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On Emelyanov's Circle Theorem

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Abstract. Given a triangle and a point T, let $\Gamma_+(T)$ be the triad of circles each tangent to the circumcircle and to a side line at the trace of T. Assuming T an interior point and each circle tangent the circumcircle internally, Lev Emelyanov has shown that the circle tangent to each of these circles is also tangent to the incircle. In this paper, we study this configuration in further details and without the restriction to interior points. We identify the point of the tangency with the incircle, and derive some interesting loci related this configuration.

Key words: Emelyanov circle, homogeneous barycentric coordinates, infinite point, nine-point circle, Feuerbach point

MSC 2000: 51M04

1. Introduction

Lev Emelyanov has established by synthetic method the following remarkable theorem:

Theorem 1 (Emelyanov [1]) *Let A_1, B_1, C_1 be the traces of an interior point T on the side lines of triangle ABC. Construct three circles Γ_+^a, Γ_+^b and Γ_+^c outside the triangle which are tangent to the sides at A_1, B_1, C_1, respectively, and also tangent to the circumcircle of ABC. The circle tangent externally to these three circles is also tangent to the incircle of triangle ABC (see Fig. 1).*

In the present paper we study this configuration from the viewpoint of geometric constructions facilitated by a computer software which allows the definitions of macros to perform the constructions efficiently and elegantly.[1] To achieve this we analyze the underlying geometry by using the language and basic results of triangle geometry. Specifically, we make use of barycentric coordinates and their homogenization (see, for example, [3, 4]). We make use of standard notations: a, b, c denote the lengths of the sides of triangle ABC, opposite to A, B, C respectively; s stands for the semiperimeter $\frac{1}{2}(a+b+c)$. Apart from the most basic

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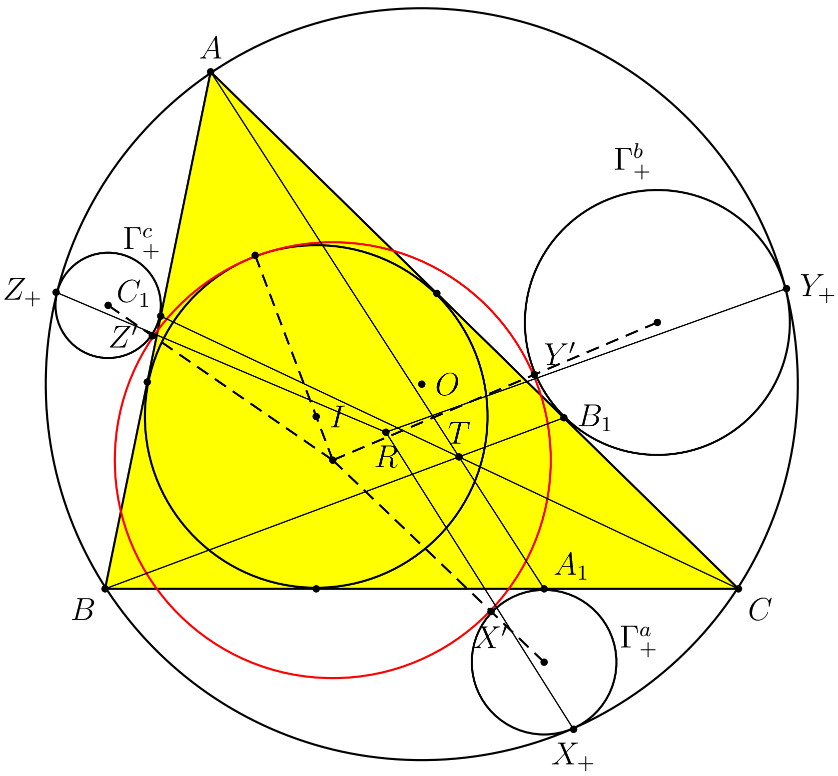

Figure 1: Emelyanov circle

Table 1: Triangle centers and their barycentric coordinates

centroid	G	$(1 : 1 : 1)$
incenter	I	$(a : b : c)$
circumcenter	O	$(a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : (c^2(a^2+b^2-c^2))$
orthocenter	H	$\left(\frac{1}{b^2+c^2-a^2} : \frac{1}{c^2+a^2-b^2} : \frac{1}{a^2+b^2-c^2}\right)$
symmedian point	K	$(a^2 : b^2 : c^2)$
Gergonne point	G_e	$\left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right)$
Nagel point	N	$(s-a : s-b : s-c)$
Mittenpunkt	M	$(a(s-a) : b(s-b) : c(s-c))$
Feuerbach point	F	$((s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2)$

triangle centers listed below with their homogeneous barycentric coordinates, we adopt the labeling of triangle centers in [2].

[1]Dynamic sketches illustrating results in this paper can be found in the author's website http://www.math.fau.edu/Yiu/Geometry.html.

The triad of circles Γ_+^a, Γ_+^b and Γ_+^c can be easily constructed with the help of the following simple lemma.

