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Inflection and Torsion Line Congruences

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Abstract. Based on E. Study's dual line coordinates, the instantaneous invariants of the relative motion between two dual unit spheres are used for deriving expressions for the velocity and the acceleration of point trajectories (dual curve). The expressions of the curvature and torsion of this curve are related to these invariants. From which the well known inflection, torsion curves and Ball's points of spherical kinematics are calculated in dual space. Then by using E. Study's map two line congruences are introduced and their spatial equivalents are examined in detail. In two ways the invariants of the relative motion are used for deriving a new proof of Disteli's formulae and concise explicit expressions of the inflection line congruence are directly obtained. The obtained explicit equations degenerate into a quadratic form, which can easily give a clear insight into the geometric properties of the inflection line congruence.

Key Words: E. Study's map, axodes, line congruence, Disteli's formulae

MSC: 53A17, 53A04, 53A05

1. Introduction

The study of the curvature theory of rigid body motions in spatial kinematics, which is the study of intrinsic properties of path trajectories, has been a subject of extensive research interest in the past years. The development of curvature theory is important in synthesizing path-generation mechanisms, i.e., a mechanism such that a point or a line in one of its members of that mechanism generates a path having the same intrinsic properties as those of a prescribed path. There exists a vast literature on the subject including several monographs, for example: O. Bottema and B. Roth [10], F.M. Dimentberg [12], J.A. Schaaf [19, 21], G.R. Veldkamp [26], H. Stachel [22, 23], J.A. Schaaf and A.T. Yang [20].

In spatial kinematics, the trajectories of oriented lines embedded in a moving rigid body are generally ruled surfaces. The curvature theory of line trajectories seeks to characterize the shape of the trajectory ruled surface and relates it to the motion of a body carrying the line that generates it. Important contributions to the curvature theory can be found for instance in [18] – [21]. The geometry of ruled surfaces has been widely applied in the design

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and manufacturing of products and many other areas such as motion analysis and simulation of rigid bodies and model-based object recognition systems [19, 20, 27, 28]. The study of line trajectories of general rigid body motions consists of two parts: the orientation and location of the moving line. The orientation of the moving line defines a cone. The intersection of the cone with a unit sphere, centered at the apex of the cone, defines a spherical curve known as the spherical image or indicatrix of the line trajectory. The location of the moving line, with respect to a reference point, is defined by a space curve known as the directrix of the line trajectory.

In this paper, we use the dual vector calculus which was introduced by E. Study [24] as a tool in geometrical investigations. The curvature properties of line trajectories are calculated in terms of the coordinates of lines and the invariants that describe kinematics of the relative motions of two rigid bodies. Analogous to kinematic theory of planar and spherical motions the well known inflection curves, torsion curves and Ball's points are calculated on dual unit sphere. Then by using E. Study's map two line congruences, which are generated by a fixed line in the moving body, are introduced and the spatial equivalent of Ball's points are named Ball's lines. In addition, the invariants of the relative motions are used for deriving a new proof of Disteli's formulae. As well as by using the Disteli's formulae, explicit equations for the inflection line congruence are obtained. The obtained equations degenerate into a quadratic form, which can easily give a clear insight into the geometric properties of the inflection line congruence.

An oriented line in Euclidean 3-space E^3 may be given by a point $\mathbf{x}$ and a unit vector $\mathbf{a}$ on it, i.e., $\|\mathbf{a}\| = 1$. A parametric equation of the line is

$$\mathbf{y} = \mathbf{x} + \mu\mathbf{a}, \ \ \mu \in \mathbb{R}. \tag{1.1}$$

Then we define the moment of the vector $\mathbf{a}$ with respect to a fixed origin point in E^3 as

$$\mathbf{a}^* = \mathbf{y} \times \mathbf{a} = \mathbf{x} \times \mathbf{a}. \tag{1.2}$$

This means that $\mathbf{a}^*$ is the same for all choices of the points on the line, and the pair $(\mathbf{a}, \mathbf{a}^*) \in E^3 \times E^3$ satisfy the following relations:

$$\langle \mathbf{a}, \mathbf{a} \rangle = 1, \ \ \langle \mathbf{a}, \mathbf{a}^* \rangle = 0. \tag{1.3}$$

The six components a_i, a_i^*, $(i = 1, 2, 3)$ of $\mathbf{a}$ and $\mathbf{a}^*$ are called the normed Plücker coordinates of the line.

An important analytical tool in the study of line trajectories are the dual numbers which were first introduced by Clifford [10]. After him Study [24] and Blaschke [8] used them as a tool for their research on differential line geometry. *Dual numbers* are the set of all pairs of real numbers written as

$$A = a + \varepsilon a^*, \quad a, a^* \in \mathbb{R}. \tag{1.4}$$

The symbol ε designates the dual unit and is subject to the rules

$$\varepsilon \neq 0, \quad \varepsilon^2 = 0, \quad \varepsilon.1 = 1.\varepsilon = \varepsilon. \tag{1.5}$$

Given the dual numbers $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$, the rules for their composition can be defined as

$$\left.\begin{array}{r} \text{equality: } A = B \iff a = b, \ a^* = b^*, \\ \text{addition: } A + B = (a + b) + \varepsilon(a^* + b^*), \\ \text{multiplication: } AB = ab + \varepsilon(a^*b + ab^*). \end{array}\right\} \tag{1.6}$$

