square root in eq. (2) is positive. This is necessary for the uniqueness of the following definition.

Definition 1 *The point of concurrency R_P is called* ***the square root of the point*** *P.*

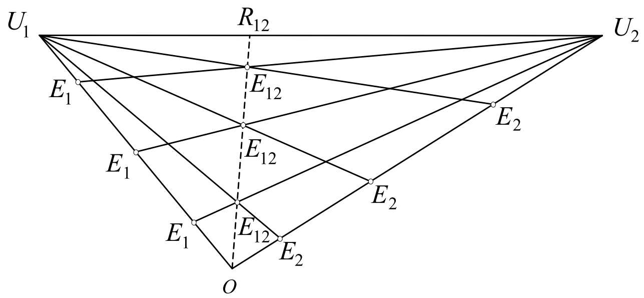

Figure 1: The line OE_{12} is independent from the positions of E_1 and E_2, i.e., from the change of the unit-length

From now on the reference system is supposed to be a CP-system. The triangle $U_1U_2U_3$ is acute. Denote its orthocenter by H. Let us consider the triangle OU_1U_2. Due to equation (1)

$$\frac{\left(\frac{e_1}{f_1}\right)^2}{\left(\frac{e_2}{f_2}\right)^2} = \frac{\tan\alpha_1}{\tan\alpha_2}$$

holds. If we denote the trace of H on the side U_1U_2 by T_{12} then we can write the right side of the equation as

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{T_{12}U_2}{U_1T_{12}}.$$

Now consider the point R_{12} of U_1U_2 for which

$$\frac{R_{12}U_2}{U_1R_{12}} = \sqrt{\frac{\tan\alpha_1}{\tan\alpha_2}}.$$

If E_{12} is the point associated to the spatial point $(1,1,0,1)$, then one can observe, that the position of the line OE_{12} is independent from the positions of E_1 and E_2 and this line intersects the side U_1U_2 at R_{12} (cf. Fig. 1).

By similar arguments one can find the points R_{13} and R_{23} on the side U_1U_3 and U_2U_3, respectively. The above mentioned definition immediately implies that the lines U_1R_{23}, U_2R_{13} and U_3R_{12} are concurrent and the point of concurrency R is nothing else but the square root of the orthocenter H of the triangle $U_1U_2U_3$. Moreover, if E_{123} is the point associated to the spatial point $(1,1,1,1)$, then the line OE_{123} passes through the point R, which is the image of the spatial point at infinity $(1,1,1,0)$. Finally we obtained the following condition (cf. Fig. 2).