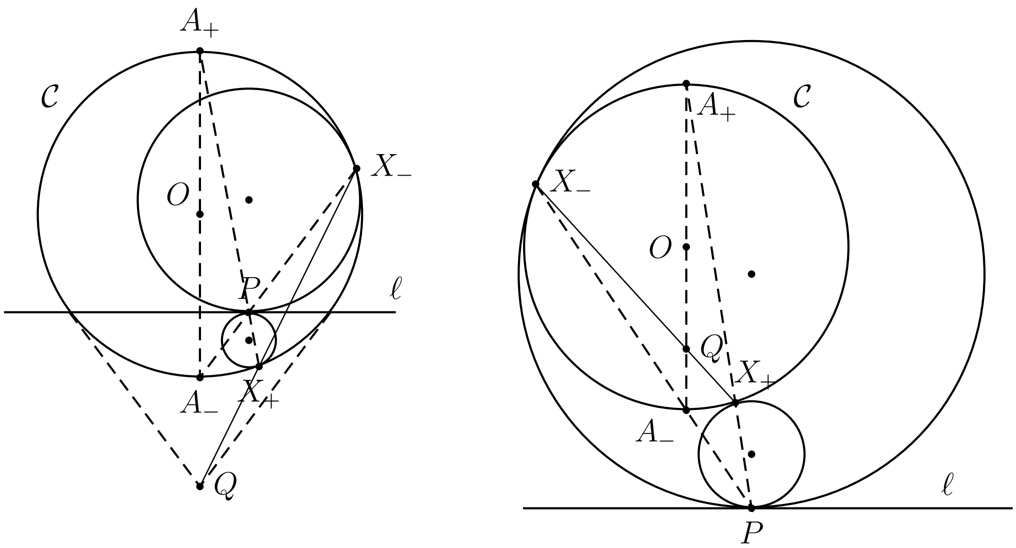

Figure 2: Construction of a circle tangent to ℓ at P and to $\mathcal{C}$

Lemma 2 *Given a line ℓ and a circle $\mathcal{C}$, let A_+ and A_- be the endpoints of the diameter of $\mathcal{C}$ perpendicular to ℓ.*

(a) If a circle touches both ℓ and $\mathcal{C}$, the line joining the points of tangency passes through one of the points $A_\pm$.

(b) If two circles are tangent to the line ℓ at the same point, the line joining their points of tangency with $\mathcal{C}$ passes through the pole of ℓ with respect to $\mathcal{C}$ (see Figs. 2A and 2B).

Computations in this paper were performed with the aids of a computer algebra system. Theorem 4 below characterizes the points of the tangency with the circumcircle. Here, we make use of the notion of the *barycentric product of two points*. Given two points $P_1 = (u_1 : v_2 : w_1)$ and $P_2 = (u_2 : v_2 : w_2)$ in homogeneous barycentric coordinates, the barycentric product $P_1 \cdot P_2$ is the point with coordinates $(u_1u_2 : v_1v_2 : w_1w_2)$. Here is a simple construction ([3]): Let X_1, X_2 be the traces of P_1 and P_2 on the line BC, distinct from the vertices B, C. For $i = 1, 2$, complete the parallelograms $AK_iX_iH_i$ with K_i on AB and H_i on AC. The line joining the intersections of BH_1 and CK_2, and of BH_2 and CK_1, passes through the vertex A, and intersects BC at a point X with homogeneous coordinates $(0 : v_1v_2 : w_1w_2)$. This is the trace of the point $P_1 \cdot P_2$ on BC. The traces of the same point on the lines CA and AB can be similarly constructed. From any two of these traces, the barycentric product $P_1 \cdot P_2$ can be determined (see Fig. 3).

2. The triads of circles $\Gamma_\pm(T)$

Consider a triangle ABC with circumcircle $\mathcal{C}$. We label $A_\pm$ the endpoints of the diameter perpendicular to the sideline BC, A_+ being the one on the same side of BC as A, A_- the antipodal point.

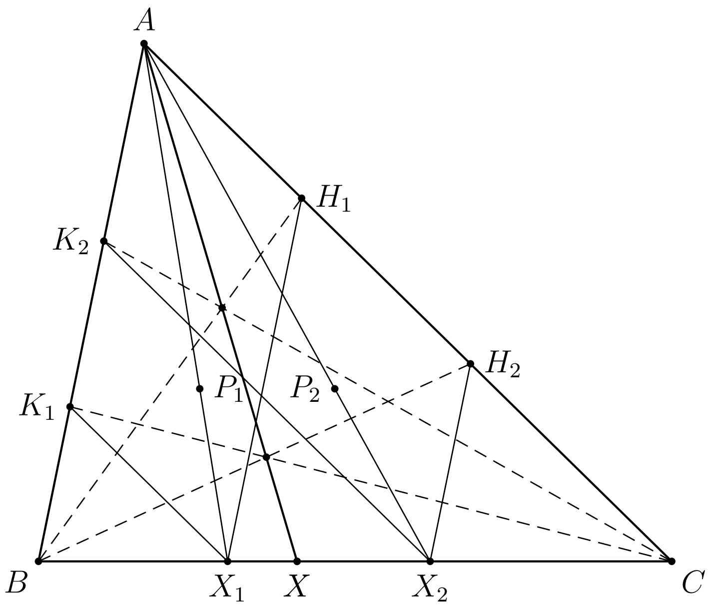

Figure 3: Construction of trace X of $P_1 \cdot P_2$ on BC

Lemma 3 *For $\varepsilon = \pm 1$,*

$$A_\varepsilon = \left(-a^2 : b(b-\varepsilon c) : c(c-\varepsilon b)\right);$$

similarly, for $B_\pm$ and $C_\pm$.