The set $D = \{A = a + \varepsilon a^*;\ a, a^* \in \mathbb{R}\}$ of dual numbers form a commutative group under addition. The associative law holds for multiplication and dual numbers are distributive. As a result, the set of dual numbers form a ring over $\mathbb{R}$. Division of dual numbers is defined as

$$\frac{A}{B} = \frac{a}{b} + \varepsilon\left(\frac{a^*}{b} - \frac{ab^*}{b^2}\right), \quad b \neq 0. \tag{1.7}$$

A dual number is called *purely dual* if

$$A = \varepsilon a^*. \tag{1.8}$$

The division by a pure dual number is not defined. A dual number $A = a + \varepsilon a^*$ is called *proper* if $a \neq 0$. An example of a dual number is the *dual angle* subtended by two skew lines in the 3-dimensional Euclidean space E^3 and defined as $\Theta = \vartheta + \varepsilon\vartheta^*$ in which ϑ and ϑ^* are, respectively, the projected angle and the minimal distance between the two lines.

The set of dual numbers can be extended to vector spaces [15]. The set

$$\left.\begin{aligned} D^3 = D\times D\times D \ &= \{\mathbf{A} :\ \mathbf{A} = (A_1,\, A_2,\, A_3);\ A_i = a_i + \varepsilon a_i^*,\ i = 1,\, 2,\, 3\} \\ &= \{\mathbf{A} = \mathbf{a} + \varepsilon\mathbf{a}^* :\ \mathbf{a},\, \mathbf{a}^* \in E^3\} \end{aligned}\right\} \tag{1.9}$$

is a module, called *dual space*. The elements of D^3 are called *dual vectors*. It is clear that any dual vector $\mathbf{A}$ in D^3-space, consists of any two real vectors $\mathbf{a}$, $\mathbf{a}^* \in E^3$ which are expressed in the natural right-handed orthonormal frame $\{\mathbf{o};\ \mathbf{i},\ \mathbf{j},\ \mathbf{k}\}$ in the 3-dimensional Euclidean space E^3 such that

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}, \quad \mathbf{a}^* = a_1^*\,\mathbf{i} + a_2^*\,\mathbf{j} + a_3^*\,\mathbf{k}, \tag{1.10}$$

and

$$\mathbf{i} = (1,\,0,\,0), \quad \mathbf{j} = (0,\,1,\,0), \quad \mathbf{k} = (0,\,0,\,1). \tag{1.11}$$

The standard operations for vectors in the 3-dimensional Euclidean space E^3 can also be defined for vectors in D^3. For given two dual vectors, $\mathbf{A} = \mathbf{a} + \varepsilon\mathbf{a}^*$ and $\mathbf{B} = \mathbf{b} + \varepsilon\mathbf{b}^*$ we have

$$\left.\begin{aligned} \text{equality: } \ \mathbf{A} = \mathbf{B} \iff \mathbf{a} = \mathbf{b} \text{ and } \mathbf{a}^* = \mathbf{b}^*, \\ \text{scalar product: } \langle\mathbf{A},\mathbf{B}\rangle = \langle\mathbf{a},\mathbf{b}\rangle + \varepsilon\left[\langle\mathbf{a}^*,\mathbf{b}\rangle + \langle\mathbf{a},\mathbf{b}^*\rangle\right], \\ \text{vector product: } \mathbf{A}\times\mathbf{B} = (\mathbf{a}\times\mathbf{b}) + \varepsilon\left[(\mathbf{a}^*\times\mathbf{b}) + (\mathbf{a}\times\mathbf{b}^*)\right]. \end{aligned}\right\} \tag{1.12}$$

If $\mathbf{a} \neq \mathbf{0}$, the *norm* $\|\mathbf{A}\|$ of $\mathbf{A}$ is defined by

$$\|\mathbf{A}\| = \|\mathbf{a}\| + \varepsilon\frac{\langle\mathbf{a},\mathbf{a}^*\rangle}{\|\mathbf{a}\|}. \tag{1.13}$$

A dual vector $\mathbf{A}$ with norm $\|\mathbf{A}\| = 1$ is called a *dual unit vector*. It is clear that

$$\|\mathbf{A}\| = 1 \iff \langle\mathbf{a},\mathbf{a}\rangle = 1, \quad \langle\mathbf{a},\mathbf{a}^*\rangle = 0. \tag{1.14}$$

It flows that relations (1.3) and (1.14) are corresponding. Via this we have the following map (*E. Study's map*): *The set of all oriented lines in Euclidean space E^3 is in one-to-one correspondence with the set of points of the dual unit sphere in the D^3-space* [13, 17].

The set

$$K = \{\mathbf{A} :\ \|\mathbf{A}\| = 1,\ \mathbf{A} \in D^3\} \tag{1.15}$$

is called the *dual unit sphere* in the D^3-space. A ruled surface is then a spherical curve on this dual unit sphere.

E. Study's map allows a complete generalization of mathematical expression from spherical point geometry to spatial line geometry by means of dual numbers extension, i.e., by replacing all ordinary quantities by the corresponding dual numbers quantities. This means that all rules of vector algebra for the kinematics of a rigid body with a fixed point (spherical kinematics) also hold for dual algebra of a free rigid body (spatial kinematics). As a result, a general rigid body motion can be described by only three dual equations rather than six real ones.

