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A Unified Proof of Ceva and Menelaus' Theorems Using Projective Geometry

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Abstract. We prove that the well known Ceva and Menelaus' theorems are both particular cases of a single theorem of projective geometry.

Key Words: Ceva's theorem, Menelaus' theorem, projective geometry

MSC: 51M04, 51N15

1. Introduction, background and notations

Ceva and Menelaus theorems are well known. However, these theorems characterize a projective property (concurrence in Ceva's theorem and collinearity in Menelaus' theorem) in terms of an affine property. The purpose of this paper is to overcome this. To be more precise, we characterize the concurrence of the cevians by using the cross ratio (a projective quantity). The dual of this latter characterization permits to state the projective version of Menelaus' theorem. Before establishing the main results, we review some simple definitions and facts about projective geometry.

A *projective point* is a line in $\mathbb{R}^3$ that passes through the origin. The *projective plane* $\mathbb{P}^2$ is the set of all projective points. If P is a projective point then there exists $\mathbf{v} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that P is the line in $\mathbb{R}^3$ that passes through $\mathbf{0}$ and $\mathbf{v}$. Thus, we can define $\pi : \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{P}^2$ as follows: $\pi(\mathbf{v})$ is the projective point through $\mathbf{0}$ and $\mathbf{v}$.

A *projective line* in $\mathbb{P}^2$ is a plane in $\mathbb{R}^3$ that passes through the origin. Given two projective points P and Q there exists a unique projective line r such that P and Q lie on r, such projective line shall be denoted by $\mathcal{L}(P,Q)$. It is easy to see that the projective point $\pi(\mathbf{u})$ lies on $\mathcal{L}(\pi(\mathbf{v}),\pi(\mathbf{w}))$ if and only if $\mathbf{u}$ is a linear combination of $\mathbf{v},\mathbf{w}$, thus $\pi(\mathbf{u})$, $\pi(\mathbf{v})$, and $\pi(\mathbf{w})$ are collinear if and only if $\det(\mathbf{u},\mathbf{v},\mathbf{w}) = 0$. Also, the projective lines with equations $\mathbf{x}^{\mathrm{T}}\mathbf{u} = 0$, $\mathbf{x}^{\mathrm{T}}\mathbf{v} = 0$, and $\mathbf{x}^{\mathrm{T}}\mathbf{w} = 0$ are concurrent if and only if $\det(\mathbf{u},\mathbf{v},\mathbf{w}) = 0$. A triangle is formed by three non collinear points or by three non concurrent lines.

Let four projective points P_1, P_2, P_3, P_4 be collinear. So, we can write $\mathbf{v}_3 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$ and $\mathbf{v}_4 = \gamma\mathbf{v}_1 + \delta\mathbf{v}_2$ for nonzero vectors $\mathbf{v}_i$ with $\pi(\mathbf{v}_i) = P_i$ for $i = 1,\dots,4$. The *cross ratio*

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of P_1, P_2, P_3, P_4 is

$$\mathrm{cr}(P_1, P_2, P_3, P_4) = \frac{\beta/\alpha}{\delta/\gamma}\,.$$

It can be proved that this definition is well done. Moreover, the cross ratio is preserved under all projective transformations (see for example [2, pp. 138–140]).

Let four projective lines a_1, a_2, a_3, a_4 be concurrent on P and let r be a projective line not passing through P. The *cross-ratio* of a_1, a_2, a_3, a_4 is

$$\mathrm{cr}(a_1, a_2, a_3, a_4) = \mathrm{cr}(a_1 \cap r, a_2 \cap r, a_3 \cap r, a_4 \cap r).$$

Since the cross-ratio of four collinear points is preserved under all projective transformations, this latter definition does not depend on the choice of the projective line r.

The fundamental theorem of projective geometry can be stated as follows: Let $A, B, C, D \in \mathbb{P}^2$ no three of which are collinear and $A', B', C', D' \in \mathbb{P}^2$ no three of which are collinear. Then there is a unique projective transformation wich maps $A \mapsto A'$, $B \mapsto B'$, $C \mapsto C'$, and $D \mapsto D'$ (see, for example, [2, p. 127]).