Proof: The infinite point of lines perpendicular to BC can be taken as

$$\left(-a^2,\ \frac{1}{2}(a^2+b^2-c^2),\ \frac{1}{2}(c^2+a^2-b^2)\right).$$

We seek t such that $t(0,1,1) + \left(-a^2, \frac{1}{2}(a^2+b^2-c^2), \frac{1}{2}(c^2+a^2-b^2)\right)$ lies on the circumcircle

$$a^2yz + b^2zx + c^2xy = 0. \tag{1}$$

This gives $t = \frac{1}{2}(b^2 + \varepsilon 2bc + c^2 - a^2)$ for $\varepsilon = \pm 1$. Thus, the perpendicular bisector of BC intersects the circumcircle A_ε given above. □

Given a point T, the triad $\Gamma_+(T)$ consists of the three circles Γ^a_+, Γ^b_+, Γ^c_+, each tangent to the circumcircle and the sidelines at the traces of T, the lines joining the points of tangency passing through A_+, B_+, C_+, respectively. This is the triad considered in [1] when T is an interior point of the triangle. We do not impose this restriction here. Similarly, we also consider the triad $\Gamma_-(T)$, consisting of the circles $\mathcal{C}'_a$, $\mathcal{C}'_b$, $\mathcal{C}'_c$.

Theorem 4 *(a) The points of tangency of the circumcircle with the circles in the triad $\Gamma_+(T)$ are the vertices of the circumcevian triangle of the barycentric product $P = I \cdot T$, where I is the incenter of triangle ABC (see Fig. 4).*

(b) Let the trilinear polar of P intersect BC, CA, AB at X', Y', Z' respectively. The points of tangency of the circumcircle with the circles in the triad $\Gamma_-(T)$ are the second intersections of AX', BY', CZ' with the circumcircle.

(c) The line joining the points of tangency of the circumcircle with the corresponding circles in the two triads passes through the corresponding vertex of the tangential triangle (see Fig. 5).

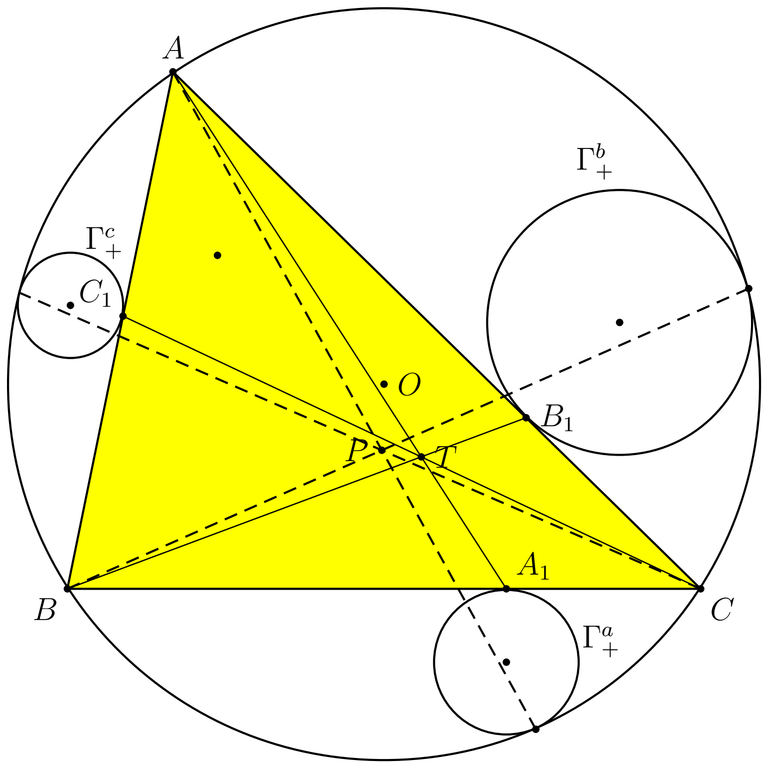

Figure 4: The triad $\Gamma_+(T)$ and $P = I \cdot T$

Proof: Let T be a point with homogeneous barycentric coordinates $(u : v : w)$.

(a) For $\varepsilon = \pm 1$, the line joining A_ε to $(0 : v : w)$ intersects the circumcircle again at a point with coordinates

$$(x, y, z) = (-a^2,\ b(b - \varepsilon c),\ c(c - \varepsilon b)) + t(0, v, w)$$

for some nonzero t satisfying (1). This gives $t = \frac{bc(v+w)}{vw}$, and

$$X_\varepsilon = \left(-a^2vw :\ bv(bw + \varepsilon cv) :\ cw(cv + \varepsilon bw)\right).$$

Similarly, we obtain analogous expressions for Y_ε and Z_ε. From this, $X_+ = \left(\frac{-a^2vw}{bw+cv} :\ bv :\ cw\right)$ is on the line joining $I \cdot T = (au : bv : cw)$ to A; similarly for Y_+ and Z_+. It follows that $X_+Y_+Z_+$ is the circumcevian triangle of $I \cdot T$.