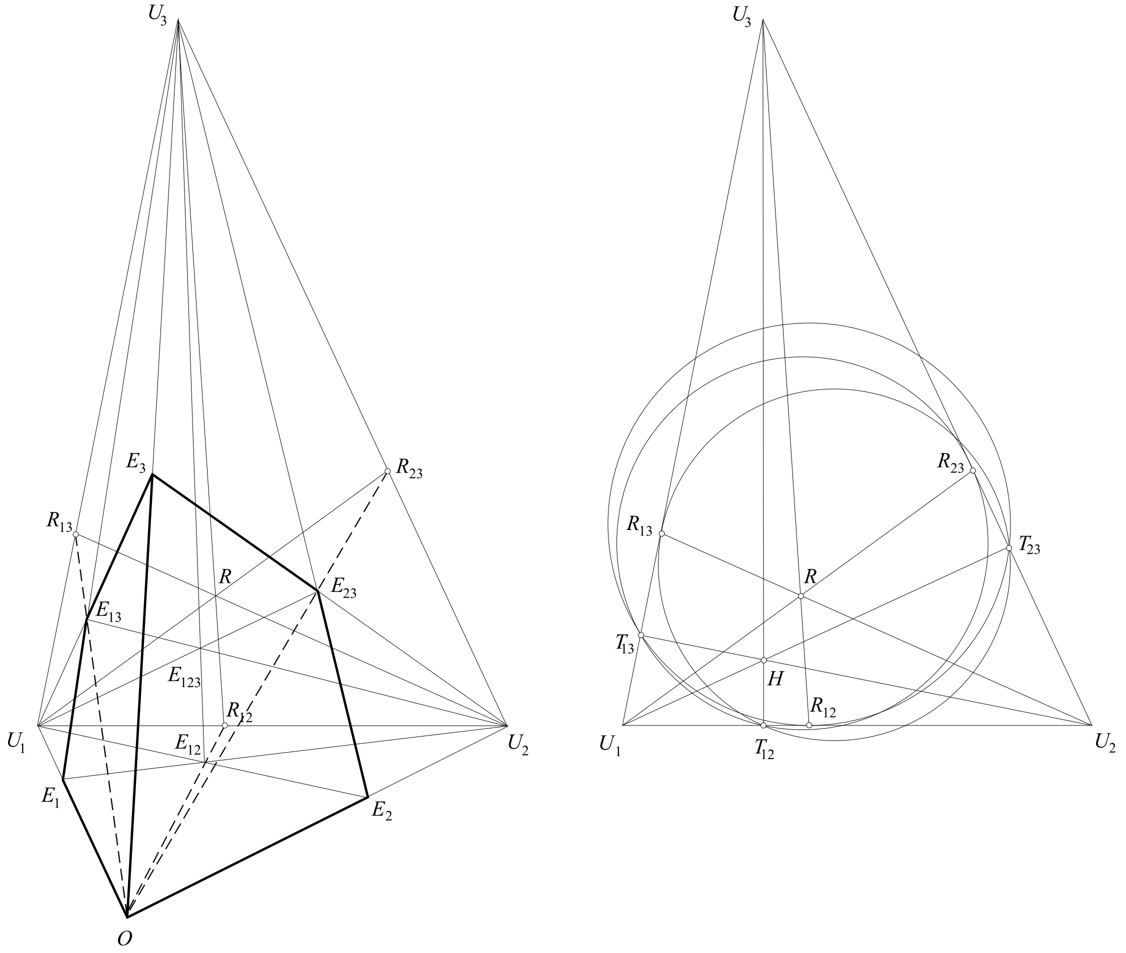

Figure 2: A central projection system and its circles

Theorem 1 *A general central axonometric reference system* $(O, E_1, E_2, E_3, U_1, U_2, U_3)$ *in* $\mathbb{P}^2$ *with unit point* E_{123} *is a central projection system iff the line* OE_{123} *passes through the square root of the orthocenter of the triangle* $U_1U_2U_3$.

This condition can be reformulated by applying the following proposition.

Lemma 2 *Let* ABC *be an acute-angled triangle and* H_a, H_b, H_c *the traces of its orthocentre* H*. There is a unique circle through* H_a, H_b *which is tangent to the side* AB*, and the touching point is an interior point of* AB*. Denote the touching point by* R_c*. Similarly finding* R_a *and* R_b *the lines* AR_a, BR_b, CR_c *are concurrent and the point of concurrency* R *is the square root of* H*.*

Proof: Consider the pencil of circles passing through H_b and H_c. These circles intersect the line BC in pairs of an involution. One special circle of this pencil splits into the line H_bH_c and the line at infinity, so the point $H'_a = H_bH_c \bigcap BC$ corresponds to the point at infinity of BC in this involution. H'_a is the harmonic conjugate of H_a with respect to B and C. The involution also yields that the power of H'_a with respect to all circles of the pencil is constant, namely $H'_aH_b \cdot H'_aH_c$. The power of H'_a also equals $H'_aB \cdot H'_aC$ as the circle with diameter BC passes through H_a and H_b as well. Consequently the touching point R_a has the distance $\sqrt{H'_aB \cdot H'_aC}$ from H'_a which gives the proof. □

Remark: In the proof of Lemma 2 we refer to a pencil of circles. The Feuerbach circle of the triangle ABC is also included in this pencil. The Feuerbach circle intersects the side BC at the pedal point H_a and the midpoint of BC, which implies that the point R_a lies between these two points.

This means that if we have a general CA-system we can try to find these circles. If they exist then the system is a CP-system and vice versa. Thus we found the following consequence.