2. One-parameter dual spherical motion

Consider two dual unit spheres K_m and K_f. Let $\mathbf{O}$ be the common center and two orthonormal dual coordinate frames $\{\mathbf{O};\, \mathbf{L}_1,\, \mathbf{L}_2,\, \mathbf{L}_3\}$ and $\{\mathbf{O};\, \mathbf{F}_1,\, \mathbf{F}_2,\, \mathbf{F}_3\}$ be rigidly linked to the dual unit spheres K_m and K_f, respectively. We suppose that $\{\mathbf{O};\, \mathbf{F}_1,\, \mathbf{F}_2,\, \mathbf{F}_3\}$ is fixed, whereas the elements of the set $\{\mathbf{O};\, \mathbf{L}_1,\, \mathbf{L}_2,\, \mathbf{L}_3\}$ are functions of a real parameter t (the time). Then we say that the dual unit sphere K_m *moves* with respect to the fixed dual unit sphere K_f. The interpretation of this is as follows: the dual unit sphere K_m rigidly connected with $\{\mathbf{O};\, \mathbf{L}_1,\, \mathbf{L}_2,\, \mathbf{L}_3\}$ moves over the dual unit sphere K_f rigidly connected with $\{\mathbf{O};\, \mathbf{F}_1,\, \mathbf{F}_2,\, \mathbf{F}_3\}$. This motion is called a *one-parameter dual spherical motion* and will be denoted by K_m/K_f. If the dual unit spheres K_m and K_f correspond to the line space H_m and H_f, respectively, then K_m/K_f corresponds to the *one-parameter spatial motion* H_m/H_f. Then H_m is the moving space with respect to the fixed space H_f.

Theorem 1 *The Euclidean motions in* E^3 *are represented in* D^3 *(the dual space) by dual orthogonal* 3×3 *matrices* $A = (A_{ij})$, *where* $AA^t = I$; A_{ij} *are dual numbers, and* I *is the* 3×3 *unit matrix.*

According to Theorem 1 the 3×3 dual matrix $A(t)$ of the motion K_m/K_f represents the one-parameter spatial motion H_m/H_f with the same parameter $t \in \mathbb{R}$.

During the motion K_m/K_f the differential velocity vector of a fixed dual point $\mathbf{X}$ on K_m, analogous to the real spherical motion [10, 17] is

$$\frac{d\mathbf{X}}{dt} = \boldsymbol{\Omega} \times \mathbf{X}, \tag{2.1}$$

where $\boldsymbol{\Omega} = \boldsymbol{\omega} + \varepsilon\, \boldsymbol{\omega}^*$ is called the instantaneous *Pfaffian* vector of the motion K_m/K_f. The *Pfaffian* dual vector $\boldsymbol{\Omega}$ at the instant t of the one-parameter dual spherical motion K_m/K_f is analogues to the *Darboux* vector in the differential geometry of space curves. In this case ω and ω^* corresponding to the instantaneous rotational differential velocity vector and the instantaneous translational differential velocity vector of the corresponding spatial motion H_m/H_f, respectively. The direction of $\boldsymbol{\Omega}$ passes through the dual poles (the instantaneous dual spherical centers of rotation) $\mathbf{R}_m$ on K_m and $\mathbf{R}_f$ on K_f. Then the dual unit vector

$$\mathbf{R} := \mathbf{R}_m = \mathbf{R}_f = \frac{\boldsymbol{\Omega}}{\|\boldsymbol{\Omega}\|}, \tag{2.2}$$

with the same orientation as $\boldsymbol{\Omega}$ is the *instantaneous screw axis* (ISA for short) of the motion K_m/K_f. The dual number $\Omega = \omega + \varepsilon\omega^* = \|\boldsymbol{\Omega}\|$ is called *dual angular speed* of the dual spherical motion K_m/K_f.

During the motion K_m/K_f, the dual unit vector $\mathbf{R}$ is a function of t. It represents the locus of the ISA on K_f and K_m. This locus is a dual curve $\mathbf{R}_m(t)$ on K_m and is called the *moving polode*. This curve corresponds to a ruled surface in H_m which is called *moving axode*. The moving axode is the locus of the ISA as viewed from the moving space H_m. The ISA on K_f is also a dual curve $\mathbf{R}_f(t)$ and is called the *fixed polode*. This polode like-wise corresponds to a ruled surface in H_f and is called the *fixed axode*. This fixed axode is made up of those lines in the fixed space H_f which at some instant coincide with a line in the moving space having zero dual velocity. Moreover, the *moving polode* contacts the fixed polode along the ISA in the first order at any instant t [1, 22].

Let K_r be the dual unit sphere generated by the right-handed dual system $\{\mathbf{O};\, \mathbf{R} = \mathbf{R}_1,\, \mathbf{R}_2,\, \mathbf{R}_3\}$. By the first order instantaneous invariants of the motion K_m/K_f we can define the orthonormal dual moving frame of K_r as follows:

$$\mathbf{R}_1(t) = \mathbf{r}_1(t) + \varepsilon \mathbf{r}_1^*(t), \tag{2.3}$$

is the ISA of the motion, and

$$\mathbf{R}_2 = \left(\frac{d\mathbf{R}_1}{dt}\right) \left\| \frac{d\mathbf{R}_1}{dt} \right\|^{-1} \tag{2.4}$$

is the common normal of two separated screw axes. A third dual unit vector is defined as

$$\mathbf{R}_3 = \mathbf{R}_1 \times \mathbf{R}_2. \tag{2.5}$$

This frame is called the *Blaschke frame*, and the corresponding lines intersect at the striction point of the axodes. The *central point* is the common point of the moving and fixed axodes (ruled surfaces) formed by the ISA. $\mathbf{R}_3$ and $\mathbf{R}_2$ are known as the *central tangent* and the *central normal* of the axodes, respectively.