(b) On the other hand, the line joining $X_- = \left(\frac{-a^2vw}{bw-cv} : bv : -cw\right)$ to A intersects BC at $(0 : bv : -cw)$; similarly for Y_- and Z_-. These three points lie on the line $\frac{x}{au} + \frac{y}{bv} + \frac{z}{cw} = 0$, the trilinear polar of $I \cdot T$.

(c) follows from Lemma 2(b). □

Remark. The three lines in Theorem 4(c) above are concurrent at a point

$$P' = \left(a^2\left(\frac{b^2}{v^2} + \frac{c^2}{w^2} - \frac{a^2}{u^2}\right)\ :\ b^2\left(\frac{c^2}{w^2} + \frac{a^2}{u^2} - \frac{b^2}{v^2}\right)\ :\ c^2\left(\frac{a^2}{u^2} + \frac{b^2}{v^2} - \frac{c^2}{w^2}\right)\right).$$

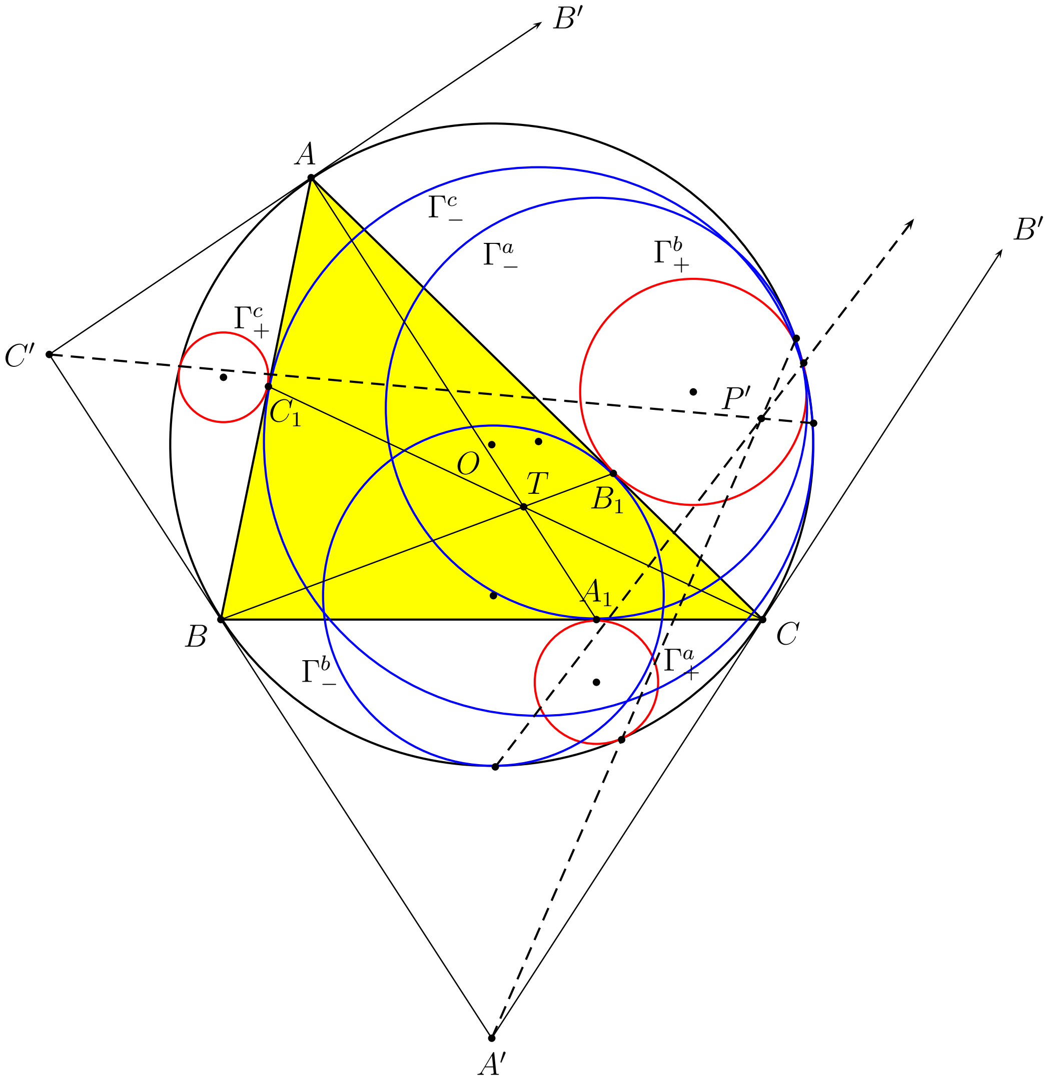

Figure 5: The triads $\Gamma_+(T)$ and $\Gamma_-(T)$

Here are two simple examples.

T	P	P'
G	I	O
I	K	K

3. Equations of the circles in $\Gamma_+(T)$

Since the points of tangency of the circles in $\Gamma_+(T)$ with the circumcircle are known, the centers of the circles can be easily located: if the circle Γ_+^a touches the circumcircle at X_+, then its center is the intersection of the lines OX_+ and the perpendicular to BC at A_1; similarly for the other two circles Γ_+^b and Γ_+^c.

