Journal for Geometry and Graphics
Volume 10 (2006), No. 1, 73–98.

Ortho-Circles of Dupin Cyclides

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Abstract. We study the set of circles which intersect a Dupin cyclide in at least two different points orthogonally. Dupin cyclides can be obtained by inverting a cylinder, or cone of revolution, or by inverting a torus. Since orthogonal intersection is invariant under Möbius transformations we first study the ortho-circles of cylinders/cones of revolution and tori and transfer the results afterwards.

Key Words: ortho-circle, double normal, dupin cyclide, torus, inversion.

MSC: 53A05, 51N20, 51N35

1. Introduction

In this paper we investigate the set of circles which intersect Dupin cyclides twice orthogonally. These circles will be called *ortho-circles*. Dupin cyclides are algebraic surfaces of degree three or four [17]. They are known to be the images of cylinders, or cones of revolution, or tori under inversions. Depending on the location of the center O of the inversion and of the choice of the input surface Φ we obtain different types of Dupin cyclides (see Fig. 1).

Dupin cyclides can be obtained by certain projections from *supercyclides* [12]. There is also a close relation between Dupin cyclides and line geometry [48] and geometric optics [34]. Dupin cyclides carry at least two one-parameter families of circles. The carrier planes of these circles form pencils with skew axes [13]. These circles comprise the set of lines of curvature on the cyclide. The normal lines of a Dupin cyclide intersect a pair of *focal conics*, i.e., a pair of conics k_1 and k_2 in planes π_1 and π_2 with $\pi_1 \perp \pi_2$. The vertices of k_1 are the foci of k_2, and vice versa. Fig. 2 shows a part of a ring shaped cyclide and its focal conics. The conics k_1 and k_2 form the degenerate *central surfaces*. Dupin cyclides can be characterized by having degenerate central surfaces [26].

Dupin cyclides are canal surfaces in two ways [10, 16]: A Dupin cyclide can be defined as the envelope Φ of a smooth one-parameter family $\mathcal{F}_1$ of spheres touching three given spheres in generic position. If we pick three arbitrary spheres from $\mathcal{F}_1$ they also define a second smooth one-parameter family $\mathcal{F}_2$ of spheres touching these three spheres. The envelope of $\mathcal{F}_2$ is the same cyclide Φ. The centers of the spheres in $\mathcal{F}_1$ and $\mathcal{F}_2$ form the above mentioned pair of focal conics (see Fig. 2).

ISSN 1433-8157/$ 2.50

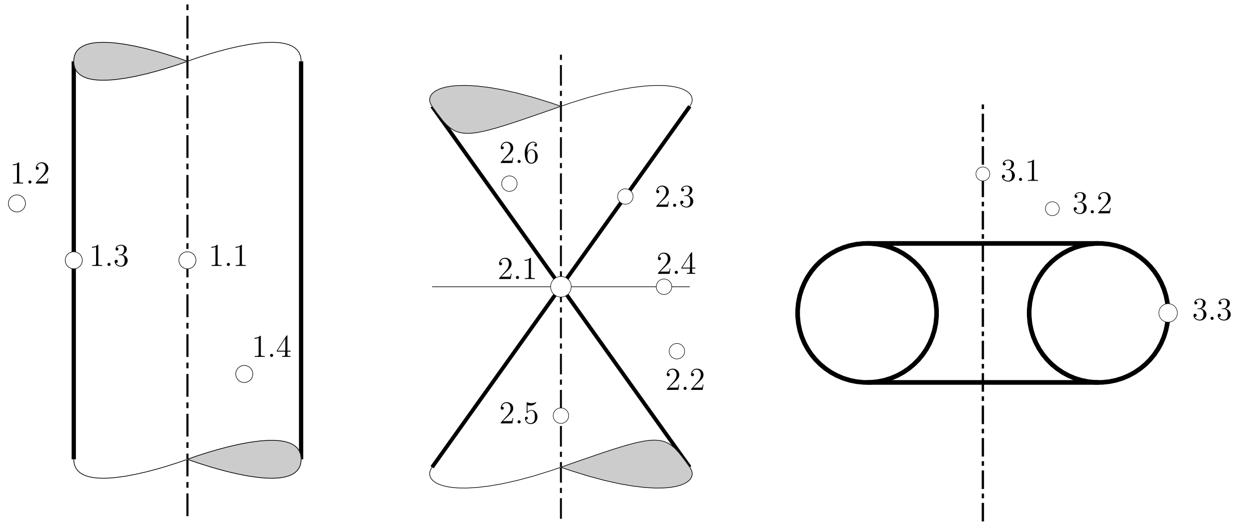

Figure 1: Dupin cyclides obtained from cylinders, cones, and tori. The possible choices of centers of inversion are labelled according to the type of resulting surface: (1.1) thorn torus, (1.2) needle cyclide, (1.3) parabolic needle cyclide (1.4) cuspidal cyclide; (2.1) cone of revolution, (2.2) horn cyclide, (2.3) parabolic horn cyclide, (2.4) symmetric horn cyclide, (2.5) spindle torus, (2.6) spindle cyclide; (3.1) ring torus, (3.2) ring cyclide, (3.3) parabolic ring cyclide. Some of these are displayed in Figs. 6, 8, or 13.

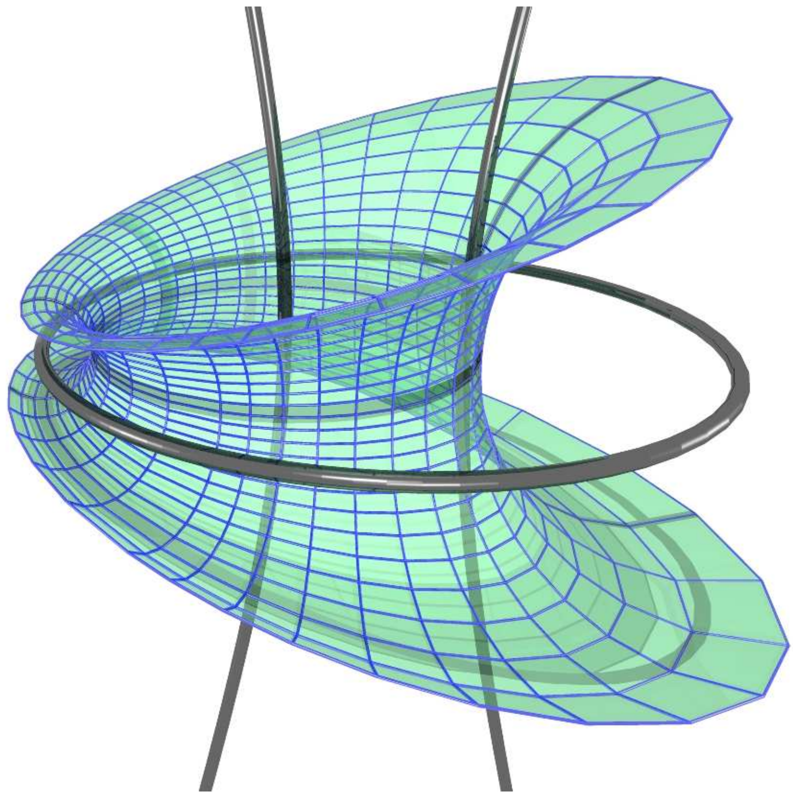

Figure 2: Part of a Dupin cyclide with its focal conics

The definition of Dupin cyclides as envelopes of spheres touching three given ones is invariant with respect to Möbius transformations, such as the inversion for example.

The CAGD community has discovered Dupin cyclides for their own purposes. These surfaces can be used as blending surfaces between pipes with different radii and canal surfaces as well [2, 3]. They also serve as blending surfaces between natural quadrics. Since Dupin cyclides are rational surfaces of relatively low degree, their parametrizations can be rewritten in rational tensor product form. This makes them accessible for CAD systems [5, 7, 11, 14, 21, 23, 27, 32, 33, 35, 36].

The huge amount of literature in the past and nowadays dealing with Dupin cyclides again indicates the importance of Dupin cyclides in theory and practice. Even Dupin cyclides in Non-Euclidean spaces have been a field of research [18, 30, 39, 40] as they serve as Clifford surfaces in conformal models of three-dimensional hyperbolic or elliptic geometry.

The study of ortho-circles arose from the generalization of *transnormal manifolds* defined by Robertson [37] and Wegner [47]. A submanifold M in Euclidean space $\mathbb{R}^n$ is called *transnormal* if for points $p, q \in M$ with corresponding normal spaces N_p, N_q, resp., $p \in N_q$ implies $q \in N_p$.

A submanifold M in Euclidean n-space $\mathbb{R}^n$ is called *manifold of constant width d* if any pair of parallel support hyperplanes has distance $d \in \mathbb{R}$. Manifolds M of constant width are known to be transnormal. Any normal of M is a double normal.

The search for circles instead of linear spaces being twice orthogonal to a surface $\Phi \subset \mathbb{R}^3$ was first started in [41]. We follow the author and give the basic definition:

Definition 1.1 *A circle $c(P_1, P_2)$ which intersects a given surface in at least two different points P_1 and P_2 orthogonally is called ortho-circle joining P_1 and P_2.*
An ortho-circle that hits Φ k-times orthogonally is called k-ortho-circle.

The thesis [41] is mainly devoted to the study of examples. The set of ortho-circles of Plücker's conoid is computed as well as the ortho-circles of the helicoid. The ortho-circles of Dupin cyclides are also investigated. Possible generalizations to φ-circles (i.e., ortho-circles which additionally intersect a given surface Φ at a given angle $\varphi \neq \pi/2$) are also discussed.

In [28] more general results on ortho-circles can be found. The ortho-circles of the wide class of *generalized surfaces of revolution*[1] can easily be described. Looking for circles which intersect a surface at least twice at the same angle $\varphi \neq \pi/2$ we find theorems similar to those given for ortho-circles [28]. Suprisingly it turned out that there is a close relation between Non-Euclidean geometries and the set of ortho-circles of spheres and planes. Spheres and planes can be characterized as those surfaces in $\mathbb{R}^3$ where any pair of points can be joined by an ortho-circle.

This paper is dedicated to the study of ortho-circles of Dupin cyclides. The ortho-circles of surfaces from this class can easily be studied by transforming the surface into a cylinder, or cone of revolution, or a torus (depending on the type of surface, as we will see later) by applying a suitable inversion. This paper summarizes some results of [41] and adds new ones.

The present paper is organized as follows: In Section 2 we collect some elementary properties of ortho-circles and summarize well known facts about inversions. In Sections 3, 4, 5 the ortho-circles of cylinders/cones of revolution and ring tori are investigated. These surfaces

[1] A generalized surface of revolution is defined in the following way [25]: Consider a developable surface Γ. Let T be a tangent plane of Γ and let further m be a curve in T. If now T is rolling on Γ without gliding, then m traces out a generalized surface of revolution. The class of generalized surfaces of revolution contains surfaces of revolution, moulding surfaces, and helical surfaces as well.

serve as primitives, i.e., afterwards we apply inversions to these and obtain the results on ortho-circles for all types of Dupin cyclides in Section 6. In Section 7 we give some ideas for future research and conclusions.

2. Preparations

2.1. Some properties of ortho-circles

Let $c(P_1, P_2)$ be the ortho-circle joining two points P_1 and P_2 on a surface Φ. With n_{P_1} and n_{P_2} we denote the respective normals of Φ at P_1 and P_2. The following observations (illustrated in Fig. 3) can be made:

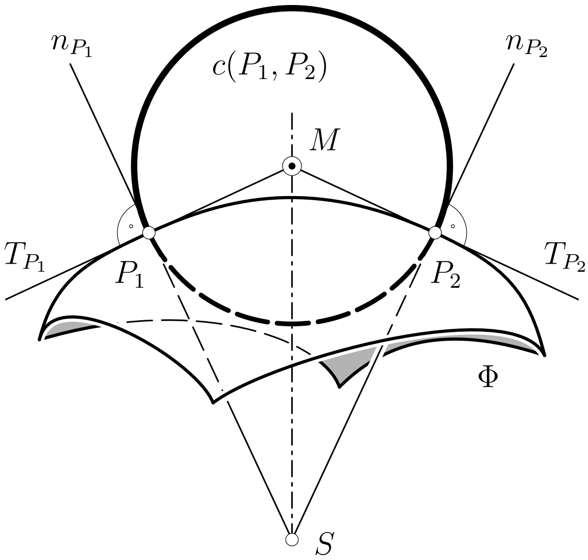

Figure 3: Ortho-circle $c(P_1, P_2)$ joining two points P_1 and P_2

1. The normals n_{P_1} and n_{P_2} intersect at a point S, or they are parallel and the line P_1P_2 is then orthogonal to both n_{P_1} and n_{P_2}.
2. The distances from S (if it exists) to points P_1 and P_2, respectively, are equal: $\overline{SP_1} = \overline{SP_2}$.
3. The axis and thus the center M of $c(P_1, P_2)$ is contained in both tangent planes T_{P_1} and T_{P_2} of Φ at P_1 and P_2, respectively. In case of parallel normals n_{P_1} and n_{P_2} we have $T_{P_1} = T_{P_2}$ and M is the midpoint of P_1P_2.
4. The four points P_1, P_2, M, and S are coplanar. They are included in the plane spanned by n_{P_1} and n_{P_2}. If the normals n_{P_1} and n_{P_2} are parallel, then the point S is their common ideal point.