Let $\mathbf{X}$ be a straight line with dual representation

$$\mathbf{X} = \mathbf{X}^t\, \mathbf{R}, \quad \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{pmatrix}. \tag{2.6}$$

Based on the results of [1], the dual velocity vector of $\mathbf{X}$ fixed in K_m is given by

$$\frac{d\mathbf{X}}{dt} = \mathbf{\Omega} \times \mathbf{X}, \quad \mathbf{\Omega} = \Omega\, \mathbf{R}_1, \quad \Omega = \omega + \varepsilon\omega^* = Q_f - Q_m. \tag{2.7}$$

The real part ω and the dual part ω^* correspond, respectively, to the rotation motions and the translation motions of the corresponding one-parameter spatial motion H_m/H_f. Hence, the one-parameter spatial motion H_m/H_f can be represented by the rotation ω about and translation ω^* along the ISA. The ratio ω^*/ω is known as the *pitch* of the motion. From equations (5.15) of [1] and (2.7), it follows that the acceleration of $\mathbf{X}$ is:

$$\frac{d^2\mathbf{X}}{dt^2} = X_3 P\Omega\, \mathbf{R}_1 - (X_2\Omega^2 + X_3\Omega')\mathbf{R}_2 + (X_2\Omega' - X_1 P\Omega - \Omega^2 X_3)\mathbf{R}_3, \tag{2.8}$$

where P, Q_f, and Q_m are the invariants of the dual spherical motion K_m/K_f (see [1]). Here the derivative with respect to t is denoted by a dash.

3. Lines with special kinematics

According to E. Study's map four independent parameters define an oriented line (line complex). So it is possible to intersect any two of line complexes and obtain a finite number of lines (line congruence) with associated properties. We are interested to research the geometrical properties of the line $\mathbf{X}$ which is adjoint with the moving axode. For this purpose, the dual curvature, the dual torsion, and the dual spherical curvature of $\mathbf{X} = \mathbf{X}(t)$ are expressed in terms of the invariants of the axodes which characterize the rigid body motion. Thus, we can define an orthonormal moving frame along $\mathbf{X} = \mathbf{X}(t)$ as follows:

$$\mathbf{E}_1 = \mathbf{X}(t), \quad \mathbf{E}_2 = \mathbf{X}'\|\mathbf{X}'\|^{-1}, \quad \mathbf{E}_3 = \mathbf{E}_1 \times \mathbf{E}_2. \tag{3.1}$$

This frame like-wise is called the Blaschke frame, and the corresponding lines intersect at the striction point on the ruling of $\mathbf{X} = \mathbf{X}(t)$. Then, combining equations (2.7), and (3.1), we have

$$\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 & X_3 \\ 0 & -\frac{X_3}{\sqrt{1-X_1^2}} & \frac{X_2}{\sqrt{1-X_1^2}} \\ \sqrt{1-X_1^2} & -\frac{X_1X_2}{\sqrt{1-X_1^2}} & -\frac{X_1X_3}{\sqrt{1-X_1^2}} \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{pmatrix}. \tag{3.2}$$

It is seen from (3.2) that the dual unit vector $\mathbf{E}_2$ intersects the ISA orthogonally. Hence the central (striction) point of the adjoint ruled surface of the moving axode lies on the common normal line between the generating line and the ISA of the motion. By construction, the *Blaschke formula* is

$$\begin{pmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \end{pmatrix} = \begin{pmatrix} 0 & P_x & 0 \\ -P_x & 0 & Q_x \\ 0 & -Q_x & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix}, \tag{3.3}$$

where

$$\left. \begin{aligned} P_x = p_x + \varepsilon p_x^* = \|\mathbf{X}'\| = \Omega\sqrt{1-X_1^2}, \\ Q_x = q_x + \varepsilon q_x^* = \frac{\det[\mathbf{X},\, \mathbf{X}',\, \mathbf{X}'']}{\|\mathbf{X}'\|^2} = \Omega X_1 + \frac{PX_3}{1-X_1^2}, \end{aligned} \right\} \tag{3.4}$$

are called the *Blaschke invariants* of the dual curve $\mathbf{X}(t)$. The *dual arc length* of the ruled surface $\mathbf{X}(t)$ is

$$dS = ds + ds^* = P_x dt. \tag{3.5}$$

In view of equation (3.5) we reparametrize $\mathbf{X}(t)$ to obtain $\mathbf{X} = \mathbf{X}(t(S))$, such that

$$\mathbf{E}_2 = \frac{d\mathbf{E}_1}{dS}. \tag{3.6}$$

Thus, as in the case of real spherical curves, the dual arc length parameter normalizes the representation of the ruled surface $\mathbf{X} = \mathbf{X}(t)$ such that its dual tangent vector $\mathbf{E}_2$ has unit magnitude. Hence, we can rewrite the Blaschke formula of $\mathbf{X} = \mathbf{X}(t)$ as