Proposition 5 *The centers of the circles in the triad $\Gamma_+(T)$ are the points*

$$\begin{aligned} O_1 &= \big(-a^2vw : v(2s(s-a)v+b(b+c)w) : w(2s(s-a)w+c(b+c)v)\big), \\ O_2 &= \big(u(2s(s-b)u+a(c+a)w) : -b^2wu : w(2s(s-b)w+c(c+a)u)\big), \\ O_3 &= \big(u(2s(s-c)u+a(a+b)v) : v(2s(s-c)v+b(a+b)u) : -c^2uv\big). \end{aligned}$$

Proposition 6 *These centers are collinear if and only if T lies on the union of the following curves:*
(i) the circumconic

$$\mathcal{E}_{\rm m}: \quad ayz + bzx + cxy = 0$$

(ii) the curve

$$\mathcal{C}_{\rm m}: \quad (2(s-a)(s-b)(s-c) + abc)\, xyz(x+y+z) + \sum_{\rm cyclic} a(s-b)(s-c)y^2z^2 = 0.$$

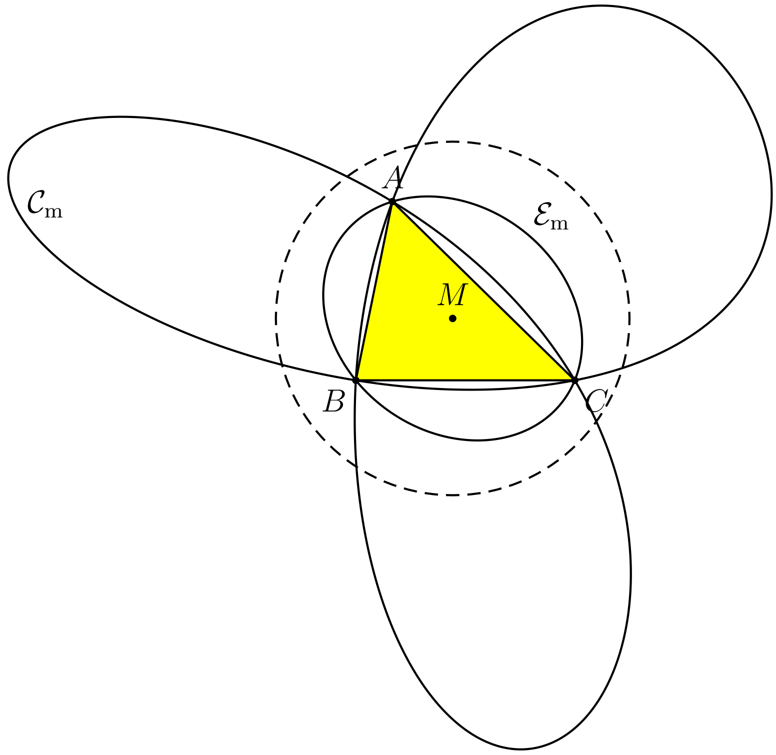

Figure 6: The curves $\mathcal{E}_{\rm m}$ and $\mathcal{C}_{\rm m}$

Remarks. (1) The circumconic $\mathcal{E}_{\rm m}$ is centered at the Mittenpunkt[2]

$$M = (a(b+c-a) : b(c+a-b) : c(a+b-c)).$$

It is an ellipse since it does not contain any interior or infinite point.
(2) The curve $\mathcal{C}_{\rm m}$ is the I-isoconjugate[3] of the circle

$$(a^2+b^2+c^2-2a-2bc-2ca)(a^2yz+b^2zx+c^2xy) + \\ +(x+y+z)\sum\nolimits_{\rm cyclic} bc(c+a-b)(a+b-c)x = 0,$$

with center at the Mittenpunkt M and radius $\frac{2s}{4R+r}\sqrt{R(R+r)}$ (see Fig. 6).

[2]The Mittenpunkt M is the intersection of the three lines each joining the midpoint of a side to the center of the excircle on that side.

[3]The I-isoconjugate of a point $P = (x : y : z)$ is the point with coordinates $\left(\frac{a}{x} : \frac{b}{y} : \frac{c}{z}\right)$. It can be constructed as the barycentric product of the incenter and the isotomic conjugate of P.

Proposition 7 *The equations of the circles in the triad $\Gamma_+(T)$ are*

$$\begin{aligned}(v+w)^2(a^2yz+b^2zx+c^2xy) &= (x+y+z)\left((cv+bw)^2x+a^2w^2y+a^2v^2z\right),\\ (w+u)^2(a^2yz+b^2zx+c^2xy) &= (x+y+z)\left(b^2w^2x+(aw+cu)^2y+b^2u^2z\right),\\ (u+v)^2(a^2yz+b^2zx+c^2xy) &= (x+y+z)(c^2v^2x+c^2u^2y+(bu+av)^2z).\end{aligned}$$

Proof: Since Γ_+^a is tangent to the circumcircle at $X_+ = (-a^2vw : bv(cv+bw) : cw(bw+cv))$, and the tangent of the circumcircle at this point is

$$(cv+bw)^2x+a^2w^2y+a^2v^2z=0,$$

the equation of Γ_+^a is

$$k(a^2yz+b^2zx+c^2xy)=(x+y+z)\left((cv+bw)^2x+a^2w^2y+a^2v^2z\right)$$

for a choice of k such that with $x=0$, the equation has only one root. It is routine to verify that this is $k=(v+w)^2$. This gives the first of the equations above. The remaining two equations of Γ_+^b and Γ_+^c are easily obtained by cyclically permuting (a,u,x), (b,v,y), and (c,w,z). □

4. The radical center Q_T and the Emelyanov circle $\mathcal{C}(T)$

The radical center of the circles in the triad $\Gamma_+(T)$ can be determined by a formula given, for example, in [4, §7.3.1].