Once we have found an ortho-circle of Φ we also have found a circle g touching Φ twice at P_1 and P_2, respectively. It is centered at S and its radius equals $\overline{SP_1} = \overline{SP_2}$. The point S is also the center of a sphere Σ touching Φ in P_1 and P_2, and g is a great circle of Σ.

Any surface normal n_P of Φ at P which is also a surface normal at another point $Q \neq P$ ($Q \in \Phi$) is called a *double normal (line)* of Φ. Throughout this paper we will regard double normal lines as *straight ortho-circles.*

2.2. Inversion

In the following we assume that points $X \in \mathbb{R}^3$ are represented by Cartesian coordinate vectors also denoted by $X = (x, y, z)$. With $\langle U, V\rangle = u_x v_x + u_y v_y + u_z v_z$ we denote the *standard scalar product* for vectors $U = (u_x, u_y, u_z)$ and $V = (v_x, v_y, v_z)$ in $\mathbb{R}^3$. Sometimes we use the projective closure of $\mathbb{R}^3$: The homogeneous coordinates of proper points thus will be given by $X = (1 : x : y : z) = (x_0 : x_1 : x_2 : x_3)$. Ideal points are characterized by $x_0 = 0$, i.e., they obey the equation of the ideal plane ω.

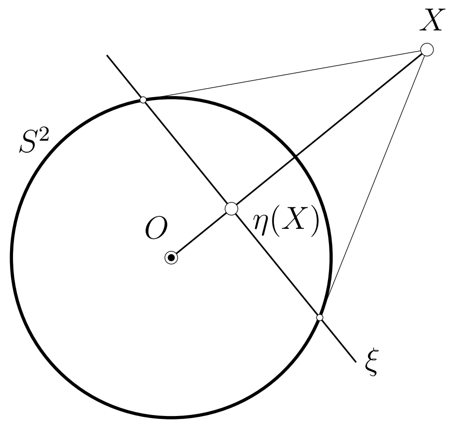

Figure 4: Inversion with respect to a circle or a sphere

Let S^2 be the Euclidean unit sphere centered at $O = (0, 0, 0)$ with the equation

$$\langle X, X\rangle - 1 = x^2 + y^2 + z^2 - 1 = 0.$$

The mapping $\eta:\ \mathbb{R}^3 \setminus \{O\} \to \mathbb{R}^3$ defined by

$$X \ \mapsto\ \eta(X) = X\langle X, X\rangle^{-1} \tag{1}$$

is called *inversion with respect to* Σ. Obviously S^2 is fixed under η and the image of O is undefined. In order to make η bijective, we perform the *conformal closure* of $\mathbb{R}^3$ by adding one single element ∞ to the set of points. This ideal element is contained in each plane and sphere [4, 6, 19]. Thus the inversion η becomes bijective if we define $\eta(O) = \infty$.

The inversion defined by (1) is birational and therefore a Cremona transformation [8, 22, 42, 43]. This means that even the coordinate functions of its inverse η^{-1} can be written as rational functions in the coordinates x, y, z. Indeed the inverse of η reads just like (1) as can easily be checked from straight forward calculations. Obviously η is involutive, i.e., $\eta = \eta^{-1}$.

The mapping η also allows a more geometric definition (see Fig. 4). Any point $X \in \mathbb{R}^3$ is first mapped to its polar plane ξ with regard to S^2 and the image point is defined by $\eta(X) = (O \vee X) \cap \xi$. This again shows that η is an involution.

For the sake of simplicity we define the *extended set $\mathcal{S}$ of spheres* (sometimes called Möbius spheres [4, 6, 19]) as the union of points, planes, and spheres in Euclidean $\mathbb{R}^3$ together with the one and only ideal element ∞.

The inversion is conformal as can be seen easily by direct computation:

$$\langle dX', dX'\rangle = \langle dX, dX\rangle\langle X, X\rangle^{-2}.$$

This holds in spaces of arbitrary dimensions.

We study the action of the inversion. For that we apply η to a sphere $\Sigma(M, r)$ with center M and radius r given by

$$\langle X - M,\ X - M\rangle - r^2 = 0.$$

With (1) an equation of $\eta(\Sigma)$ reads

$$\langle X, X\rangle(\langle M, M\rangle - r^2) - 2\langle X, M\rangle + 1 = 0.$$

The image surface thus is a sphere with center $M/(\langle M, M\rangle - r^2)$ and radius $\rho = (\langle M, M\rangle - 1)/(\langle M, M\rangle - r^2)$. Note that $\eta(M) = M\langle M, M\rangle^{-1}$ and the center of $\eta(\Sigma)$ do in general not coincide. For $M = O$ and $r = 1$ we find $\eta(\Sigma) = S^2$. Spheres with $M = O$ are mapped to concentric ones; this is the only case where the center of the image sphere and the image of the center are identic. In case of $\langle M, M\rangle - r^2 = 0$, i.e., the center O of inversion is contained in Σ, the equation of $\eta(\Sigma)$ simplifies to $1 = 2\langle X, M\rangle$. Consequently $\eta(\Sigma)$ is a plane perpendicular to OM. The latter fact together with the property that η is involutive shows that planes which do not pass through O are mapped to spheres containing O. Moreover: The restriction of η to any plane through O acts like a planar inversion as shown in Fig. 4. Obviously the elements of $\mathcal{S}$ are mapped to elements of $\mathcal{S}$.

These properties make it possible to describe the set of ortho-circles of Dupin cyclides: The cylinder of revolution, the cone of revolution, and the torus are envelopes of smooth one-parameter families of spheres. By applying an inversion to this one-parameter family of spheres we obtain another one-parameter family of spheres. The envelope Φ of the original family is thus mapped to the envelope $\eta(\Phi)$ of the family of image spheres. Since η is conformal ortho-circles of Φ are mapped to ortho-circles of $\eta(\Phi)$.

Because of the conformality of η orthogonal intersection is invariant with respect to inversions. Moreover, if c is a line of curvature on a C^2-surface $\Phi \in \mathbb{R}^3$ then its inverse $\eta(c)$ is a line of curvature on the η-image $\eta(\Phi)$ of Φ [38].

Later the following theorem will be of importance when we are going to characterize surfaces formed by ortho-circles. It is due to F. Joachimsthal [26] and is sometimes ascribed to O. Bonnet [25]:

Theorem 2.1 *Let Φ_1 and Φ_2 be two C^2-surfaces in $\mathbb{R}^3$ which intersect along a curve c. Any two of the following three statements imply the third:*

1. *The curve c is a line of curvature on Φ_1.*
2. *The curve c is a line of curvature on Φ_2.*
3. *Φ_1 and Φ_2 intersect along c at constant angle.*

Recalling the properties of an inversion or a general Möbius transformation we see that this theorem also applies to the inverse images of Φ_1, Φ_2, and c.

3. Dupin cyclides with a cuspidal point

3.1. Ortho-circles of cylinders of revolution

Let Φ be a cylinder of revolution. For short we will call Φ a cylinder. Now we are going to describe its set of ortho-circles. As can be seen in Fig. 5 we have:

Lemma 3.1 1. *Each pair of points on the same generator of a cylinder Φ can be joined by an ortho-circle.*

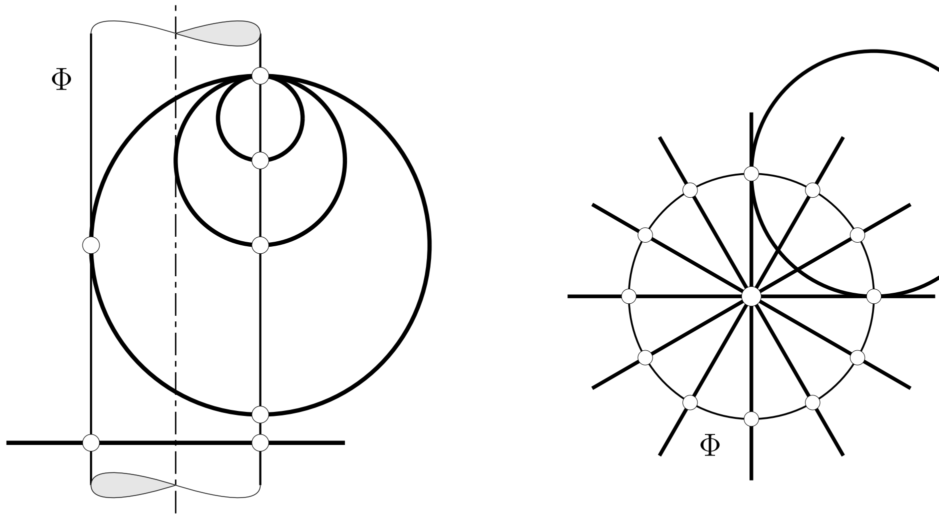

Figure 5: Ortho-circles of a cylinder of revolution

2. *Each pair of points on the same parallel circle of a cylinder* Φ *can be joined by an ortho-circle.*

There are no circles that intersect a cylinder more than twice orthogonally.

The set of straight ortho-circles of Φ is the congruence $\mathcal{N}$ of its surface normals. $\mathcal{N}$ can be decomposed into one-parameter families of pencils of lines in two ways:
(1) pencils with its vertex on the cylinder's axis (normals along parallel circles), and
(2) pencils of parallel lines (normals along generators).

Let c be an ortho-circle of Φ joining two points on the same generator. Applying a rotation about Φ's axis, Φ is transformed into itself and c sweeps a torus $\overline{\Phi}$. Depending on the radius of c we obtain all types of tori: the ring shaped, the thorn shaped, and the spindle shaped torus as well. Any $\overline{\Phi}$ shares at least two lines of curvature with Φ, moreover they intersect along these curves at constant angle $\pi/2$. We have:

Lemma 3.2 *For any cylinder* Φ *of revolution there exists a two-parameter family* $\mathcal{F}^2$ *of tori* $\overline{\Phi}$ *such that each torus* $\overline{\Phi} \in \mathcal{F}^2$ *consists of ortho-circles of* Φ *(each of which joining points on the same generator). Each* $\overline{\Phi} \in \mathcal{F}^2$ *intersects* Φ *along two lines of curvature (parallel circles).* $\mathcal{F}^2$ *contains a one-parameter family* $\mathcal{F}^1$ *of tori touching* Φ *along a line of curvature (parallel circle).*

Let now c be an ortho-circle joining points on the same parallel circle of Φ. We can apply the translation in the direction of Φ's generators. The ortho-circle c generates a cylinder $\overline{\Phi}$ of revolution. Obviously $\overline{\Phi}$ and Φ share two lines of curvature, i.e., two generators. So we have:

Lemma 3.3 *For any cylinder* Φ *of revolution there exists a two-parameter family* $\mathcal{F}^2$ *of cylinders of revolution such that each cylinder* $\overline{\Phi} \in \mathcal{F}^2$ *consists of ortho-circles of* Φ *(each of which joining points on the same parallel circle). Each* $\overline{\Phi} \in \mathcal{F}^2$ *intersects* Φ *orthogonally along two straight lines of curvature.*

3.2. Images of cylinders of revolution under inversions

A cylinder Φ of revolution can be seen as the envelope of a one-parameter family of spheres with constant radius and their centers located on a straight line. The family is determined by three congruent spheres with collinear centers. Thus the inverse $\eta(\Phi)$ of Φ is also an envelope of a one-parameter family of spheres defined by the η-images of the three given spheres in $\mathcal{S}$.