$$\frac{d}{dS}\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \Delta_x \\ 0 & -\Delta_x & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix}, \tag{3.7}$$

where Δ_x is the *dual spherical curvature* of $\mathbf{X} = \mathbf{X}(t)$. It is clear that the axis of curvature of the ruled surface $\mathbf{X} = \mathbf{X}(t)$ is

$$\mathbf{U} = \cos\Theta_x \mathbf{E}_1 + \sin\Theta_x \mathbf{E}_3, \tag{3.8}$$

where $\Theta_x = \vartheta_x + \varepsilon\vartheta_x^*$ is a dual angle between the lines $\mathbf{X}$ and $\mathbf{U}$. Then the following relation exists:

$$\cot\Theta_x = \frac{Q_x}{P_x}. \tag{3.9}$$

As in the case of a real spherical curve we may write for the dual curve $\mathbf{X} = \mathbf{X}(t)$ the following relations:

$$K_x = \frac{1}{\sin\Theta_x}, \quad T_x = -\frac{d\Theta_x}{dS}, \quad K_x = \sqrt{1+\Delta_x^2}\,, \tag{3.10}$$

where K_x is the dual curvature, and T_x is the *dual torsion* of the dual curve $\mathbf{X} = \mathbf{X}(t)$.

3.1. Inflection line congruence

Generally, as the motion proceeds, the lines in the moving space H_m trace ruled surfaces in the fixed space H_f. As a result of the instantaneous motion, lines adjoint with the ISA trace trajectories which are ruled surfaces. Under certain conditions the trajectories are referred to as the *inflection lines* at an instant of the motion. Now, we show that the spatial equivalent of the inflection circle of planar kinematics is a line congruence which we consider as *inflection line congruence*. Therefore, we deduce that the set of lines with zero dual geodesic curvature is the spatial equivalent of the inflection circle of planar kinematics. Then, from equations (3.4), and (3.9), we have

$$\Delta_x = \cot\Theta_x = \frac{\Omega X_1(1-X_1^2) + PX_3}{\Omega(1-X_1^2)^{\frac{3}{2}}}\,. \tag{3.11}$$

Equation (3.11) can be easily rewritten in the following form:

$$\Delta_x = \frac{X_1}{\sqrt{X_2^2+X_3^2}} + \frac{1}{\Delta}\frac{X_3}{(X_2^2+X_3^2)^{\frac{3}{2}}}\,; \qquad \Delta = \frac{\Omega}{P} \iff \delta + \varepsilon\delta^* = \frac{\omega+\varepsilon\omega^*}{p+\varepsilon p^*}\,. \tag{3.12}$$

From eqs. (3.10) and (3.12) we have

$$\Delta_x = 0 \iff K_x = 1. \tag{3.13}$$

This equation is considered as the case for the inflection path. Since the osculating circle of this path is a great circle on the sphere K_f and intersects the path in three consecutive points. It can be shown that the axis of curvature $\mathbf{U}$ can only be as an inflection axis when $T_x = 0$, which does not hold in the general case. It follows from equation (3.11) that for all points with $\Delta_x = 0$ their trajectories lie on a dual great circle up to third order. Also, from equation (3.11), we can see that

$$\Delta_x = 0 \iff \cot\Theta_x = 0 \iff \vartheta_x = \frac{\pi}{2}, \ \ \vartheta_x^* = 0. \tag{3.14}$$

In this instant, the lines $\mathbf{X}$, $\mathbf{E}_2$ and $\mathbf{U}$ constitute the Blaschke frame and are intersected at the striction point of the ruled surface $\mathbf{X} = \mathbf{X}(t)$. Furthermore, we notice that the lines $\mathbf{U}$ and $\mathbf{E}_3$ are coincident, which means that the line $\mathbf{X}$ moves about $\mathbf{U}$ with constant pitch equal to the distribution parameter of the ruled surface $\mathbf{E}_2 = \mathbf{E}_2(t)$, and the ruled surface $\mathbf{X} = \mathbf{X}(t)$ is a right helicoid [4]. According to equation (3.12), points with $\Delta_x = 0$ obey the equation

$$C_G: \quad \Delta X_1(X_2^2+X_3^2) + X_3 = 0, \tag{3.15}$$

which is the dual spherical equivalent of the inflection circle of planar kinematics. As we see from equation (3.15) this spherical equivalent of the inflection circle is a dual spherical curve of third degree. The real part of equation (3.15) identifies the inflection cone for the spherical part of the motion H_m/H_f and is given by

$$C_g: \quad \delta x_1(x_2^2+x_3^2)+x_3=0. \tag{3.16}$$

The intersection of the inflection cone with a real unit sphere centered at the apex of the cone defines a spherical curve. There is a plane for each line, associated with each direction of a line of the inflection cone, defined by the following dual part of equation (3.15):

$$\pi_g: \quad (x_2^2+x_3^2)(\delta x_1^*+\delta^* x_1)+2x_1\delta(x_3x_3^*+x_2x_2^*)+x_3^*=0, \tag{3.17}$$

where x_1, x_2, and x_3 are the direction cosines of the line $\mathbf{X}$ and x_1^*, x_2^*, and x_3^* are given by

$$x_1^*=(p_2x_3-p_3x_2), \quad x_2^*=(p_3x_1-p_1x_3), \quad x_3^*=(p_1x_2-p_2x_1), \tag{3.18}$$

where p_1, p_2, and p_3 are the components of the position vector from the origin in H_f to a line belong to the associated plane of lines. Since the equation (3.15) is of third degree, this line congruence consists of all common lines of two cubic line complexes (eqs. (3.16) and (3.17)). Hence the Plücker coordinates of the lines $\mathbf{X}\in C_G$ satisfy the equations (3.16), (3.17) and the relations (1.3), and in general represent a ruled surface in the fixed space H_f.