Proposition 8 *If $T=(u:v:w)$, the radical center of the triad $\Gamma_+(T)$ is the point*[4]

$$Q_T=\Big(\,a\left(2a(s-a)uvw(u+v+w)+bcu^2(v+w)^2-a(-av^2w^2+bw^2u^2+cu^2v^2)\right) \;:\; \cdots \;:\; \cdots\Big).$$

Corollary 9 *The radical center Q_T is*
(1) a point on the circumcircle if and only if T is an infinite point or lies on the ellipse $\mathcal{E}_{\rm m}$,
(2) an infinite point if T lies on the curve $\mathcal{C}_{\rm m}$.

For $T=G$, this radical center is the point

$$X_{1001}=\left(a(a^2-a(b+c)-2bc):\cdots:\cdots\right),$$

which divides the centroid G and X_{55}, the internal center of similitude of the circumcircle and incircle, in the ratio $GX_{1001}:X_{1001}X_{55}=R+r:3R$.

The circle $\mathcal{C}(T)$ Emelyanov established in [1] is the one for which each of the circles in the triad $\Gamma_+(T)$ is oppositely tangent to both $\mathcal{C}(T)$ and the circumcircle. This circle can be easily constructed since the radical center Q_T is known. If the line joining Q_T to X_+ (respectively Y_+, Z_+) intersects the circle Γ_+^a (respectively Γ_+^b, Γ_+^c) again at X' (respectively Y', Z'), then the circle through X', Y' and Z' is the one Emelyanov constructed (see Fig. 1).

[4] The second and third components are obtained from the first by cyclically permuting (a,b,c) and (u,v,w).

Proposition 10 *The equation of the Emelyanov circle $\mathcal{C}(T)$ is*

$$(u+v+w)(a^2yz+b^2zx+c^2xy)-(x+y+z)\left(\frac{f}{u}\,x+\frac{g}{v}\,y+\frac{h}{w}\,z\right)=0,$$

where

$$\begin{aligned} f=f(u,v,w) &= -kvw+(s-a)(avw+bwu+cuv), && (2)\\ g=g(u,v,w) &= -kwu+(s-b)(avw+bwu+cuv), && (3)\\ h=h(u,v,w) &= -kuv+(s-c)(avw+bwu+cuv), && (4)\end{aligned}$$

where

$$k=(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a). \qquad (5)$$

Examples (1) For the Gergonne point G_e, the Emelyanov circle $\mathcal{C}(G_e)$ clearly coincides with the incircle.

(2) For the centroid G, the Emelyanov circle $\mathcal{C}(G)$ has equation

$$12(a^2yz+b^2zx+c^2xy)-(x+y+z)\sum_{\text{cyclic}}(a^2+2a(b+c)-(3b^2+2bc+3c^2))x=0.$$

This has center [5]

$$C_G=\left(a^4-a^3(b+c)-2a^2(b^2-bc+c^2)+a(b+c)(b-c)^2+(b^2-c^2)^2:\ \cdots\ :\ \cdots\right).$$

This point divides the segment between the incenter and the nine-point center in the ratio $2:1$.

(3) From the equation of the circle, it is clear that $\mathcal{C}(T)$ degenerates into a line if and only if T is an infinite point.

5. Point of tangency of the Emelyanov circle and the incircle

The main result of [1] states that the circle $\mathcal{C}(T)$ touches the incircle of the reference triangle. Here we identify the point of tangency explicitly. Note that every point on the incircle has coordinates

$$\left(\frac{x^2}{s-a}:\frac{y^2}{s-b}:\frac{z^2}{s-c}\right)$$

for an *infinite* point $(x:y:z)$.

Theorem 11 *The circle $\mathcal{C}(T)$ is tangent to the incircle at the point*

$$F_T=\left(\frac{1}{s-a}\left(\frac{1}{(s-b)v}-\frac{1}{(s-c)w}\right)^2\ :\ \cdots\ :\ \cdots\right). \qquad (6)$$

Note that F_T corresponds to the infinite point of the line

$$\frac{x}{(s-a)u}+\frac{y}{(s-b)v}+\frac{z}{(s-c)w}=0.$$

Here is an alternative construction of the point F_T.

[5]The center of a circle with given equation can be determined by a formula given in, for example, [4, §10.7.2].