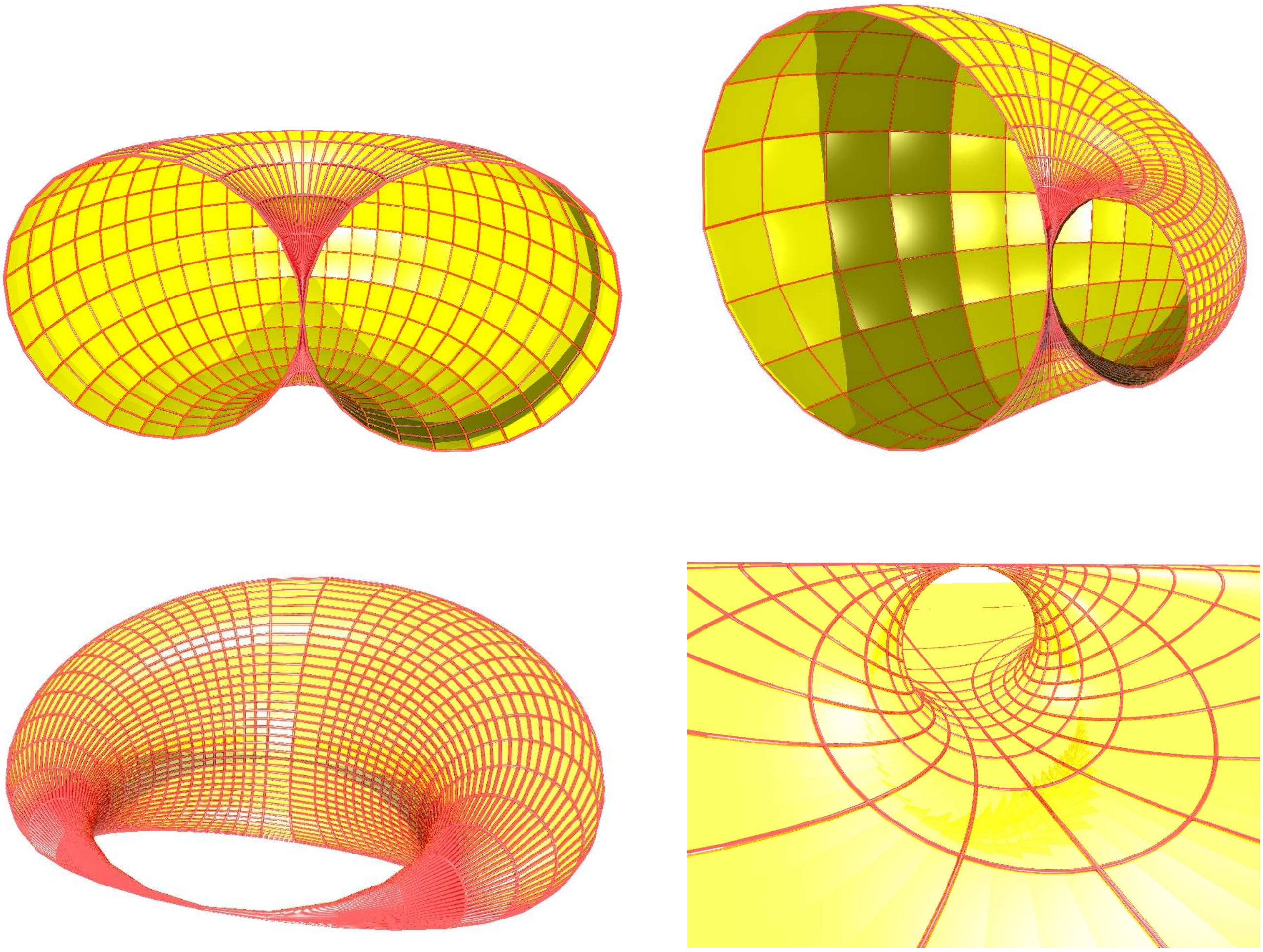

Figure 6: Dupin cyclides: first row: thorn torus, cuspidal cyclide;
second row: needle cyclide, and parabolic needle cyclide

The lines of curvature of Φ are the straight generators together with the parallel circles. Any inversion will transform them onto two one-parameter families of circles in the image surface $\eta(\Phi)$. Lines of curvature on Φ passing through O are mapped to straight lines (of curvature) on $\eta(\Phi)$.

Let the cylinder Φ of revolution be given by its equation

$$(x-e)^2 + y^2 - 1 = 0 \tag{2}$$

with constant $e \in \mathbb{R}$. It means no restriction to assume that the cylinder's radius equals 1. With $N = \langle X, X \rangle$ and (1) the inverse $\eta(\Phi)$ of Φ is given by

$$N^2(e^2-1) - 2exN + e^2 - 1 = 0. \tag{3}$$

Eq. (3) describes a *needle cyclide* if $|e| > 1$ and a *spindle cyclide* if $|e| < 1$. If $e = 0$ Eq. (3) is that of a *thorn torus* as a special case of a cuspidal cyclide. The degree of $\eta(\Phi)$ reduces to 3 if $e = \pm 1$; in this case the center O of inversion is contained in Φ and $\eta(\Phi)$ is called *parabolic needle cyclide*. Examples of η-images of cylinders are shown in Fig. 6.

4. Dupin cyclides with nodes

4.1. Ortho-circles of cones of revolution

Consider a cone Φ of revolution, from now on a *cone* for short. As can be seen from Fig. 7, the following facts are obvious:

Lemma 4.1 *1. Each pair of points on the same generator of a cone Φ can be joined by an ortho-circle.*

2. *Each pair of points on the same parallel circle of a cone Φ can be joined by an ortho-circle.*
3. *The cone admits 4-ortho-circles.*
4. *The 4-ortho-circles of a cone joining points on the same parallel circle form a sphere Σ centered at the cone's apex.*
5. *There exists a one-parameter family of such spheres.*
6. *For each generator g there exists a one-parameter family of ortho-circles joining points on g and touching the opposite generator.*

Remarks: 1. Assume that the angle of aperture of the cone Φ equals $\pi/2$ and let c be an ortho-circle joining two points on g such that c is in contact with the opposite generator. Then c is in contact with Φ and intersects Φ at V orthogonally at the same time.

2. The vertex of a quadratic cone is its only singular point. This makes it necessary to say a few words about ortho-circles through singular points. Let c be an ortho-circle joining any two (regular) points P_1 and P_2 on the same generator g of Φ. All points on g share the same tangent plane. The center of c is the midpoint of P_1P_2. If now P_1 is moving along the generator towards V, the center of the ortho-circle is again defined as the midpoint of VP_2, and together with P_2 and the normal n_{P_2} the ortho-circle c is uniquely defined.

Let c be an ortho-circle of Φ joining two points on the same generator of Φ. Applying a rotation about Φ's axis, Φ is transformed into itself and c sweeps a torus $\overline{\Phi}$. Depending on the radius of c we obtain all types of tori: the ring shaped, the thorn shaped, and the spindle shaped torus as well. Any $\overline{\Phi}$ intersects Φ at least along two common lines of curvature (parallel circles) at constant angle $\pi/2$. We have:

Lemma 4.2 *For any cone Φ of revolution there exists a two-parameter family $\mathcal{F}^2$ of tori such that each torus $\overline{\Phi} \in \mathcal{F}^2$ consists of ortho-circles of Φ (each of which joining points on the same generator). Each $\overline{\Phi} \in \mathcal{F}^2$ intersects Φ along two lines of curvature (parallel circles) orthogonally.*

Assume Φ's angle of aperture is different from $\pi/2$. Then there is a one-parameter family $\mathcal{F}^1 \subset \mathcal{F}^2$ of tori which touch Φ along a line of curvature (parallel circle) and intersect Φ twice orthogonally along two lines of curvature (parallel circles) at the same time.

In case of an right angle of aperture the curve of contact shrinks to the vertex of Φ.

Assume c is an ortho-circle of a cone Φ of revolution joining two points on the same parallel circle p. Applying the dilatation with the cone's vertex for its center, the circle c generates a cone $\overline{\Phi}$. Since c and p intersect in two points they are contained in a sphere Σ. Σ is centered at the common point V of the axes of both c and p, respectively. V is Φ's vertex since c's axis is the intersection of two tangent planes of Φ. Thus $\overline{\Phi}$ is also a cone of revolution. The cones Φ and $\overline{\Phi}$ share two lines of curvature (generators) and we have:

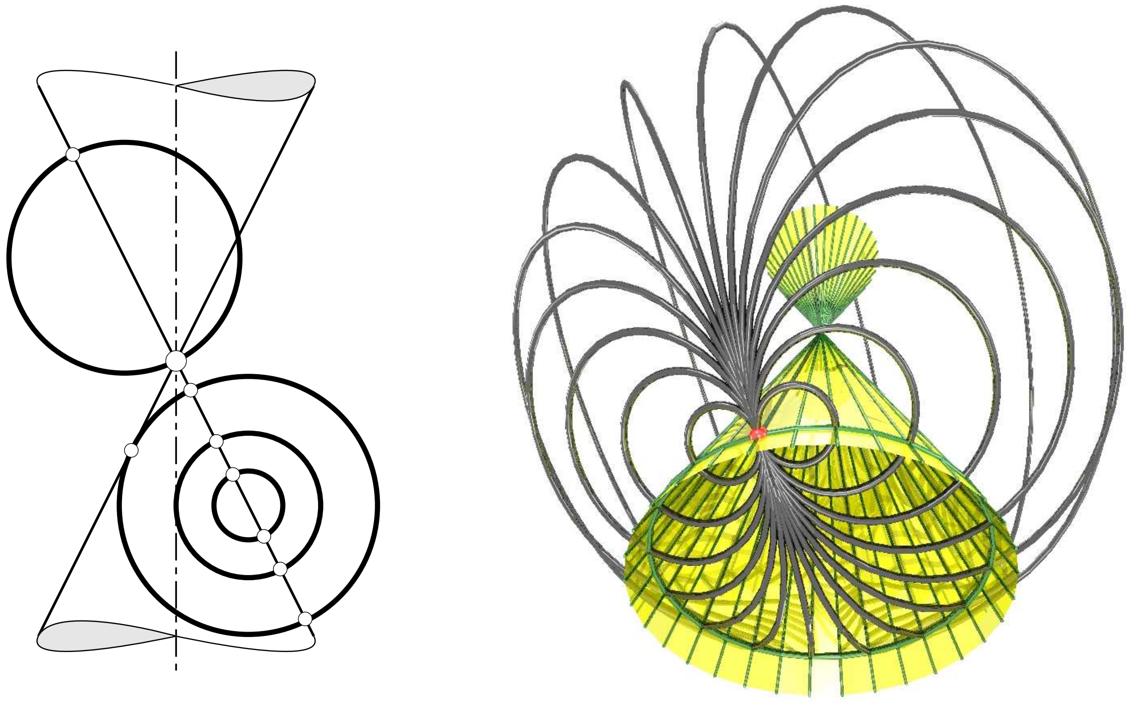

Figure 7: Ortho-circles joining points on a cone of revolution

Lemma 4.3 *For any cone Φ of revolution there exists a two-parameter family $\mathcal{F}^2$ of cones of revolution such that the parallel circles of each cone $\overline{\Phi} \in \mathcal{F}^2$ are ortho-circles of Φ.*
Each cone $\overline{\Phi} \in \mathcal{F}^2$ shares the vertex with Φ and intersects Φ along two lines of curvature (generators) orthogonally.

The aforementioned facts on intersecting axes of circles c and p imply:

Lemma 4.4 *The ortho-circles of a cone Φ of revolution joining points on the same parallel circle form a sphere Σ centered in the cones vertex. Σ carries a one-parameter family of 4-ortho-circles of Φ which are located in the meridian planes of Φ.*

Fig. 7 shows some ortho-circles of a cone Φ of revolution joining a point with points on the same parallel circle.

Remark: The normals of a cone Φ of revolution are not ortho-circles of Φ though their images under inversions are ortho-circles of the image surface.

4.2. Images of cones of revolution under inversions

Similarly to the case of cylinders of revolution we can treat cones Φ of revolution. Like any surface of revolution, cones of revolution can be seen as the envelopes of one-parameter families of spheres. The radius is growing linearly while the centers traverse the cone's axis. The inverse $\eta(\Phi)$ of Φ will therefore be an envelope of a one-parameter family of spheres.

We note that on Φ there are straight generators and circles as well. These curves comprise the set of lines of curvature on Φ and so do their η-images on $\eta(\Phi)$. Let Φ be given by the equation