3.2. Torsion line congruence

On the dual unit sphere the locus of points with $T_x = 0$ is the spatial equivalent of the cubic of stationary curvature of spherical kinematics [10]. If a dot denotes the derivation with respect to the dual arc length of $\mathbf{X}=\mathbf{X}(t)$, then from equations (3.9) and (3.10) we get

$$T_x=\frac{-\dot{\Delta}_x}{1+\Delta_x^2} \tag{3.19}$$

in view of equation (3.10). It is clear that for all points with $T_x = 0$ the trajectories lie on a dual great circle up to third order. Furthermore, one can easily find

$$T_x=0 \iff \Delta_x=C,\ C=c+\varepsilon c^*\in D. \tag{3.20}$$

Using equation (3.12) we can obtain the trajectories of all points with $T_x = 0$ as follows:

$$C_T:\ \Delta\big[X_1(1-X_1^2)-C(1-X_1^2)^{\frac{3}{2}}\big]+X_3=0. \tag{3.21}$$

The equation (3.21) is a line congruence of degree six. The spatial equivalent of all lines which satisfy eq. (3.21) can be called *torsion line congruence*. The real part of equation (3.21) identifies the torsion cone for the spherical part of the motion H_m/H_f and is given by

$$C_t:\ \delta\big[x_1(1-x_1^2)-c(1-x_1^2)^{\frac{3}{2}}\big]+x_3=0. \tag{3.22}$$

The intersection of the torsion cone with a real unit sphere centered at the apex of the cone defines a spherical curve. As in above discussion, there are associated planes of lines with each direction of the torsion cone, defined by

$$\pi_t:\ \left(\delta-2x_1^2+4c\delta x_1\sqrt{1-x_1^2}\right)x_1^*+x_3^*-c(\delta^*+c^*)(1-x_1^2)^{\frac{3}{2}}=0. \tag{3.23}$$

3.3. Ball's lines

At every instant of the motion K_m/K_f there are some points having $\Delta_x = 0$ and $T_x = 0$. These points yield the dual spherical analogs of the Ball's points of planar and spherical motion. The common lines $C_G \cap C_T$ of the moving space H_m, with direction lying at $C_g \cap C_t$ and which are located at $\pi_g \cap \pi_t$, will satisfy both equations (3.16) and (3.23). These lines can be called the *Ball's lines* (ruled surface) and have trajectories on great circle up to third order.

4. Disteli formulae of spatial kinematics

In 1914 M. Disteli [9] succeeded in determining a curvature axis for the generating line of a ruled surface and extended the famous planar construction of Euler-Savary to spatial kinematics. The Disteli formula may be obtained directly by computing the dual spherical curvature of $\mathbf{X}(t)$ in terms of spherical coordinates. Since $\mathbf{X}$ is a dual unit vector, we can write out the components of $\mathbf{X}$ in the following form:

$$\mathbf{X} = \cos\Theta\,\mathbf{R}_1 + \sin\Theta\cos\Phi\,\mathbf{R}_2 + \sin\Theta\sin\Phi\,\mathbf{R}_3. \tag{4.1}$$

This choice of coordinates is such that $\Phi = \varphi + \varepsilon\varphi^*$ is the dual angle between the central normal of $\mathbf{X}(t)$ and $\mathbf{R}_2$ measured about the ISA — this means a screw motion of angle φ about the ISA and distance φ^* along it carries $\mathbf{R}_2$ to the central normal $\mathbf{E}_2$ of $\mathbf{X}(t)$. The dual angle $\Theta = \theta + \theta^*$ defines the position of $\mathbf{X}$ relative to the ISA of the motion H_m/H_f.

A similar set of coordinates may be used to identify the axis of curvature $\mathbf{U}$. Since the central normal $\mathbf{E}_2$ is also normal to $\mathbf{U}$, it is identified by the same dual angle Φ about the ISA of the motion H_m/H_f (see Fig. 1 below). Denoting its dual angle with the ISA by $\Theta_c = \theta_c + \varepsilon\theta_c^*$ we have

$$\mathbf{U} = \cos\Theta_c\,\mathbf{R}_1 + \sin\Theta_c\cos\Phi\,\mathbf{R}_2 + \sin\Theta_c\sin\Phi\,\mathbf{R}_3. \tag{4.2}$$

The dual spherical radius of curvature Θ_x can be given by

$$\Theta_x = \Theta - \Theta_c \iff \theta_x = \theta - \theta_c,\ \theta_x^* = \theta^* - \theta_c^*. \tag{4.3}$$