Theorem 2 *If a general* central axonometric reference system $(O, E_1, E_2, E_3, U_1, U_2, U_3)$ *in* $\mathbb{P}^2$ *is given, denote the traces of the orthocentre of the triangle* $U_1U_2U_3$ *by* T_{12}, T_{13}, T_{23}*. Find the intersection point* R_{12} *of* U_1U_2 *and* OE_{12}*. Similarly find* R_{13} *and* R_{23}*. The system is a central projection system iff the following circles exist: one through* T_{12}, T_{13} *and touching* U_2U_3 *at* R_{23}*, one through* T_{12}, T_{23} *and touching* U_1U_3 *at* R_{13} *and one through* T_{13}, T_{23} *and touching* U_1U_2 *at* R_{12}*.*

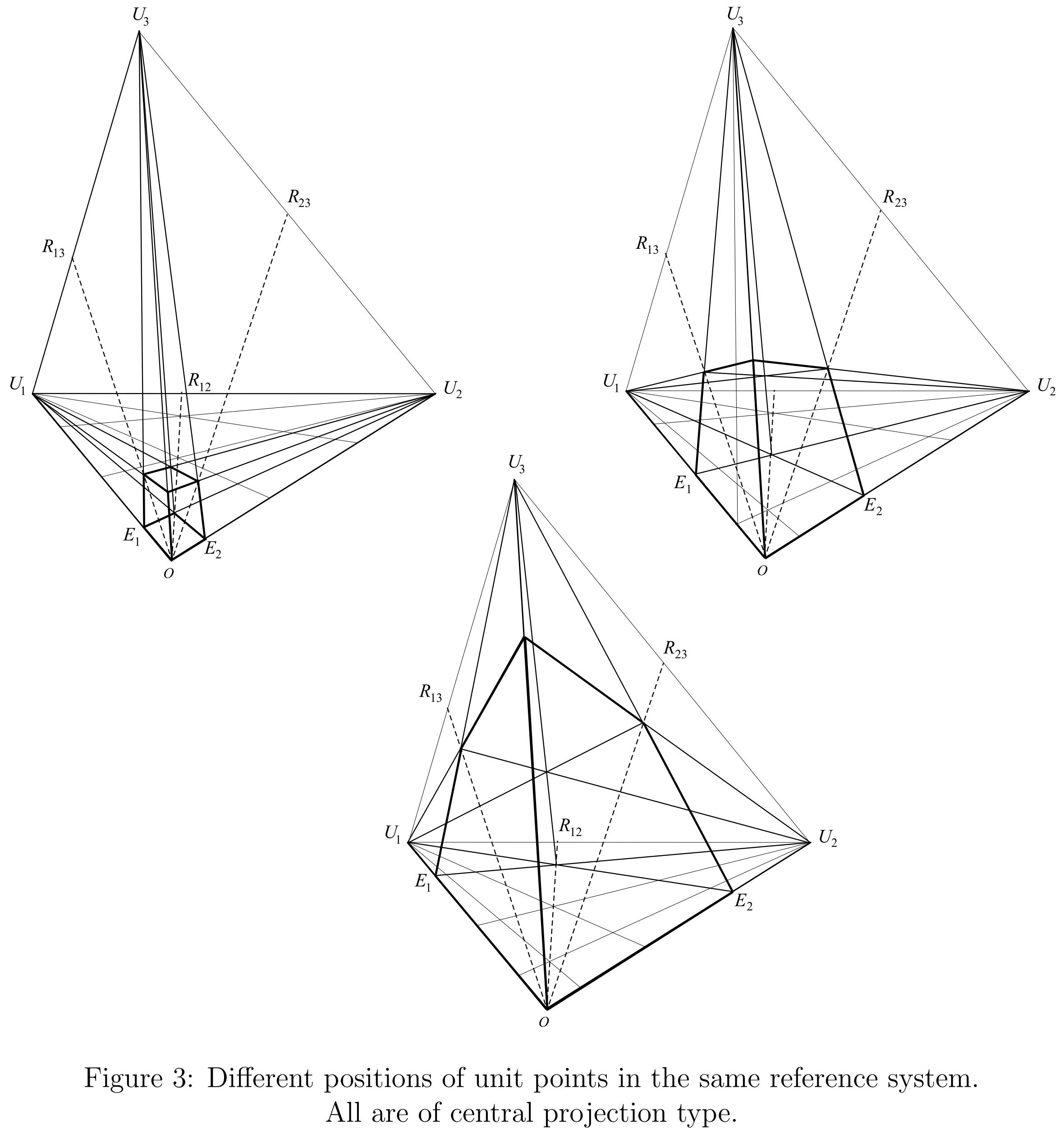

Figure 3: Different positions of unit points in the same reference system.
All are of central projection type.