$$(x-e)^2 + y^2 - \frac{(z-f)^2}{k^2} = 0, \tag{4}$$

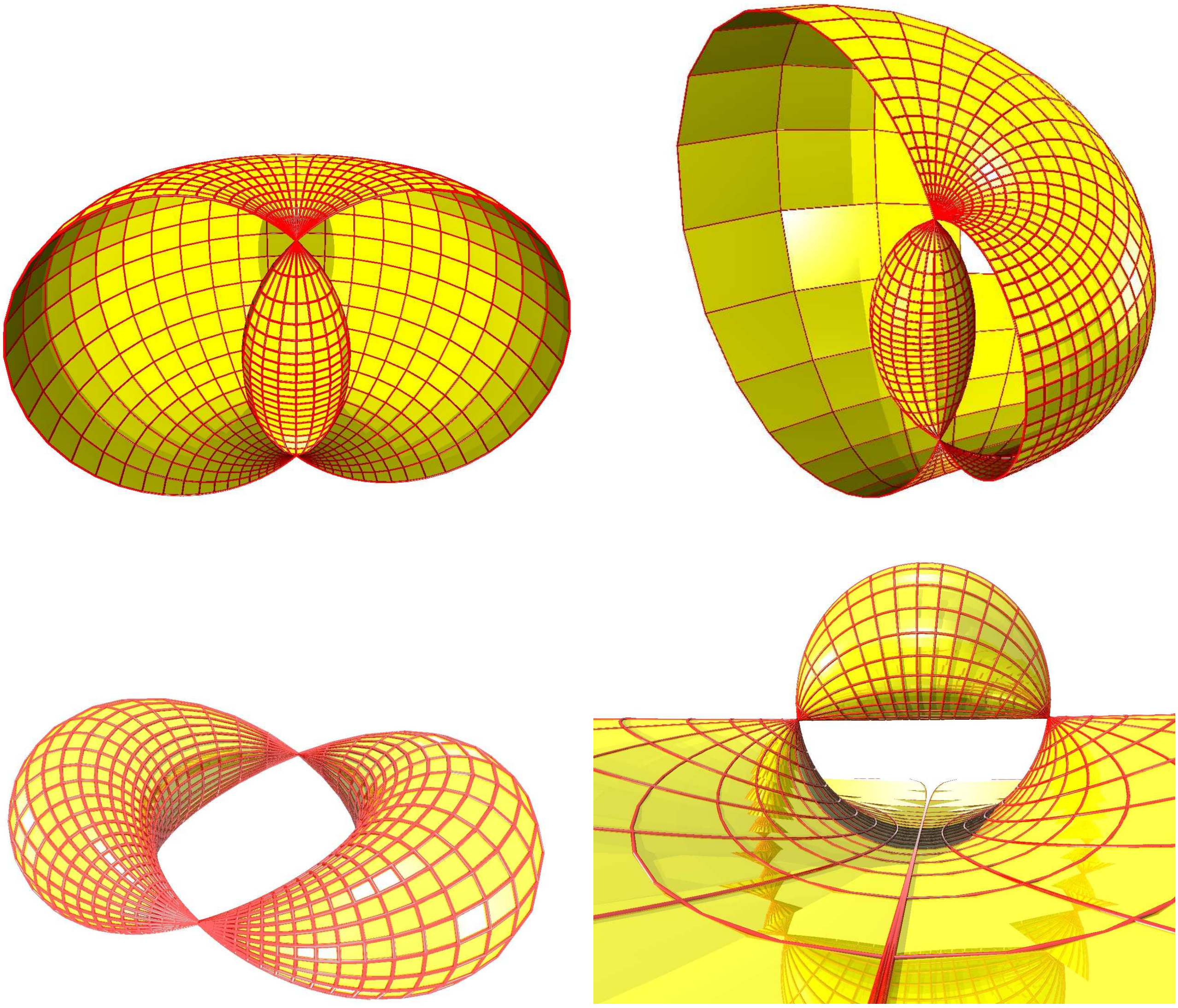

Figure 8: Dupin cyclides: first row: spindle torus, spindle cyclide;
second row: horn cyclide, and parabolic horn cyclide

where the constant $k \in \mathbb{R} \setminus \{0\}$ equals the cotangent of the half angle of aperture. The real constants e, f determine the position of the apex of Φ. With (1) and $N = \langle X, X \rangle$ we get the equation of $\eta(\Phi)$ as

$$N^2\left(e^2 - \frac{f^2}{k^2} - 1\right) + 2N\left(-ex + \frac{f}{k^2}\,z\right) + x^2 + y^2 - \frac{z^2}{k^2} = 0. \tag{5}$$

Eq. (5) is of degree 4. It is the equation of a *spindle cyclide* if $e^2k^2 - f^2 < 0$ (i.e., O is an interior point of Φ) and a *horn cyclide* if $e^2k^2 - f^2 > 0$ (i.e., O is an exterior point of Φ). As a special case we obtain the *spindle torus* for $e = 0$ and $f \neq 0$. The horn cyclide ($\neq$ spindle torus) becomes symmetric if $f = 0$ and $e \neq 0$ [1]. All these are quartic surfaces. In case of $ek = \pm f$ we find $O \in \Phi$ and $\eta(\Phi)$ is called *parabolic horn cyclide*. The latter is of degree 3. Fig. 8 shows some examples: a spindle torus, a spindle cyclide, horn cyclide, and a parabolic horn cyclide.

Under $e = f = 0$ the center O of inversion is the apex of Φ. Since lines and planes through O are transformed into themselves under η we have $\Phi = \eta(\Phi)$.

5. Dupin cyclides without nodes

5.1. Ortho-circles of ring tori

Let m be a circle in a plane μ and let further $A \subset \mu$ be a line. The surface Φ generated by m while μ is rotating about A is called a *torus*. Depending on the number of real intersection points the following names are common: For $\#A \cap m = 2$ the surface Φ is a *spindle torus* (see Fig. 8), for $\#A \cap m = 1$ Φ is a *thorn torus* (see Fig. 6), and for $\#A \cap m = 0$ Φ is a *ring torus* (see Fig. 13).

Note that from the view point of the complex extension the ring and spindle shaped surfaces do not differ. We observe that the above definition of a torus includes the case of spheres when A happens to be a diameter of m. The appearing spheres are of multiplicity two.

The following facts are obvious and illustrated in Fig. 9:

Lemma 5.1 *1. Any two points on the same parallel circle of a torus Φ can be joined by an ortho-circle.*

2. *Any two points on the same meridian circle of a torus Φ can be joined by an ortho-circle.*
3. *The torus admits 4-ortho-circles joining points on the same meridian circle.*
4. *The 4-ortho-circles of a torus joining points on the same parallel circle p form a sphere Ω centered at the axis of Φ. Ω is centered at the vertex of Φ's tangent cone along p.*
5. *There exists a pencil of such spheres including the plane of Φ's circular spine curve.*
6. *There exists a one-parameter family of ortho-circles joining points on the same meridian circle and touching the opposite one.*

Let now m be a meridian circle. Each point $P_1 \in m$ can be joined with each point $P_2 \neq P_1$ on m by an ortho-circle c. Under the rotation about the axis of the torus Φ the ortho-circle c sweeps a torus $\overline{\Phi}$ which intersects Φ orthogonally at least along two lines of curvature (parallel circles). All types of tori (the ring, thorn, and spindle shaped) show up as surfaces $\overline{\Phi}$. If P_1 and P_2 are joined by a straight ortho-circle (diameter of m) the surface $\overline{\Phi}$ is the cone of revolution formed by Φ's normals along the paths of P_1 and P_2, respectively. So we can state:

Lemma 5.2 *For each ring torus Φ there exists a two parameter family $\mathcal{F}^2$ of coaxial tori such that the meridian circles of each torus $\overline{\Phi} \in \mathcal{F}^2$ are ortho-circles of Φ (each of which joining points on the same meridian circle).*

$\mathcal{F}^2$ contains a one-parameter family $\mathcal{F}^1$ of cones of revolution comprising the set of straight ortho-circles of Φ.

$\mathcal{F}^2$ contains a one-parameter family $\mathcal{F}^1$ of tori which intersect Φ orthogonally along two lines of curvature (parallel circles) and touch Φ along another line of curvature (parallel circle) at the same time.

Consider p is an arbitrary parallel circle of a torus Φ. The normals as well as the meridian tangents along p form a cone of revolution. Thus we can apply Lemmas 4.3 and 4.4 and state:

Lemma 5.3 *The ortho-circles of any torus Φ joining points on the same parallel circle form a sphere.*

There exists a one-parameter family $\mathcal{F}^1$ of such spheres. Each sphere $\overline{\Phi} \in \mathcal{F}^1$ intersects Φ along two lines of curvature (parallel circles) orthogonally.

Each sphere in $\mathcal{F}^1$ contains a one-parameter family $\mathcal{F}^1$ of 4-ortho-circles contained in the meridian planes of Φ.

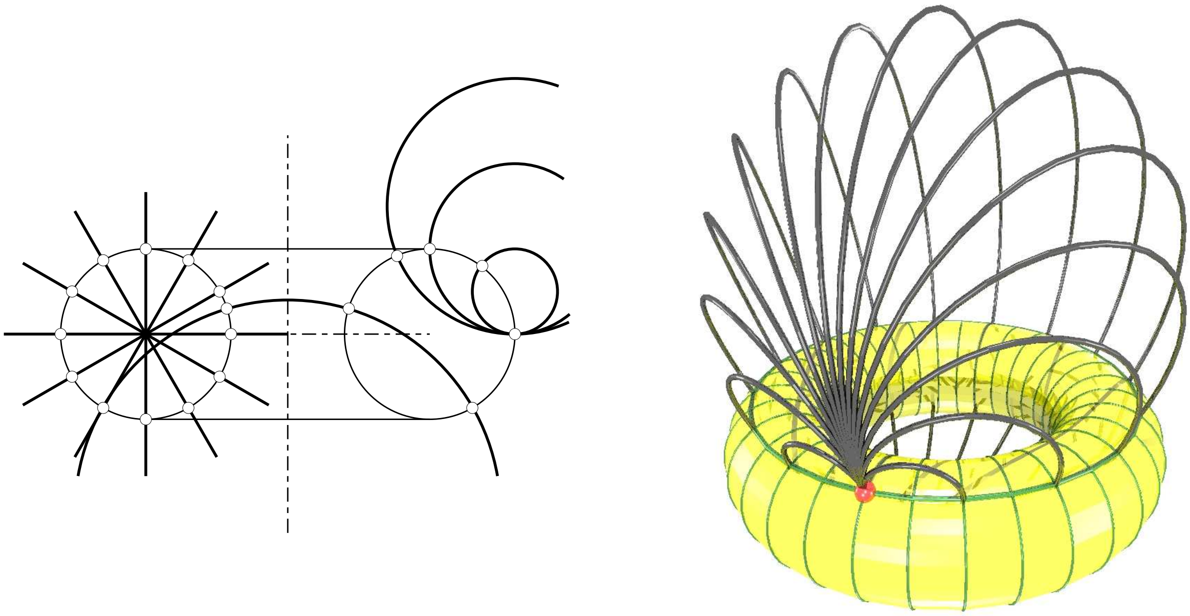

Figure 9: Ortho-circles joining points of a ring torus

The normals of any torus Φ are at least double normals of Φ, and thus the congruence $\mathcal{N}$ of surface normals of Φ equals the set of straight ortho-circles of Φ as is the case for the cylinder of revolution as well. As illustrated in Fig. 10, $\mathcal{N}$ can be decomposed into a one-parameter family of pencils of lines carried by meridian planes and with vertices on the circular spine curve. On the other hand $\mathcal{N}$ is the set of generators of cones of revolution, all of them passing through the circular spine curve and share the axis A with the torus.

Any ring shaped torus Φ carries a third one-parameter family of circles different from the meridian and parallel circles [15, 20, 44, 45, 46, 49]. These circles are called Villarceau *circles* and can be found either by intersecting Φ with double tangent planes τ or by intersecting with spheres Ψ touching Φ twice without being in line contact. As an example, consider a doubly touching sphere Ψ which is centered in the plane π_1 of Φ's circular spine curve and touches Φ in two different points, at one from inside and at the other from outside (see Fig. 10).

Remark: A sphere touching a ring torus Φ twice in points $P_1 \neq P_2$ without being in line contact with Φ can be called Villarceau *sphere.*
Any such sphere Ψ intersects Φ in a real spherical quartic q with two double points. (Note that the curve $x_1^2+x_2^2+x_3^2 = x_0 = 0$ in the ideal plane is a double curve of Φ and thus splits from $\Phi\cap\Psi$ with multiplicity 2.) As q has two double points it is the union of two conics. Since q is entirely contained in a sphere, these conics are circles. Thus *any Villarceau sphere shares two Villarceau circles with the ring torus* Φ. Unfortunately this is not mentioned in [45].
For any torus (indeed for any C^1-surface) Φ there exists a two-parameter family $\mathcal{F}^2$ of doubly touching spheres. The centers of these spheres form the *medial axis* of Φ [9, 31]. The medial axis of a ring torus with Eq. (6) consists of its axis, its circular spine curve, and the one-sheet hyperboloid of revolution with equation $(x-e)^2/r^2 + y^2/r^2 - z^2/(R^2 - r^2) = 1$.