Then, we have the identity

$$\Delta_x = \cot\Theta_x = \cot(\Theta - \Theta_c). \tag{4.4}$$

Substituting equations (4.4) and (4.1) into (3.12), we obtain the following relation:

$$\Delta\left(\cot(\Theta - \Theta_c) - \cot\Theta\right) = \frac{\sin\Phi}{\sin^2\Theta}\,, \tag{4.5}$$

which can be easily reduced to

$$\cot\Theta_c - \cot\Theta = \frac{\Delta}{\sin\Phi}\,. \tag{4.6}$$

Equation (4.6) is the *dual spherical Euler-Savary equation* (compare with [22], Theorem 6). By separating the real and the dual parts, respectively, we get

$$(\cot\theta_c - \cot\theta)\sin\varphi = \delta, \tag{4.7}$$

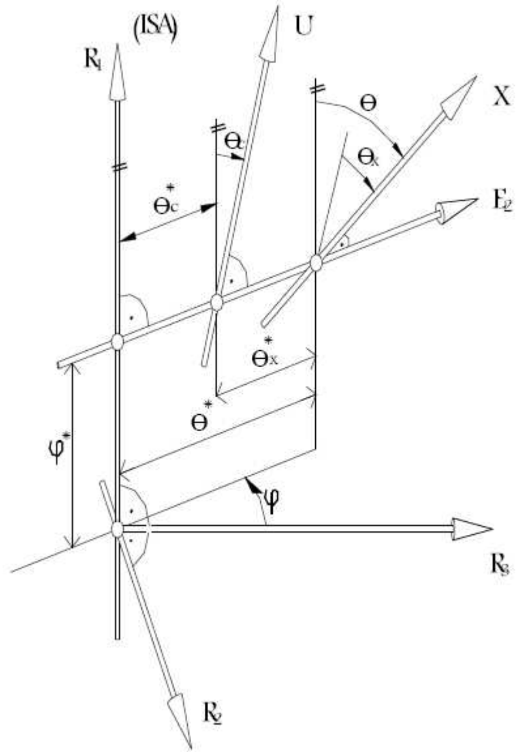

Figure 1: The moved line X and its Disteli-axis U

and

$$\varphi^*(\cot\theta_c - \cot\theta)\cos\varphi - \left(\frac{\theta_c^*}{\sin^2\theta_c} - \frac{\theta^*}{\sin^2\theta}\right)\sin\varphi = \delta^*. \tag{4.8}$$

The spherical Euler-Savary equation (4.7) together with (4.8) are called the *Disteli formulae* of spatial kinematics (compare [10]).

On the other hand, we can derive a second dual version of the Euler-Savary equation as follows: From equations (3.4), (3.5), and (4.1) one finds easily

$$dS = \Omega\sin\Theta dt. \tag{4.9}$$

The combination of eqs. (3.2) and (4.1) gives

$$\mathbf{R}_1 = \cos\Theta\,\mathbf{E}_1 + \sin\Theta\,\mathbf{E}_3. \tag{4.10}$$

Then, by equations (3.7), the first derivative of $\mathbf{R}_1$ with respect to dS is

$$\frac{d\mathbf{R}_1}{dS} = (-\sin\Theta\,\mathbf{E}_1 + \cos\Theta\,\mathbf{E}_3)\frac{d\Theta}{dS} + (\cos\Theta - \Delta_x\sin\Theta)\mathbf{E}_2. \tag{4.11}$$

On the other hand, a simple calculation shows that

$$\frac{d\mathbf{R}_1}{dS} = \frac{dt}{dS}\,\frac{d\mathbf{R}_1}{dt}\,. \tag{4.12}$$

It follows that

$$\frac{d\mathbf{R}_1}{dS} = \frac{P}{\Omega\sin\Theta}\,\mathbf{R}_2. \tag{4.13}$$

The validity of equation (4.13) is proved in [1]. The combination of eqs. (3.2) and (4.1) leads to

$$\frac{d\mathbf{R}_1}{dS} = \frac{P}{\Omega \sin\Theta}(\sin\Theta\cos\Phi\,\mathbf{E}_1 - \sin\Phi\,\mathbf{E}_2 - \cos\Theta\sin\Phi\,\mathbf{E}_3). \tag{4.14}$$

We have found, by equating the coefficients of $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ in equations (4.11) and (4.14), that

$$\frac{d\Theta}{dS}\sin\Theta + \frac{P}{\Omega}\cos\Phi = 0 \tag{4.15}$$

and

$$\cos\Theta - \Delta_x \sin\Theta = -\frac{P\sin\Phi}{\Omega\sin\Theta}\,. \tag{4.16}$$

For the dual spherical curvature Θ_x we have (4.3). Substituting this into the left hand side of equation (4.16) one finds

$$\cos\Theta - \Delta_x \sin\Theta = \frac{1}{\sin\Theta}\,\frac{1}{\cot\Theta - \cot\Theta_c}\,. \tag{4.17}$$

The combination of the eqs. (4.17) with (4.16) yields (4.6). Moreover, by substituting

$$\frac{Q_f}{P} = \cot\Theta_f \quad \text{and} \quad \frac{Q_m}{P} = \cot\Theta_m$$

into equation (4.15) one obtains

$$\frac{(\cot\Theta_f - \cot\Theta_m)}{\cos\Phi} = \Omega\big[\dot{\Theta}\big]^{-1}. \tag{4.18}$$

This is the *second Euler-Savary equation* for dual spherical motion. Here, $\cot\Theta_f$ and $\cot\Theta_m$ are called the *dual spherical curvatures* of the fixed and moving axodes, respectively [1].