Proposition 12 *Let DEF be the intouch triangle of triangle ABC, i.e., the cevian triangle of the Gergonne point. Given a point $T = (u : v : w)$ with cevian triangle $A_1B_1C_1$, let*

$$X = B_1C_1 \cap EF, \quad Y = C_1A_1 \cap FD, \quad Z = A_1B_1 \cap DE.$$

The triangles XYZ and DEF are perspective at F_T with coordinates given in (6) (see Fig. 7).

We note an immediate corollary of Theorem 11.

Corollary 13 *The Emelyanov circle $\mathcal{C}(G)$ touches the incircle at the Feuerbach point*

$$F = X_{11} = ((s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2),$$

the point of tangency of the incircle with the nine-point circle.

Examples: (4) If T is an infinite point, the Emelyanov circle degenerates into a line tangent to the incircle. Now, a typical infinite point has coordinates

$$\Big((b-c)(a+t) : (c-a)(b+t) : (a-b)(c+t)\Big).$$

The corresponding Emelyanov circle degenerates into a line tangent to the incircle at the point

$$\Big((s-a)(b-c)^2(a+t)^2(a(a(b+c)-(b^2+c^2))+t(2a^2-a(b+c)-(b-c)^2))^2 : \cdots : \cdots\Big).$$

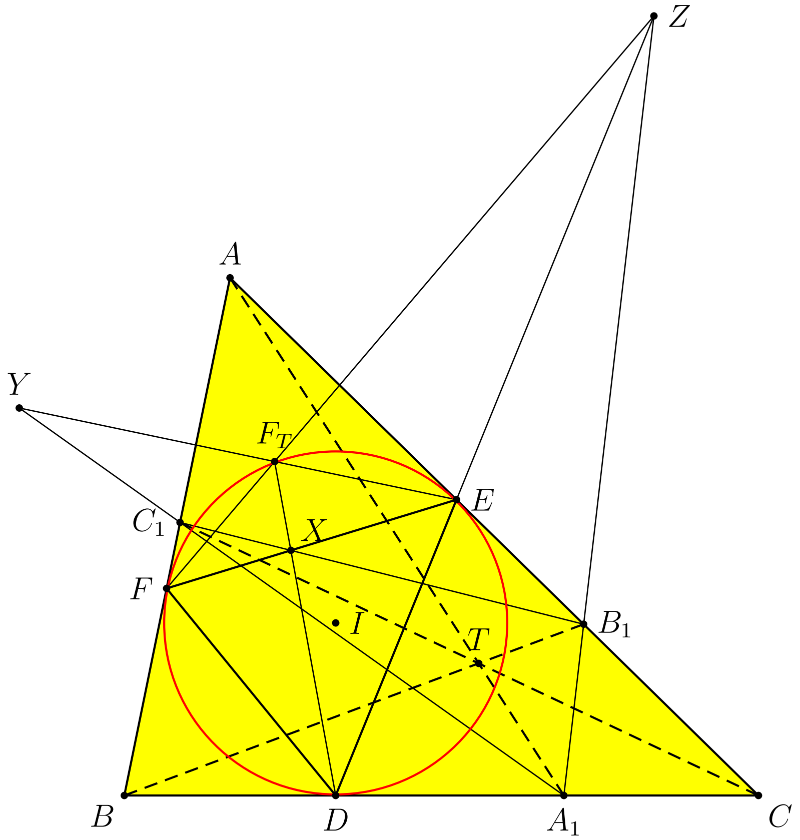

Figure 7: Construction of F_T

(5) If T is a point on the curve $\mathcal{C}_{\rm m}$, we have seen in Proposition 6 and Corollary 9 that the centers of the circles in $\Gamma_+(T)$ are collinear and the radical center Q_T is an infinite point. If ℓ is the line containing the centers, and if the circles in $\Gamma_+(T)$ touch the circumcircle at X, Y, Z respectively, then the Emelyanov circle contains the reflections of X, Y, Z in ℓ. It is therefore the reflection of the circumcircle in ℓ.

6. The locus of T for which $\mathcal{C}(T)$ is tangent to the incircle at a specific point

Theorem 14 *Let T be a point other than the Gergonne point. The locus of T' for which $\mathcal{C}(T')$ touches the incircle at the same point as $\mathcal{C}(T)$ does is the circum-hyperbola through T and the Gergonne point.*

Proof: By the remark following Theorem 11, for $T = (u : v : w)$ and $T' = (x : y : z)$, the circles $\mathcal{C}(T)$ and $\mathcal{C}(T')$ touch the incircle at the same point if and only if

$$\frac{\frac{1}{(s-b)v} - \frac{1}{(s-c)w}}{(s-a)x} + \frac{\frac{1}{(s-c)w} - \frac{1}{(s-a)u}}{(s-b)y} + \frac{\frac{1}{(s-a)u} - \frac{1}{(s-b)v}}{(s-c)z} = 0.$$

This defines a circumconic, which clearly contains T and the Gergonne point. Since the circumconic contains an interior point (the Gergonne point), it is a hyperbola. □

Taking T to be the centroid, we obtain the following corollary.