Let v be a Villarceau circle on a ring torus Φ and let further τ be the double tangent plane of Φ carrying v. The plane τ touches Φ in two different points, say D_1 and D_2, respectively. There are exactly two pairs of points on v which can be joined by an ortho-circle:

1. Φ's surface normals at D_1 and D_2 are parallel and the plane τ is tangent to Φ at both

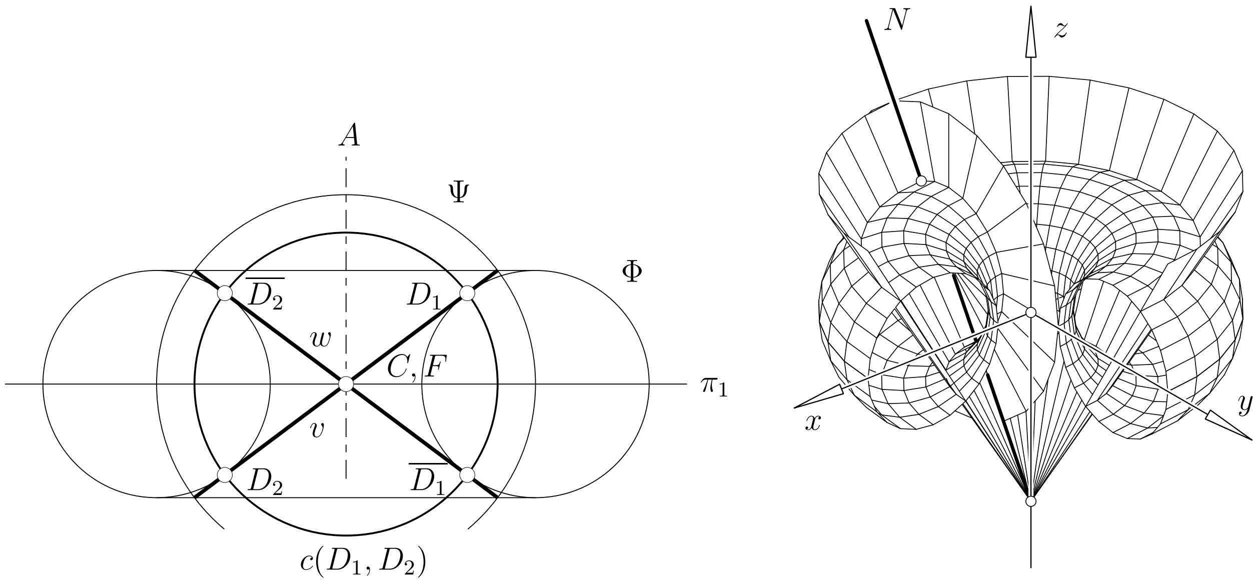

Figure 10: Left: Villarceau sections of a ring torus.
Right: Congruence of surface normals of a torus

points. Thus there is an ortho-circle $c(D_1, D_2)$ joining D_1 with D_2. Points D_1 and D_2 are located in the same meridian plane μ of Φ, the line D_1D_2 is tangent to both meridian circles $\mu \cap \Phi$. Obviously $c(D_1, D_2)$ is a 4-ortho-circle of Φ.
The circle v can also be obtained as a part of the intersection of Φ with a Villarceau sphere Ψ centered in the plane π_1 of Φ's circular spine curve. (Ψ is touching Φ from the inside and the outside.) The remaining real part of the intersection curve $\Psi \cap \Phi$ is a second Villarceau circle w which can be obtained from v by reflection with respect to π_1. The carrier plane of w is Φ's tangent plane in points $\overline{D_1}$ and $\overline{D_2}$, respectively. This again shows that $c(D_1, D_2) = c(\overline{D_1}, \overline{D_2})$ is a 4-ortho-circle.

2. Let $C \in v$ be the point closest to the axis A of Φ and let $F \in v$ denote the point with maximal distance to A. The line $CF \subset \pi_1$ is the surface normal of Φ at both C and F. Indeed it is four times a normal of Φ. Thus CF is a straight 4-ortho-circle.

We can state

Lemma 5.4 *A pair (P_1, P_2) of points on a Villarceau circle can be joined by an ortho-circle $c(P_1, P_2)$ if and only if $c(P_1, P_2)$ is a 4-ortho-circle.*

Proof: The only points on a Villarceau circle v which can be joined by an ortho-circle are those which can also be joined with a parallel circle or a meridian circle. Since any meridian circle and any parallel circle intersects v exactly once, there are no further points on v different from D_1, D_2 and C, V which can be joined by an ortho-circle. □

Figure 11 shows a ring torus with points on a Villarceau circle joined by 4-ortho-circles. An obvious consequence of Lemma 5.4 is

Lemma 5.5 *The 4-ortho-circles of a ring torus Φ joining points on Villarceau circles form a sphere Ω and the plane π_1 of Φ's circular spine curve. Ω is concentric with Φ, i.e., it is centered at $A \cap \pi_1$. Ω and π intersect at right angles.*

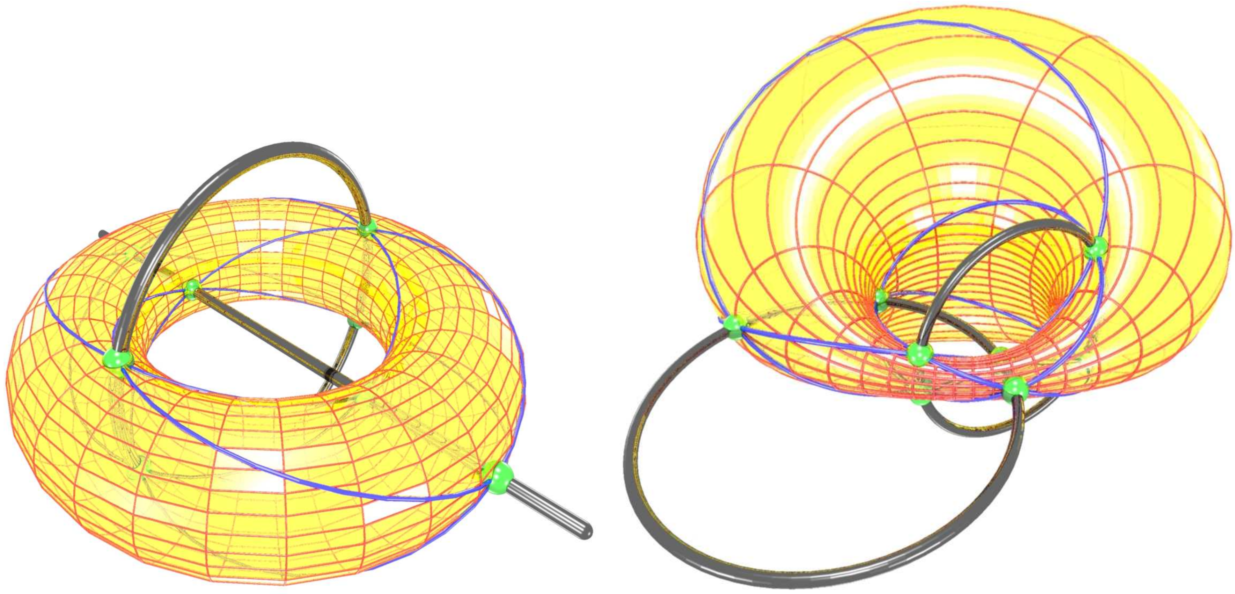

Figure 11: Left: Ring torus with a Villarceau circle and points joined by a 4-ortho-circle (Lemma 5.4). Right: Ring shaped Dupin cyclide with 4-ortho-circles joining points on a Villarceau circle (Lemma 6.8)

Remark: The plane π_1 carries the straight 4-ortho-circles of Φ.

Villarceau circles are known to be *loxodromes*, i.e., curves intersecting another family of curves on a surface at constant angle [15, 46, 49]. In this case the curves of reference are the lines of curvature. Since angles, circles, and lines of curvature are invariant with respect to inversions, the ring shaped Dupin cyclides carry a third set of circles (also called Villarceau circles) corresponding to the Villarceau circles of a ring torus (see Fig. 11).

Remark: The Villarceau circles on a ring cyclide Φ can also be found by intersection. The double tangent planes of Φ as well as the Villarceau spheres share two circles v and w, respectively, with Φ. Any double tangent plane τ of a ring cyclide Φ is the η-image of a sphere containing the center O of inversion. The set of doubly touching spheres of the preimage of Φ containing the center of inversion form a one-parameter subfamily of the two-parameter family of doubly touching spheres of $\eta^{-1}(\Phi)$.

Finally we give the following theorem dealing with the crossratio of the four common points of a 4-ortho-circle and a ring torus.

Lemma 5.6 *Let Φ be a ring or spindle torus with major radius R and minor radius r, respectively. Let further c be an arbitrary 4-ortho-circle of Φ intersecting in points S_1, S_2, S_3, and S_4. Then the four points S_i can be arranged such that their crossratio equals r^2/R^2.*

Proof: We use the notations given in Fig. 12. Let c be a 4-ortho-circle of a ring or spindle torus Φ. Its center Z is contained in the axis A of Φ. With m we denote one meridian circle of Φ in the plane μ of c.

1. In order to compute the crossratio of the four points S_i, we use the stereographic projection σ from one of the intersection points of c and A onto the straight 4-ortho-circle $n \in \mu$, say P. The crossratio remains invariant under σ.

Let $C, F \in n$ be the σ-images of S_1 and S_2, respectively. We have to show that $C, F \in m$. Assume $F = \sigma(S_1)$. Clearly $\sphericalangle ZS_1M = \pi/2$. Further we observe that

$$\sphericalangle MS_1F = \pi/2 - \sphericalangle ZS_1P \quad \text{and} \quad \sphericalangle MFS_1 = \pi/2 - \sphericalangle ZPS_1.$$

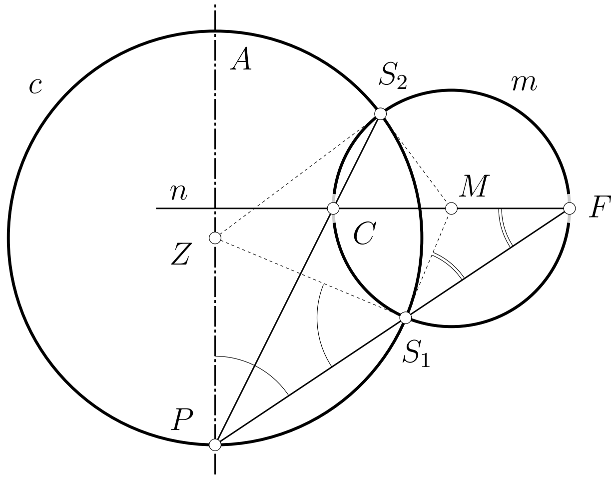

Figure 12: Crossratio of intersection points of a torus and a 4-ortho-circle c

Consequently we have

$$\sphericalangle\, MS_1F = \sphericalangle\, MFS_1 \quad \text{and therefore} \quad \overline{MS_1} = \overline{MF_1}$$

which shows $F \in m$. Similarly we show $\sigma(S_2) = C$.
If we use the other point of $c \cap A$ as center of projection, S_1 is mapped to C and S_2 is mapped to F.

2. Let $\overline{F}$ and $\overline{C}$ denote the points symmetric to F and C with respect to A. Obviously they are the stereographic images of the remaining intersection points S_3 and S_4 of c and Φ.
Now it is elementary to verify that the crossratio of F, $\overline{F}$, $\overline{C}$, and C equals r^2/R^2 and we find $\mathrm{cr}(F, \overline{F}, \overline{C}, C) = \mathrm{cr}(S_1, S_3, S_4, S_2) = r^2/R^2$. □

Remark: Four points can be permuted in 24 ways, but there appear only six different values of crossratios: If $\mathrm{cr}(A, B, C, D) = \delta$, we have $\mathrm{cr}(A, C, B, D) = 1 - \delta$, $\mathrm{cr}(D, A, B, C) = \delta/(\delta - 1)$, $\mathrm{cr}(A, D, B, C) = (\delta - 1)/\delta$, $\mathrm{cr}(A, B, D, C) = 1/\delta$, and $\mathrm{cr}(A, C, D, B) = 1/(\delta - 1)$. Thus we can also assign the five values R^2/r^2, $R^2/(R^2 - r^2)$, $r^2/(R^2 - r^2)$, $1 - r^2/R^2$, and $1 - R^2/r^2$ as crossratios to certain arrangements of points S_i.

Lemma 5.6 holds for straight ortho-circles as well, which is clear from the proof.

5.2. Images of ring tori under inversions

Like any torus the ring torus Φ can be seen as envelope of two one-parameter families of spheres: The first family consists of congruent spheres with their centers located at the circular spine curve. The second family comprises the spheres touching Φ along its parallel circles. These spheres are centered at the straight spine curve, i.e., the axis of Φ. So we can expect $\eta(\Phi)$ to be the envelope of two one-parameter families of spheres.

Let $R, r \in \mathbb{R}$ denote the radii of the spine curve and the meridian of a ring torus Φ. With the real constant e we denote the distance of Φ's center to the origin of a Cartesian coordinate system. Then Φ is given by the equation

$$\big((x - e)^2 + y^2 + z^2 - R^2 - r^2\big)^2 - 4R^2(r^2 - z^2) = 0. \tag{6}$$

Now we apply an inversion to Φ, i.e., we substitute (1) in (6) and obtain the equation of $\eta(\Phi)$:

$$
\begin{gathered}
N^2\left((e^2-R^2-r^2)-4R^2r^2\right)\\
+2N(1-2ex)(e^2-R^2-r^2)+(1-2ex)^2+4R^2z^2=0,
\end{gathered}
\tag{7}
$$

where $N=\langle X,X\rangle$. This is the equation of a *ring cyclide* or a *ring torus* if $e=0$. For $e=\pm(R\pm r)$ the center O of inversion is contained in Φ and the degree of the image surface reduces to 3. The surface $\eta(\Phi)$ is then called *parabolic ring cyclide*. Fig. 13 shows a ring torus and two possible images under inversion: a (quartic) ring cyclide and a (cubic) parabolic ring cyclide.