It is well known the duality principle in projective geometry: for any projective result established using points and lines, while incidence is preserved, a symmetrical result holds if we interchange the roles of lines and points.

2. Main results

In this section we state the main results of this paper only using terms from projective geometry.

Theorem 2.1 *Let ABC be a triangle and r a projective line with $A, B, C \notin r$. Let $A' = \mathcal{L}(B, C) \cap r$, $B' = \mathcal{L}(C, A) \cap r$, and $C' = \mathcal{L}(A, B) \cap r$. Let A'', B'', and C'' be three projective points distinct from A, B, C such that $A'' \in \mathcal{L}(B, C)$, $B'' \in \mathcal{L}(C, A)$, and $C'' \in \mathcal{L}(A, B)$ (see Fig. 1, left). Then $\mathcal{L}(A, A'')$, $\mathcal{L}(B, B'')$, and $\mathcal{L}(C, C'')$ are concurrent if and only if*

$$\mathrm{cr}(B, C, A'', A') \cdot \mathrm{cr}(C, A, B'', B') \cdot \mathrm{cr}(A, B, C'', C') = -1.$$

Proof: Denote $\alpha = \mathrm{cr}(B, C, A'', A')$, $\beta = \mathrm{cr}(C, A, B'', B')$, and $\gamma = \mathrm{cr}(A, B, C'', C')$. Because this theorem is concerned exclusively with the projective geometry, by the dual of the fundamental theorem of projective geometry, we can suppose that the equations of $\mathcal{L}(A, B)$, $\mathcal{L}(B, C)$, $\mathcal{L}(C, A)$, and r are $z = 0$, $x = 0$, $y = 0$, and $x + y + z = 0$, respectively. It is easy to deduce that

$$A = \pi(1, 0, 0), \quad B = \pi(0, 1, 0), \quad C = \pi(0, 0, 1),$$

and

$$A' = \pi(0, -1, 1), \quad B' = \pi(1, 0, -1), \quad C' = \pi(-1, 1, 0).$$

Now, $A'' = \pi(0, 1, \lambda)$ for some $\lambda \neq 0$ because A'' lies on $\mathcal{L}(B, C)$ and $B \neq A'' \neq C$. Since $\alpha = \mathrm{cr}(B, C, A'', A') = -\lambda$ we get $A'' = \pi(0, 1, -\alpha)$ and thus, the equation of $\mathcal{L}(A, A'')$ is $\alpha y + z = 0$. Analogously, the equations of $\mathcal{L}(B, B'')$ and $\mathcal{L}(C, C'')$ are $x + \beta z = 0$ and $\gamma x + y = 0$, respectively. Now, $\mathcal{L}(A, A'')$, $\mathcal{L}(B, B'')$, and $\mathcal{L}(C, C'')$ are concurrent if and only if

$$0 = \begin{vmatrix} 0 & \alpha & 1 \\ 1 & 0 & \beta \\ \gamma & 1 & 0 \end{vmatrix} = 1 + \alpha\beta\gamma.$$