The existence of these circles (cf. Fig. 2) can easily be controlled, so this latter theorem gives us a simple Euclidean construction to verify if an existing drawing is a central projection system. Note, that the existence of all three circles is necessary, because each circle is "responsible" for the fulfillment of the equality of one ratio in eq. (1). For a Euclidean construction of the square root of an interior point, see [9].

Here we have to remark, that allowing negative signs in eq. (2) we obtain alternative solutions for R_{12}, R_{13} and R_{23}, which yield alternative circles and three more possibilities for the point R as well. For example in Fig. 2 these new alternatives for R would be the images of the spatial points at infinity $(1, 1, -1, 0)$, $(1, -1, 1, 0)$ and $(1, -1, -1, 0)$, respectively. Similar statements could also be formulated by these additional solutions.

3. Moving the unit points along the axes

A simple way to change a CP-system is to move one of the unit points, say E_1, along its axis. The points O, U_1, U_2, U_3 remain unchanged which yields a constant right side of eq. (1). To preserve the ratios of the left side of the equation as well (and thus preserve the central projection type of the system) the other two unit points E_2 and E_3 will be forced to move along their axes as well. This movement can easily be calculated analytically, but it can also be constructed in a simple way: drawing OE_{12}, OE_{13} and OE_{23} of the existing system, the points E_{12}, E_{13}, E_{23} have to be moved along these lines. Fig. 3 shows different positions of E_1 and the other two unit points preserving the system to be of central projection type. This continuous change of the system can easily be constructed and calculated as well, and gives the impression to be getting closer and closer to the object, more precisely, to the origin. This is a similar effect to that one which can be achieved by decreasing the distance of the spatial origin and the centre of the projection but preserving the distance of the centre and the image plane.

On the other hand, our condition can also be applied in the correction of a CA-system. Suppose we have a general CA-system which does not fulfill the requirements of being a CP-system. If we would like to correct the system to satisfy the conditions, some of its base points have to be moved. In our current case the modification of the CA-system will be performed by modifying its unit points and preserving the position of the points O, U_1, U_2, U_3. Once the CA-system is given, by elementary methods one can draw the three circles in the triangle $U_1U_2U_3$ passing through two traces of the orthocentre and touching the third side each. Thus we find the points R_{12}, R_{13}, R_{23} as touching points (cf. Theorem 2). In general none of the points E_{12}, E_{13}, E_{23} will be on the lines $OR_{12}, OR_{13}, OR_{23}$, hence at least two of them have to be repositioned along their axis. If we decide to preserve the position of one unit point, say E_1, then the positions of the other two unit points E_2 and E_3 are uniquely determined and can be found by a simple construction. In Fig. 4 three different cases of correction can be seen by preserving O, U_1, U_2, U_3, E_1, then O, U_1, U_2, U_3, E_2 and finally O, U_1, U_2, U_3, E_3 of the original system.

A further possibility could be the modification of all three unit points of the CA-system to be a CP-system. Among the infinitely many possible positions one may use the one with minimum distortion comparing with the original system. This problem can be solved by calculating the movement of the unit points and find the minimum of the squared distances between the new and the original unit points.