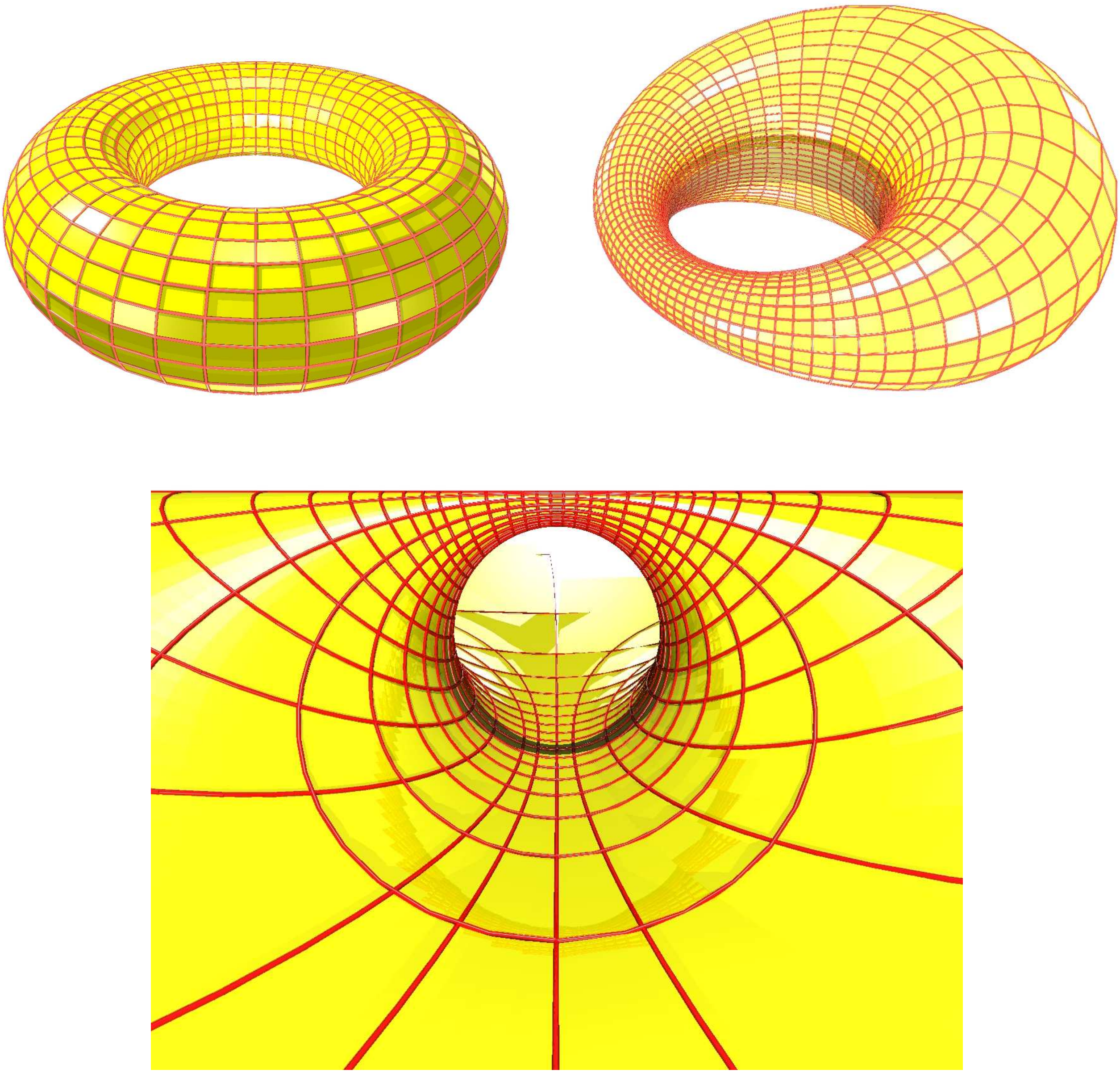

Figure 13: Dupin cyclides: ring torus, ring cyclide, parabolic ring cyclide

6. Surfaces formed by the ortho-circles of Dupin cyclides

So far we have studied the set of ortho-circles of cylinders, and cones of revolution, and the ring torus as well. Lemmas 3.2, 3.3, 4.3, 4.4, 5.2, and 5.3 clarify the shape of the surfaces built by certain ortho-circles of the respective surfaces. We have summarized well known facts about inversions and concerning the images of cones, cylinders, and ring tori under inversions. These preparations enable us to describe the set of ortho-circles of Dupin cyclides and surfaces formed by these ortho-circles.

6.1. Ortho-circles of cyclides with a cuspidal point

As an immediate consequence of Lemma 3.1 we find:

Lemma 6.1 *Each pair of points on the same line of curvature on a thorn torus, cuspidal cyclide, needle cyclide, or parabolic needle cyclide can be joined by an ortho-circle.*

In the case of cuspidal cyclides it is useful to distinguish between two types of lines of curvature: those passing through the cuspidal point will be called *lines of curvature of 1st kind*. The others will be referred to as the lines of curvature of *2nd kind*. The latter do not pass through the cuspidal point.

Under any inversion the straight lines (of curvature) on the cylinder are mapped to the lines of curvature of 1$^{\text{st}}$ kind, the parallel circles are mapped to the lines of curvature of 2$^{\text{nd}}$ kind.

If we apply an inversion to the surfaces described in Lemma 3.2 we get:

Theorem 6.1 *For any cuspidal Dupin cyclide Φ (including all special cases) there exists a two-parameter family $\mathcal{F}^2$ of Dupin cyclides such that each cyclide $\overline{\Phi} \in \mathcal{F}^2$ consists of ortho-circles (each of which joining points on the same line of curvature of 1st kind) of Φ.*

According to Theorem 2.1 each $\overline{\Phi} \in \mathcal{F}^2$ intersects Φ along two lines of curvature of 2nd kind orthogonally.

The family $\mathcal{F}^2$ contains a one-parameter family $\mathcal{F}^1$ of Dupin cyclides which touch Φ along a line of curvature of 2nd kind.

Remark: When ever we use the term line of curvature in this section, we keep in mind that these are circles, except a few ones which are straight lines in the parabolic cyclides or the generators of cylinders and cones.

Lemma 3.3 can now be formulated as follows:

Theorem 6.2 *For any cuspidal Dupin cyclide Φ (including all special cases) there exists a two-parameter family $\mathcal{F}^2$ of cuspidal Dupin cyclides such that each Dupin cyclide $\overline{\Phi} \in \mathcal{F}^2$ consists of ortho-circles (each of which joining points on the same line of curvature of 2nd kind) of Φ.*

According to Theorem 2.1 each $\overline{\Phi} \in \mathcal{F}^2$ intersects Φ along lines of curvature of 1st kind orthogonally.

Remark: In this family $\mathcal{F}^2$ we cannot find cyclides which are in line contact with Φ like those mentioned in Theorem 6.1.

Note that all ortho-circles appearing here are themselves lines of curvature on the surfaces $\overline{\Phi}$ formed by them.

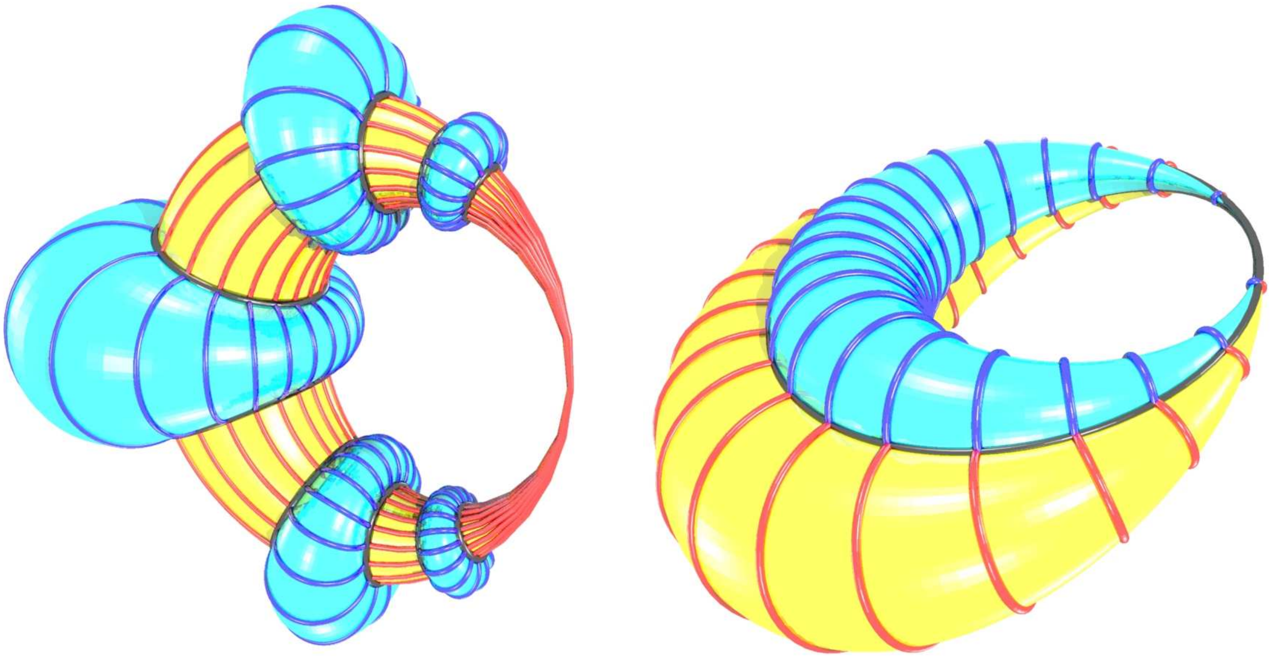

Figure 14: Cuspidal cyclides with cyclides formed by ortho-circles. Left: Cuspidal cyclide and ring cyclides (cf. Theorem 6.1). Right: Cuspidal cyclides carrying each other's ortho-circles while sharing the cuspidal point (cf. Theorem 6.2)

6.2. Ortho-circles of cyclides with nodes

In Section 4 we had to distinguish between generators and parallel circles. Again there are two kinds of lines of curvature: The straight lines of curvature on the cone Φ are transformed to those lines of curvature on $\eta(\Phi)$ passing through the nodes of $\eta(\Phi)$ and will henceforth be called of 1^{st} kind. The η-images of the parallel circles will be called lines of curvature of 2^{nd} kind. Unfortunately the theorems of this section seem to be formulated circumstantially.

We apply an inversion to the curves and surfaces described in Lemma 4.1 and find:

Lemma 6.2 *1. Each pair of points on the same line of curvature on a Dupin cyclide with nodes can be joined by an ortho-circle.*

2. *Each Dupin cyclide with nodes admits 4-ortho-circles.*
3. *The 4-ortho-circles of a Dupin cyclide with nodes joining points on the same line of curvature of 2^{nd} kind form a sphere.*
4. *There exists a one-parameter family of such spheres.*
5. *For any line of curvature of 1^{st} kind on a Dupin cyclide with nodes there exist two one-parameter families of ortho-circles touching the surface.*

We can reformulate Lemmas 4.2 and 4.3 and obtain

Theorem 6.3 *For any Dupin cyclide Φ with nodes there exists a two-parameter family $\mathcal{F}^2$ of Dupin cyclides with nodes such that each cyclide $\overline{\Phi} \in \mathcal{F}^2$ consists of those ortho-circles of Φ joining points on the same line of curvature of 1^{st} kind.*

According to Theorem 2.1 each cyclide $\overline{\Phi} \in \mathcal{F}^2$ intersects Φ along two lines of curvature of 2^{nd} kind orthogonally.

There is a one-parameter family $\mathcal{F}^1 \subset \mathcal{F}^2$ of Dupin cyclides touching Φ along a line of curvature of 2^nd^ kind, if the angle of aperture of the tangent cone at the nodes is different from $\pi/2$ (see Fig. 16).

If the angle of aperture of the tangent cone at the node of the cyclide Φ is a right one, then the ortho-circles touch the cone and thus the cyclide at the node and intersect there at right angles at the same time.

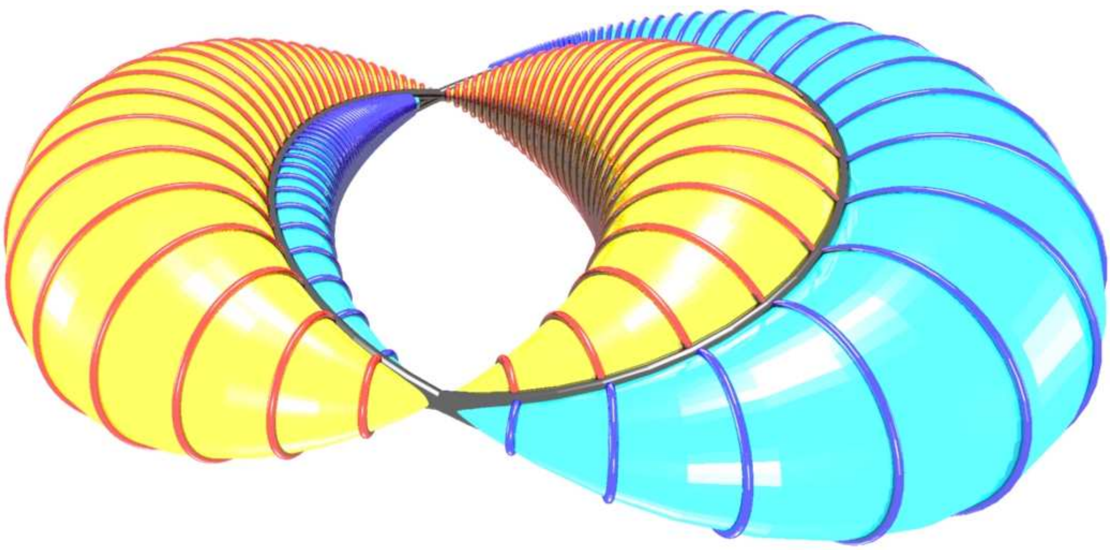

Figure 15: Two horn cyclides carrying some of each other's ortho-circles and sharing the nodes (cf. Theorem 6.4)

Theorem 6.4 *For any Dupin cyclide Φ with nodes there exists a two-parameter family $\mathcal{F}^2$ of Dupin cyclides with the same nodes such that the lines of curvature of 2^{nd} kind on each $\overline{\Phi} \in \mathcal{F}^2$ are ortho-circles of Φ joining points of the same line of curvature of 2^{nd} kind.*

Each cyclide $\overline{\Phi} \in \mathcal{F}^2$ intersects Φ along two lines of curvature of 1^{st} kind orthogonally, according to Theorem 2.1.