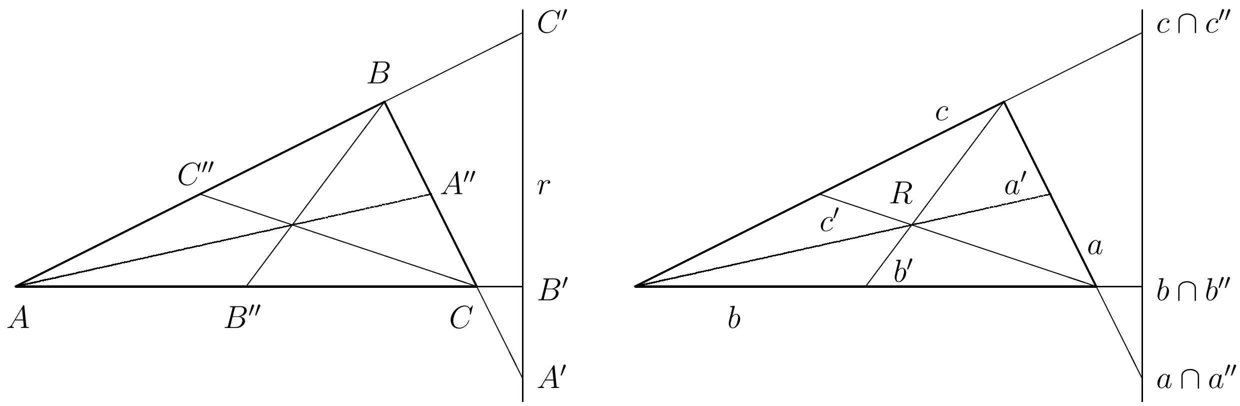

Figure 1: Theorem 2.1 and its dual

This finishes the proof. □

We can dualize Theorem 2.1: *Let abc be a triangle and R a projective point with* $R \notin a \cup b \cup c$. *Let* $a' = \mathcal{L}(b \cap c, R)$, $b' = \mathcal{L}(c \cap a, R)$, *and* $c' = \mathcal{L}(a \cap b, R)$. *Let* a'', b'', *and* c'' *be three projective lines distinct of* a, b, c *such that* $b \cap c \in a''$, $c \cap a \in b''$, *and* $a \cap b \in c''$ *(see Fig. 1, right). Then* $a \cap a''$, $b \cap b''$, *and* $c \cap c''$ *are collinear if and only if*

$$\operatorname{cr}(b, c, a'', a') \cdot \operatorname{cr}(c, a, b'', b') \cdot \operatorname{cr}(a, b, c'', c') = -1.$$

In order to state this theorem clearer, notice that under the hypothesis of the dual of Theorem 2.1, if we denote $A = b \cap c$, $B = c \cap a$, $C = a \cap b$, $A' = a \cap a''$, $B' = b \cap b''$, $C' = c \cap c''$, $A'' = a \cap a'$, $B'' = b \cap b'$, and $C'' = c \cap c'$, then

$$\operatorname{cr}(b, c, a'', a') = \operatorname{cr}(b \cap a, c \cap a, a'' \cap a, a' \cap a) = \operatorname{cr}(C, B, A', A'') = \operatorname{cr}(B, C, A'', A')$$

and analogously,

$$\operatorname{cr}(c, a, b'', b') = \operatorname{cr}(C, A, B'', B'), \qquad \operatorname{cr}(a, b, c'', c') = \operatorname{cr}(A, B, C'', C').$$

Thus, we get the following result:

Theorem 2.2 *Let ABC be a triangle and let R be a projective point. Let* $A'' = \mathcal{L}(B, C) \cap \mathcal{L}(A, R)$, $B'' = \mathcal{L}(C, A) \cap \mathcal{L}(B, R)$, *and* $C'' = \mathcal{L}(A, B) \cap \mathcal{L}(C, R)$. *Let* A', B', *and* C' *be three projective points distinct from* A, B, C *such that* $A' \in \mathcal{L}(B, C)$, $B' \in \mathcal{L}(C, A)$, *and* $C' \in \mathcal{L}(A, B)$. *Then* A', B', C' *are collinear if and only if*

$$\operatorname{cr}(B, C, A'', A') \cdot \operatorname{cr}(C, A, B'', B') \cdot \operatorname{cr}(A, B, C'', C') = -1.$$

Recall that two projective points C, D are said to be *harmonic conjugates* with respect to the projective points A, B when $\operatorname{cr}(A, B, C, D) = -1$. By Theorems 2.1 and 2.2, the following result holds.