Corollary 15 *The locus of T' for which $\mathcal{C}(T')$ touches the incircle at the Feuerbach point is the circum-hyperbola through the centroid and the Gergonne point.*

Remark. This circumconic has equation $\frac{b-c}{x} + \frac{c-a}{y} + \frac{a-b}{z} = 0$. Its center is the point

$$X_{1086} = \left((b-c)^2 : (c-a)^2 : (a-b)^2\right)$$

on the Steiner in-ellipse. The tangent to the ellipse at this point is also tangent to the nine-point circle (at the Feuerbach point).

More generally, for a given point $Q = \left(\frac{p^2}{s-a} : \frac{q^2}{s-b} : \frac{r^2}{s-c}\right)$ on the incircle, corresponding to an infinite point $(p : q : r)$, the locus of T for which the Emelyanov circle $\mathcal{C}(T)$ touches the incircle at Q is the circum-hyperbola

$$\mathcal{H}(Q): \quad \frac{p}{(s-a)x} + \frac{q}{(s-b)y} + \frac{r}{(s-c)z} = 0.$$

To construct the hyperbola, we need only locate one more point. This can be easily found with the help of Proposition 12. Let the line DQ intersect AC and AB at Y_0 and Z_0 respectively. Then $T_0 = BY_0 \cap CZ_0$ is a point on the locus.

The hyperbola $\mathcal{H}(Q)$ is a rectangular hyperbola if it contains the orthocenter. This is the Feuerbach hyperbola with center at the Feuerbach point. For T on this hyperbola, the point of tangency of the Emelyanov circle $\mathcal{C}(T)$ with the incircle is the point

$$X_{3022} = \left(a^2(b-c)^2(s-a)^3 : \cdots : \cdots\right).$$

7. The involution τ

Theorem 16 *Let $f(u,v,w)$, $g(u,v,w)$, and $h(u,v,w)$ be the functions given in (1). The mapping*

$$\tau(u:v:w) = \left(\frac{1}{f(u,v,w)} : \frac{1}{g(u,v,w)} : \frac{1}{h(u,v,w)}\right)$$

is an involution such that the Emelyanov circles $\mathcal{C}(T)$ and $\mathcal{C}(\tau(T))$ are identical.

Proof: Let $\phi(T) = \tau(T^\bullet)^\bullet$.[6] It is enough to show that ϕ is an involution. If $T = (u : v : w)$, it is easy to check that

$$\begin{aligned}\phi(u:v:w) \;=\; & \big(-ku + (s-a)(au+bv+cw) \;:\; -kv + (s-b)(au+bv+cw) \\ & \;:\; -kw + (s-c)(au+bv+cw)\big),\end{aligned} \tag{7}$$

where k is given in (5). The mapping ϕ is an involution. In fact, it N denotes the Nagel point, and P the intersection of the lines NT and $ax+by+cz=0$, then $\phi(T)$ is the harmonic conjugate of T with respect to N and P. Since $\tau(T) = \phi(T^\bullet)^\bullet$, τ is also an involution. □

If T is the centroid, $\tau(T)$ is the point

$$\left(\frac{1}{a^2+2a(b+c)-(3b^2+2bc+3c^2)} \;:\; \cdots \;:\; \cdots\right).$$

7.1. The fixed points of τ

The fixed points of τ are the isotomic conjugates of the fixed points of ϕ. From (7), it is clear that the fixed points of ϕ are the Nagel point $(s-a : s-b : s-c)$ and the points on the line $ax+by+cz=0$. Therefore, the fixed points of τ are the Gergonne point and points on the circum-ellipse $\mathcal{E}_\mathrm{m}$. A typical point on $\mathcal{E}_\mathrm{m}$ is

$$T = \left(\frac{a}{(b-c)(a+t)} \;:\; \frac{b}{(c-a)(b+t)} \;:\; \frac{c}{(a-b)(c+t)}\right).$$

For such a point T, the circles in the triad $\Gamma_+(T)$ are all tangent to the circumcircle at

$$P_T = \left(\frac{a^2}{(b-c)(a+t)} \;:\; \frac{b^2}{(c-a)(b+t)} \;:\; \frac{c^2}{(a-b)(c+t)}\right).$$

Every circle tangent to the circumcircle at this point is clearly tangent to the three circles of the triad $\Gamma_+(T)$. There are two circles tangent to the incircle and also to the circumcircle at P_T. One of them is the Emelyanov circle $\mathcal{C}(T)$. This touches the incircle at the point

$$\begin{aligned}&\big(a^2(s-a)(bc(2a^2-a(b+c)+(b+c)^2)\\ &+t(a^2(b+c)-2a(b^2-bc+c^2)+(b+c)(b-c)^2)) \;:\; \cdots \;:\; \cdots\big).\end{aligned}$$

In particular, the point of tangency is the Feuerbach if[7]

$$T = X_{673} = \left(\frac{1}{a(b+c)-(b^2+c^2)} \;:\; \frac{1}{b(c+a)-(c^2+a^2)} \;:\; \frac{1}{c(a+b)-(a^2+b^2)}\right),$$

The circles $\mathcal{C}(T)$ and those in the triad $\Gamma_+(T)$ are tangent to the circumcircle at X_{105}.

[6] Here, $()^\bullet$ stands for isotomic conjugation.

[7] X_{673} can be constructed as the intersection of the lines joining the centroid to the Feuerbach point, and the Gergonne point to the symmedian point.

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