Figure 15 shows two horn cyclides carrying some of each orther's ortho-circles while sharing the nodes.

The result corresponding to Lemma 4.4 follows at once:

Theorem 6.5 *The ortho-circles of a Dupin cyclide Φ with nodes joining points on the same line of curvature of 2^{nd} kind form a sphere. This sphere carries a one-parameter subfamily of the two-parameter family of 4-ortho-circles of Φ.*

Again we observe that the ortho-circles forming the surfaces $\overline{\Phi}$ are lines of curvature on $\overline{\Phi}$.

6.3. Ortho-circles of ring cyclides

Now we are going to describe the set of ortho-circles of Dupin cyclides without nodes, i.e., the ring torus, the ring cyclide, and the parabolic ring cyclide. We only have to apply an inversion to curves and surfaces mentioned in the lemmas of the previous section.

It is useful to summarize the following facts (see Lemma 5.1):

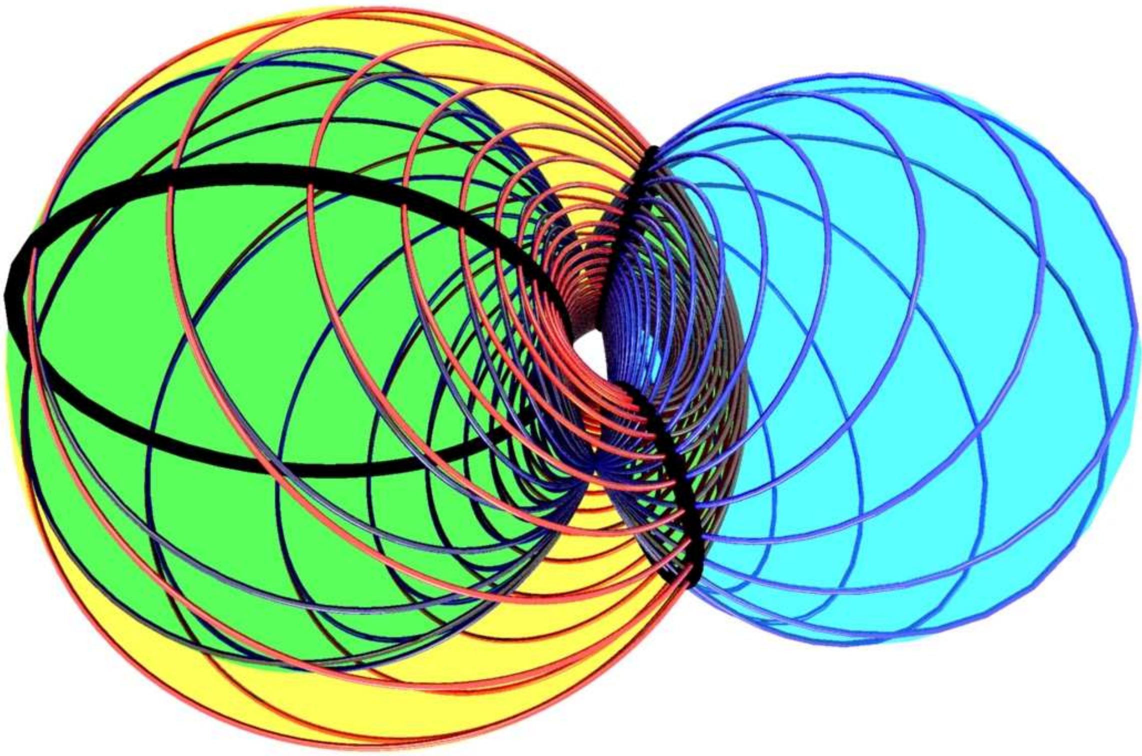

Figure 16: Ring cyclide and a horn cyclide of ortho-circles sharing three lines of curvature (cf. Theorem 6.3)

Lemma 6.3 *1. Each pair of points on the same line of curvature of a ring cyclide (including the ring torus and the parabolic ring cyclide) can be joined by an ortho-circle.*

2. The ring cyclide allows 4-ortho-circles.

3. The 4-ortho-circles of a ring cyclide can be arranged in a one-parameter family of spheres.

In order to formulate our results precisely, we call those lines of curvature on a ring cyclide (ring torus, parabolic ring cyclide) lines of curvature of *1st kind* which are the η-images of the meridian circles of the ring torus mentioned in Section 5. The η-images of the parallel circles will be referred to as the lines of curvature of *2nd kind*.

The more general versions of Lemmas 5.2 and 5.3 now read:

Theorem 6.6 *For any ring cyclide Φ there exists a two-parameter family $\mathcal{F}^2$ of Dupin cyclides such that the lines of curvature of 1st kind on each cyclide $\overline{\Phi} \in \mathcal{F}^2$ are ortho-circles of Φ (each of which joining points on the same line of curvature of 1st kind).*

Each Dupin cyclide $\overline{\Phi} \in \mathcal{F}^2$ intersects Φ along two lines of curvature of 2nd kind orthogonally, cf. Theorem 2.1.

For each ring cyclide Φ there exists a one-parameter family $\mathcal{F}^1$ of Dupin cyclides with nodes such that each cyclide $\overline{\Phi} \in \mathcal{F}^1$ intersects Φ along two lines of curvature at right angles. (The cyclides $\overline{\Phi}$ can be obtained by inverting the cones formed by the normals at points of parallel circles of a ring torus.)

There exists a one-parameter family $\mathcal{F}^1 \subset \mathcal{F}^2$ of Dupin cyclides such that each cyclide $\overline{\Phi} \in \mathcal{F}^2$ touches Φ along a line of curvature of the 2nd kind.

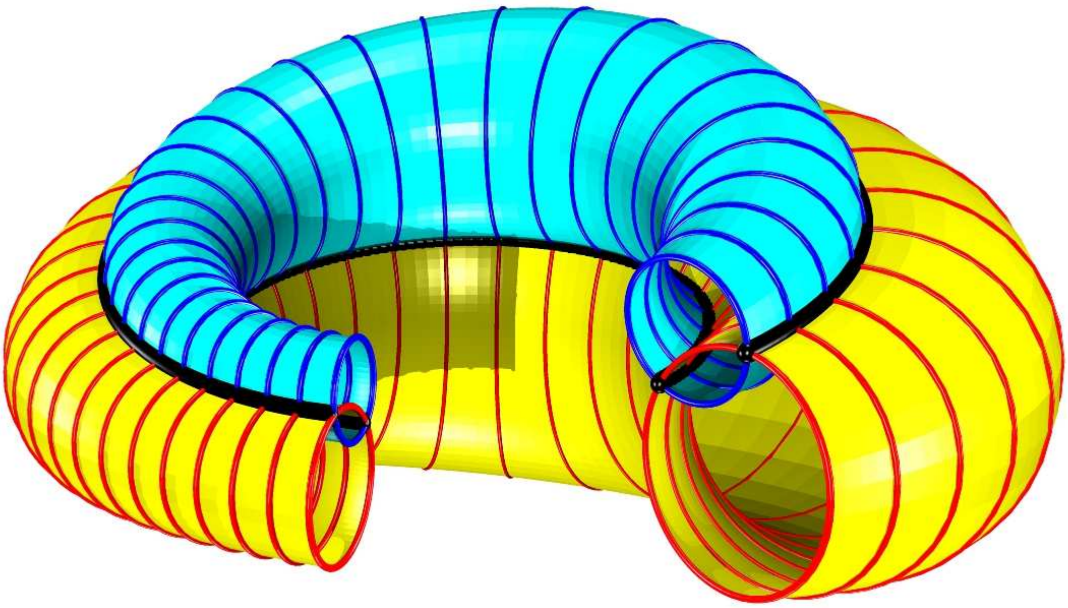

Figure 17: Ring cyclides carrying ortho-circles of each other (cf. Theorem 6.6)

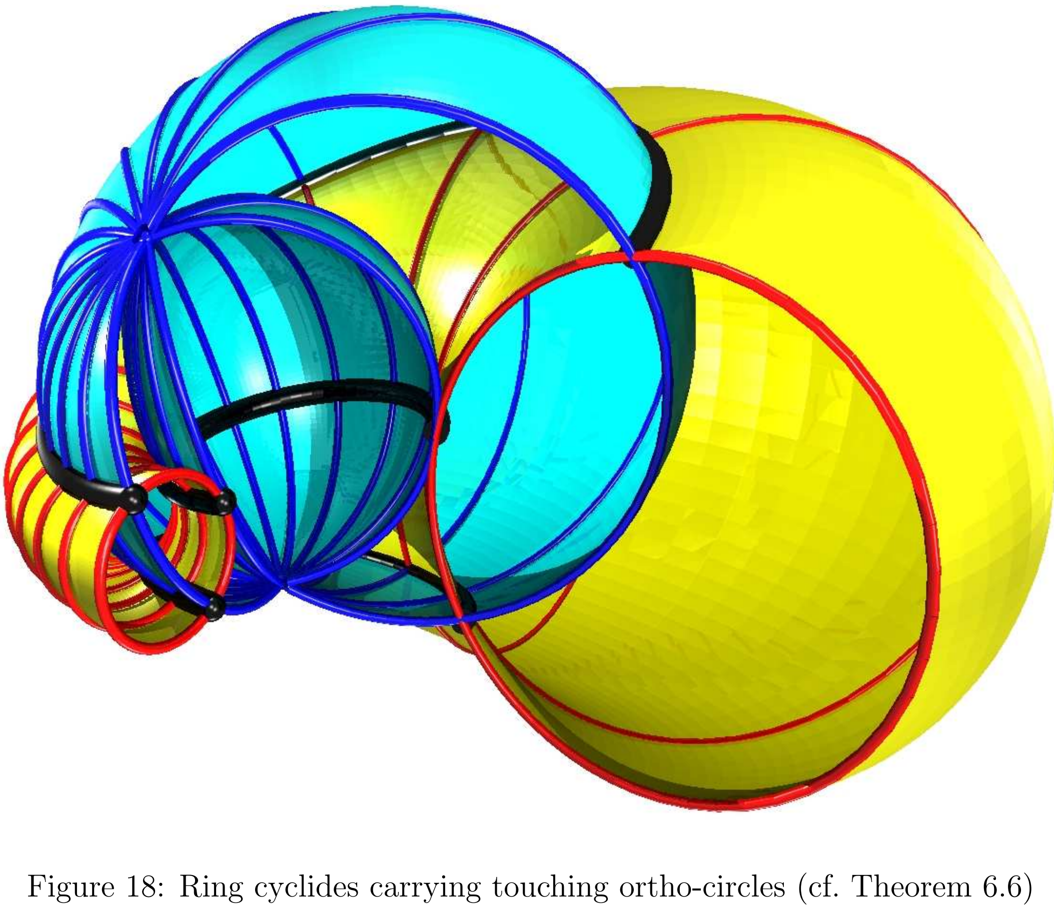

Figure 18: Ring cyclides carrying touching ortho-circles (cf. Theorem 6.6)

Fig. 17 shows two ring cyclides Φ and $\overline{\Phi}$. The lines of curvature of 1^{st} kind of $\overline{\Phi}$ are ortho-circles of Φ and vice versa. Fig. 18 shows a ring cyclide and a spindle cyclide carrying some of each other ortho-circles. The surfaces are in line contact along a common line of curvature and intersect along two lines of curvature at right angles.

Theorem 6.7 *The ortho-circles of any ring cyclide Φ joining points on lines of curvature of 2^{nd} kind form a sphere which intersects Φ along two lines of curvature of 2^{nd} kind orthogonally (cf. Theorem 2.1).*

Since ring shaped Dupin cyclides carry Villarceau circles, we have in analogy to Lemma 5.4

Theorem 6.8 *Two points P_1 and P_2 on a Villarceau circle of a ring shaped Dupin cyclide can be joined by an ortho-circle $c(P_1, P_2)$ if and only if $c(P_1, P_2)$ is a 4-ortho-circle.*

The facts described in Theorem 6.8 are illustrated in Fig. 11. Since crossratios are invariant with respect to inversions we can state Lemma 5.6 in a more general form:

Theorem 6.9 *On all 4-ortho-circles of a quartic ring, spindle, or horn cyclide the four intersection points have the same crossratio.*

Remark: For any needle and cuspidal cyclide Φ there exists a two-parameter of 3-ortho-circles. Each of them is passing through the cuspidal point. This two-parameter family of 3-ortho-circles can be arranged in a parabolic pencil of spheres touching at the cuspidal point. The 3-ortho-circles of Φ are the η-preimages of the normals of the cylinder $\eta^{-1}(\Phi)$.