4.1. Identification of lines of the inflection line congruence

In order to identify the lines of the inflection line congruence, from eqs. (3.14) and (4.3) we have

$$\theta - \theta_c = \frac{\pi}{2}, \qquad \theta^* = \theta_c^*, \tag{4.19}$$

Substituting the first equation of (4.19) into eq. (4.7), we obtain

$$\sin\varphi\tan^2\theta + \delta\tan\theta + \sin\varphi = 0, \tag{4.20}$$

which is a quadratic equation for the parameter $\tan\theta$. Therefore, the explicit equation of θ is expressed as

$$\tan\theta = \frac{-\delta \pm \sqrt{\Lambda}}{2\sin\varphi}\,, \tag{4.21}$$

where $\Lambda = \delta^2 - 4\sin^2\varphi$. Then, according to equation (4.7), the explicit equation of θ_c is expressed as

$$\tan\theta_c = \frac{\sin\theta\tan\varphi}{\delta\tan\theta + \sin\theta}\,, \tag{4.22}$$

where $\tan\theta$ is obtained from equation (4.21). Eqs. (4.21) and (4.22) not only represent the concise explicit expressions of the inflection cone for the real spherical part of the motion, but also provide a convenient tool for investigating the special cases of the inflection line congruence. According to the value of Λ in (4.21), the geometric properties of the explicit equation are discussed as follows:

1. If $\Lambda > 0$, then the two values of θ can always be determined and the two lines L^+, and L^- are defined. Therefore, there exists four solutions for φ^* in eq. (4.8).
2. If $\Lambda = 0$, then from eqs. (4.8) and (4.21) we have

$$\theta = \frac{3\pi}{4}, \quad \varphi^* = -\frac{\delta^*}{2\cos\varphi}, \tag{4.23}$$

and there is a rang of values of φ for which θ^* is undefined.

By substituting (4.21) into (4.8), with attention to equation (4.19), we get

$$\varphi^*\delta\cos\varphi + \delta^*\sin\varphi \pm \frac{\theta^*}{2}\sqrt{\Lambda} = 0. \tag{4.24}$$

Eq. (4.24) is linear in the position coordinates φ^* and θ^* of the line $\mathbf{X}$. Therefore, the lines in a given fixed direction in the moving space H_m satisfying equation (4.24) lie on a plane. Therefore, the inflection line congruence consists of a set of planes π_g parallel to $\mathbf{X}$, each of which is associated with a direction of the cubic spherical inflection cone defined in equation (3.16). The angle φ identifies the direction of the central normal $\mathbf{E}_2$, thus equation (4.21) defines two lines L^+, and L^- in the plane spanned by $\mathbf{E}_2$ and the ISA. Note that the lines L^+ and L^- are associated with the roots for a given θ in eq. (4.21). All the lines L of the moving space H_m parallel to $\mathbf{X}$ and also in the plane spanned by $\mathbf{E}_2$ and the ISA satisfying equation (4.19). The cubic spherical inflection cone and the associated plane of lines defines the inflection line congruence.

If the parameters φ, δ, and δ^* in (4.24) are given and θ^* (the distance along the central normal $\mathbf{E}_2$ from the ISA) is chosen as the independent parameter, then equation (4.24) takes the form

$$\varphi^* = \mp(\frac{\sqrt{\Lambda}}{2\delta\cos\varphi})\theta^* - \frac{\delta^*}{\delta}\tan\varphi, \tag{4.25}$$

It follows that the two lines L^+ and L^- intersect the ISA at the distance $-\frac{\delta^*}{\delta}\tan\varphi$. For the direction $\varphi = 0$ these lines passing through the origin achieve their minimal slope $\mp\frac{1}{2}$. For $\varphi = \frac{\pi}{2}$, these lines are parallel and on either side of the ISA at the distance $\mp\delta^*/\sqrt{\delta^2 - 4}$.

These results provide a simple geometrical means for the geometrical properties of lines of the inflection line congruence (Fig. 2). Thus, we have arrived into the well-known theorem

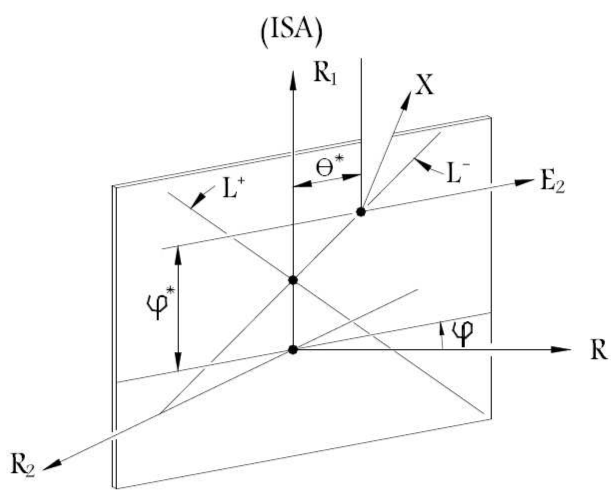

Figure 2: The lines L^+ and L^- for a given value of φ

[10]: Any line in the space is intersected orthogonally by three lines of the inflection line congruence.

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