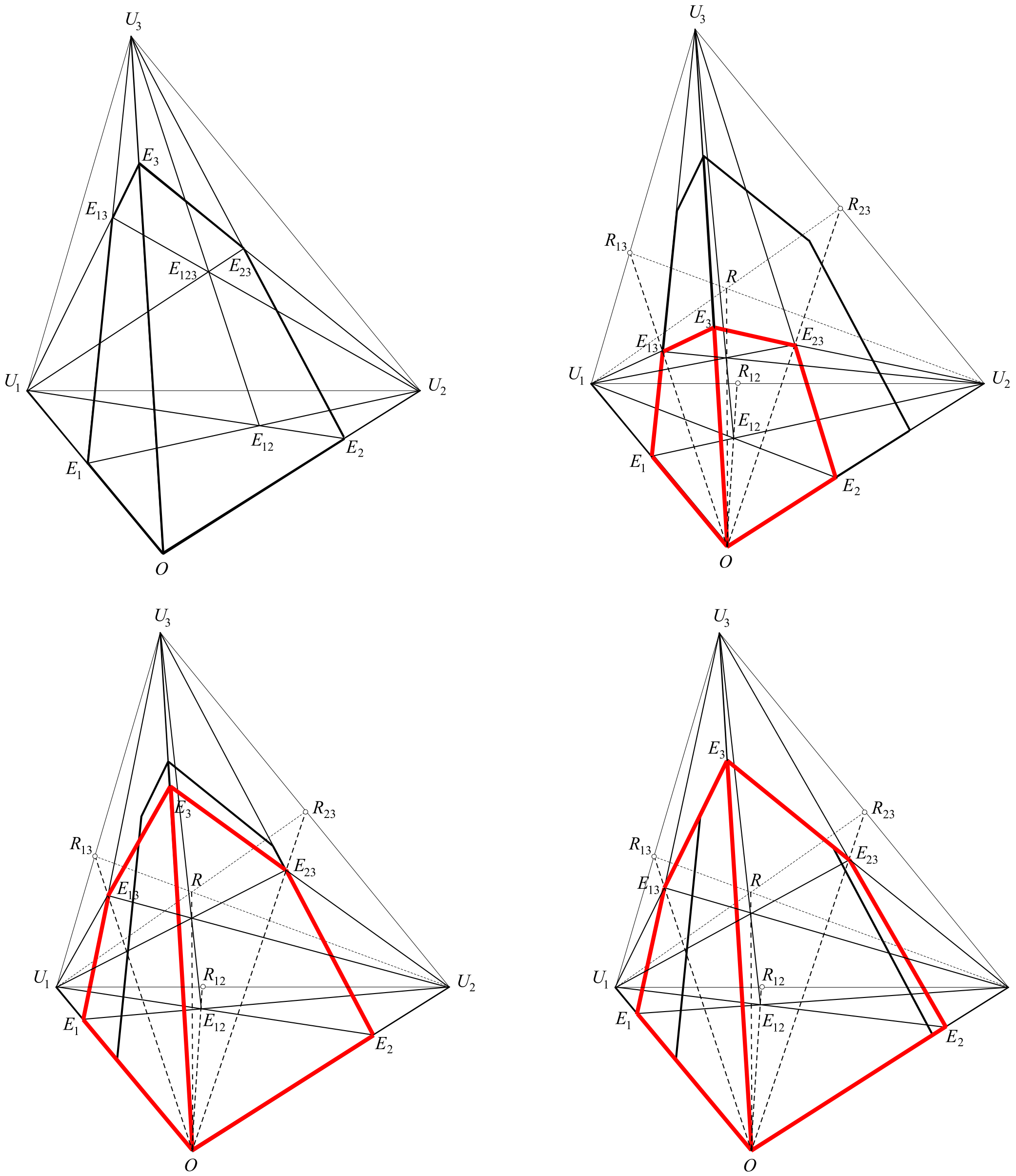

Figure 4: a) the original CA-system and three different corrections to be a CP-system, preserving O, U_1, U_2, U_3 and b) E_1, c) E_2, d) E_3

4. Future work

Applying the new synthetic condition for a CA-system to be of central projection type, modification of unit points of the reference system has been discussed. Further questions naturally arise about changing the positions of other points of the system, effects of the alteration of the origin and especially the points at infinity.

Here we applied the Szabó-Stachel-Vogel condition, which requires finite points in the reference system. Other conditions — like the one by Dür [5] — do not assume finite base points, thus the application of these theorems may lead more general description of moving central axonometric reference systems.

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