The set of 4-ortho-circles joining points on Villarceau circles of a ring cyclide can be described by (cf. Lemma 5.5)

Theorem 6.10 *The 4-ortho-circles joining points on Villarceau circles of a ring cyclide form a pair of orthogonally intersecting spheres. Each of them intersects Φ along two lines of curvature (circles) at right angles.*

Proof: We start with the ring torus in Lemma 5.5 and apply an inversion. The sphere Ω is mapped to a sphere and the plane π_1 containing Ω's center is also mapped to a sphere. Ω and π_1 intersect Φ orthogonally along parallel circles; therefore their η-images do the same. □

7. Final remarks

A circle that intersects a given surface Φ at least twice at the same angle $\varphi \neq 0, \pi/2$ will be called a *φ-circle*. Most of the theorems concerning ortho-circles presented so far in this paper can be reformulated if we replace the term *ortho-circle* by *φ-cirlce*. Some of them need minor corrections: There are no 4-ortho-circles for a cylinder Φ of revolution. For any $\varphi \neq 0, \pi/2$ we can find 4-φ-circles of Φ.

The fact that being an ortho-circle and being a Dupin cyclide (including cylinders and cones of revolution) is invariant with respect to Möbius transformations is used throughout the paper. The ortho-circles of a Dupin cyclide Φ can somehow be seen as its Möbius geometric normals. With this normalization Dupin cyclides are *transnormal manifolds* in the sense of Wegner [47]. Unfortunately the normals of a cone of revolution are not double normals, they only intersect once at right angle. This blemish can be removed using the conformal closure as performed in Section 2.2 by defining an intersection at the ideal element as an intersection at any angle we need.

We do not answer the question whether Dupin cyclides are the only transnormal manifolds in the sense of Möbius geometry or not. This could be a topic of future research.

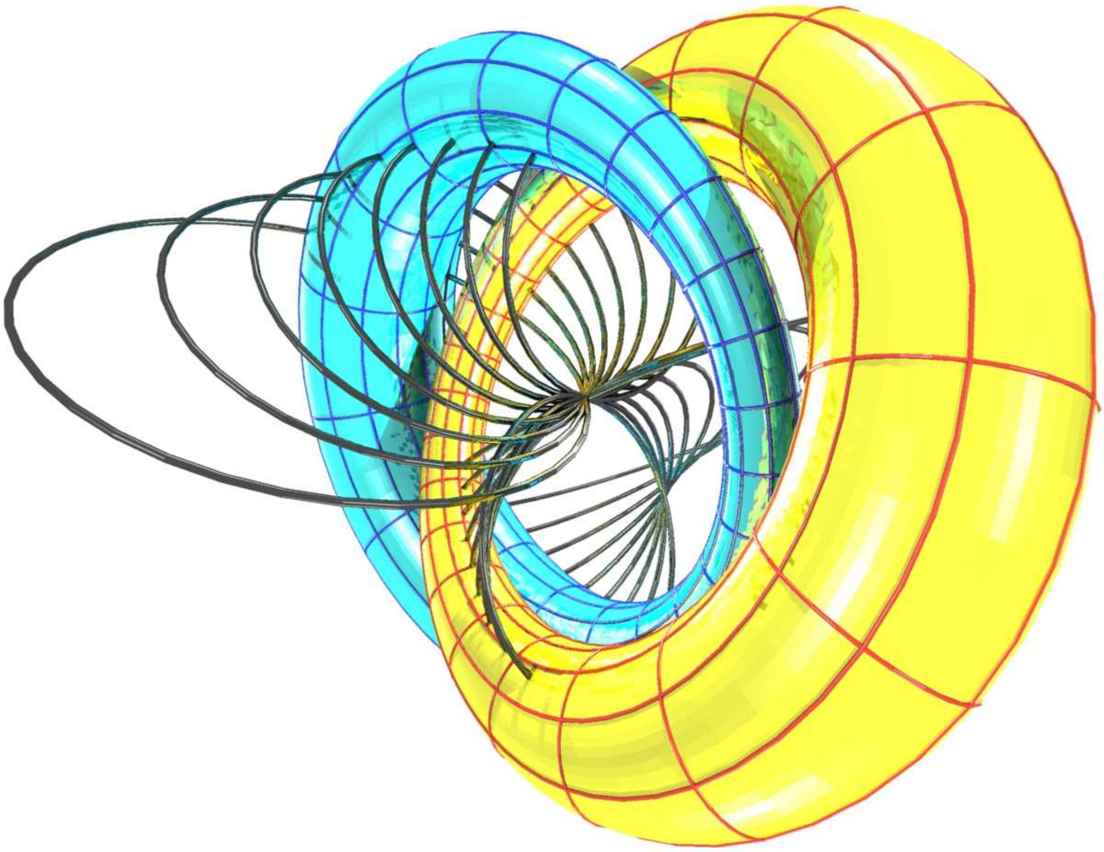

Figure 19: Two ring cyclides with common ortho-circles

We skip the study of the set of centers of ortho-circles of Dupin cyclides as they are not Möbius invariant; the interested reader is referred to [41]. Dupin cyclides with special metric properties such as additional symmetries may lead to interesting results in this context (see [24]).

A further topic of future research could be the study of common normals and common ortho-circles of a pair of Dupin cyclides. One could ask for configurations of two Dupin cyclides with common ortho-circles. Are there in general finitely many? Fig. 19 shows two ring cyclides with infinitely many common ortho-circles. (Only a few of them are shown.) They are the images of two ring tori with skew axes and infinitely many common normals (see [29]).

References

[1] G. Albrecht, W.L.F. Degen: *Construction of Bézier triangles on the symmetric Dupin horn cyclide by means of inversion.* Comput.-Aided Geom. Design **14**, 349–375 (1997).

[2] S. Allen, D. Dutta: *Cyclides in pure blending I.* Comput.-Aided Geom. Design **14**, 51–75 (1997).

[3] S. Allen, D. Dutta: *Cyclides in pure blending II.* Comput.-Aided Geom. Design **14**, 77–102 (1997).

[4] W. Benz: *Geometrie der Algebren.* Springer-Verlag, Berlin-Heidelberg-New York 1973.

[5] W. Boehm: *On cyclides in geometric modeling*. Comput.-Aided Geom. Design **7**, 243–255 (1990).

[6] T.E. Cecil: *Lie Sphere Geometry*. Springer-Verlag, Berlin-Heidelberg-New York 1992.

[7] V. Chandru, D. Dutta, C.M. Hoffmann: *On the geometry of Dupin cyclides*. The Visual Computer **5**, 277–290 (1989).

[8] L.-F. Cheong, K.-O. Ng: *Geometry of Distorted Visual Space and Cremona Transformation*. Int. J. of Comp. Vision **32**/3, 195–212 (1999).

[9] H.L. Choi, S.W. Choi, H.P. Moon: *Mathematical theory of medial axis transform*. Pacific J. Math. **181**, 57–88 (1997).

[10] G. Darboux: *Sur une class remarquable de courbes et de surfaces algébraiques et sur la théorie des imaginaires*. Paris 1873.

[11] W.L.F. Degen: *Generalized cyclides for use in CAGD*. In A. Bowyer (ed.), *The Mathematics of Surfaces IV,* Oxford Univ. Press 1994, pp. 349–363.

[12] W.L.F. Degen: *On the origin of supercyclides*. In R. Cripps (ed.), *The Mathematics of Surfaces VIII,* Information Geometers, Winchester 1998, pp. 297–312.

[13] W.L.F. Degen: *Die zweifach Blutelschen Kegelschnittflächen*. Manusc. Math. **55**, 9–38 (1986).

[14] W.L.F. Degen: *Cyclides*. In G. Farin, J. Hoschek, K.-S. Kim (eds.), *Handbook of Computer Aided Geometric Design,* North Holland – Elsevier 2002, pp. 575–601.

[15] F. Dorau: *Über die Loxodromen der Dupinschen Zykliden*. PhD Thesis, Vienna University of Technology, 1967.

[16] C. Dupin: *Application de Géométrie*. Paris 1822.

[17] K. Fladt, A. Baur: *Analytische Geometrie spezieller Flächen und Raumkurven*. Vieweg, Braunschweig 1975.

[18] A. Gfrerrer: *Dupinsche Zykliden im dreidimensionalen elliptischen Raum*. Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **204**, 59–80 (1995).

[19] O. Giering: *Vorlesungen über höhere Geometrie*. Vieweg, Braunschweig-Wiesbaden 1982.

[20] A. Hirsch: *Extension of the 'Villarceau-Section' to surfaces of revolution with a generating cone*. J. Geometry Graphics **6**/2, 121–132 (2002).

[21] J. Hoschek, D. Lasser: *Fundamentals of Computer Aided Geometric Design*. A. K. Peters, 1993.

[22] H.P. Hudson: *Cremona Transformations in Plane and Space*. 2nd edition, Cambridge Univ. Press, 1996.

[23] M. Kaps: *Teilflächen einer Dupinschen Zyklide in Bézierdarstellung*. PhD Thesis, TU Braunschweig, 1990.

[24] J. Krames: *Ein anallagmatisches Büschel Dupinscher Ringzykliden*. Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **178**, 45–60 (1969).

[25] E. Kruppa: *Analytische und konstruktive Differentialgeometrie*. Springer Verlag, Wien 1957.

[26] R. v. Lilienthal: *Besondere Flächen*. In: Encykl. d. Math. Wiss. III, D5, 269–354.

[27] C. Mäurer: *Rationale Bézier-Kurven und Bézier-Flächenstücke auf Dupinschen Zykliden*. PhD Thesis, Darmstadt University of Technology, 1997.

[28] B. Odehnal: *Circles which intersect a given surface at least two times orthogonally.* Geometry Preprint No. 155, Vienna University of Technology, Feb. 2006.

[29] B. Odehnal: *Common normals of two tori.* J. Geometry Graphics **9**/1, 51–65 (2005).

[30] D. Palman: *Dupinsche Zykliden des einfach isotropen Raumes.* Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **190**, 427–443 (1982).

[31] M. Peternell: *Geometric properties of bisector surfaces.* Graphical Models **62**, 202–236 (2000).

[32] H. Pottmann, M. Peternell: *Applications of Laguerre-Geometry in CAGD.* Comput.-Aided Geom. Design **15**, 165–186 (1998).

[33] H. Pottmann, M. Peternell: *A Laguerre geometric approach to rational offsets.* Comput.-Aided Geom. Design **15**, 223–249 (1998).

[34] H. Pottmann, J. Wallner: *Computational Line Geometry.* Springer Verlag, Heidelberg, Berlin u.a. 2001.

[35] M.J. Pratt: *Cyclides in computer aided geometric design.* Comput.-Aided Geom. Design **7**, 221–242 (1990).

[36] M.J. Pratt: *Cyclides in computer aided geometric design II.* Comput.-Aided Geom. Design **12**, 131–152 (1995).

[37] S.A. Robertson: *Generalized constant width for manifolds.* Mich. Math. J. II, 67–105 (1964).

[38] R. Rothe: *Über die Inversion einer Fläche und die konforme Abbildung zweier Flächen aufeinander mit Erhaltung der Krümmungslinien.* Math. Ann. **72**, 57–77 (1912).

[39] H. Sachs: *Zur Geometrie der Zykliden des einfach isotropen Raumes* $I_3^{(1)}$. Glasnik mat. **24**, 561–582 (1989).

[40] H. Sachs: *Parabolische Schiebzykliden des einfach isotropen Raumes.* Geometriae Dedicata **31**, 301–320 (1989).

[41] M. Schrott: *Über Kreise, die eine Fläche mehrmals orthogonal schneiden.* PhD Thesis, Vienna University of Technology, 1996.

[42] J. Semple, L. Roth: *Introduction to Algebraic Geometry.* Oxford Univ. Press, 1949.

[43] I.R. Shafarevich: *Basic Algebraic Geometry I.* Springer-Verlag, New York 1994.

[44] H. Stachel: *Remarks on A. Hirsch's Paper concerning Villarceau Sections.* J. Geometry Graphics **6**/2, 133–139 (2002).

[45] K. Strubecker: *Über die Kugelschnitte des Torus.* Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **180**, 137–175 (1971).

[46] Y. Villarceau: *Note concernant un troisième système de sections circulaires qu'admet le tore circulaire ordinaire.* Comptes rendus hebdomadaires des Séances de l'Académie de Sciences **27**, 246 (1848).

[47] B. Wegner: *Beiträge zur Differentialgeometrie transnormaler Mannigfaltigkeiten.* PhD Thesis, TU Berlin, 1970.

[48] E.A. Weiss: *Einführung in die Liniengeometrie und Kinematik.* B.G. Teubner, Leipzig 1937.

[49] W. Wunderlich: *Über die Torusloxodromen.* Monatsh. Math. **56**/4, 313–334 (1952).

Received May 30, 2006; final form July 7, 2006