Corollary 2.1 *Given a triangle ABC, let* C'', C' *be harmonic conjugates with respect to* A, B; *let* A'', A' *be harmonic conjugates with respect to* B, C; *and let* B'', B' *be harmonic conjugates with respect to* C, A. *Then* A', B', *and* C' *are collinear if and only if* $\mathcal{L}(A, A'')$, $\mathcal{L}(B, B'')$, *and* $\mathcal{L}(C, C'')$ *are concurrent.*

The above corollary is indeed classical and it deals with the classical "polarity with respect to a triangle". A variant of a Desargues' theorem says that if in the triangle ABC, the cevians AA'', BB'', and CC'' meet in point R, the lines AB, $A''B''$ meet in C'; AC, $A''C''$ meet in B'; and BC and $B''C''$ meet in A', then the points A', B', C' are collinear. By the construction, the pairs A' and A'', B' and B'', C' and C'' are harmonic conjugate with respect to (B, C), (C, A), (A, B). This aforementioned polarity maps the common points of the lines AA'', BB'', and CC'' onto the line passing through $A'B'C'$. See [5] for a comprehensive summary on projective geometry at the beginning of twentieth century. See [7, Ch. 7] (for example) for another proof of the former corollary.

3. Ceva and Menelaus' theorems

In this section we explain the way in which the affine plane is involved in Theorems 2.1 and 2.2. The *ideal line* is the projective line whose equation is $z = 0$. An *ideal point* is a projective point that belongs to the ideal line. If $\mathcal{A}(\mathbb{P}^2)$ is the set of all non-ideal points, we can establish two bijective maps in the following way:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{i} & \mathcal{A}(\mathbb{P}^2) \\ (x,y) & \mapsto & \pi(x,y,1) \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{A}(\mathbb{P}^2) & \xrightarrow{j} & \mathbb{R}^2 \\ \pi(x,y,z) & \mapsto & (x/z, y/z) \end{array}$$

We can easily check that $j \circ i = I_{\mathbb{R}^2}$ and $i \circ j = I_{\mathcal{A}(\mathbb{P}^2)}$, where I denotes the identity map. In the following, we say that P is an *affine point* (or simply *point*) when $P \in \mathbb{R}^2$.

Let A, B, C, D be four collinear points. It is easy to see (see, for example, [2]) that

$$\operatorname{cr}(i(A), i(B), i(C), i(D)) = \frac{\overrightarrow{AC}/\overrightarrow{BC}}{\overrightarrow{AD}/\overrightarrow{BD}}.$$

Let A, B, C be three collinear points. Let D be the intersection of the ideal line and $\mathcal{L}(i(A), i(B))$. It can be easily checked that

$$\operatorname{cr}(i(A),\, i(B),\, i(C),\, D) = \frac{\overrightarrow{AC}}{\overrightarrow{BC}}.$$

Ceva's and Menelau's theorems are classical results. We present here a proof of these theorems based on the former section. For an analytical proof of Ceva's and Menelau's theorems with coordinates, it can be consulted [2]. For a analytical proof without coordinates, see, for example, [1]. Another standard idea for Ceva's theorem is to study the areas of sub-triangles, see [6]. In this same web-site, one can consult a proof of Menelau's theorem. Grünbaum and Shepard show that Ceva's theorem and Menelaus' theorem are both corollaries of a single result based on areas, see [3]. In [7, Ch. 7], the Menelaus' theorem was proved before than the Ceva's theorem. One can consult also [4] and references therein.

Corollary 3.1 (Ceva's Theorem) *Let ABC be a triangle and let X, Y, and Z be points on the sides AB, BC, and CA respectively. Then CX, BZ, and AY are concurrent if and only if*

$$\frac{\overrightarrow{BY}}{\overrightarrow{YC}} \cdot \frac{\overrightarrow{CZ}}{\overrightarrow{ZA}} \cdot \frac{\overrightarrow{AX}}{\overrightarrow{XB}} = 1$$

Proof: We shall apply Theorem 2.1 when r is the ideal line. Let us define $A' = r \cap \mathcal{L}(i(B), i(C))$, $B' = r \cap \mathcal{L}(i(C), i(A))$, and $C' = r \cap \mathcal{L}(i(A), i(B))$. Since

$$\operatorname{cr}(i(B),\, i(C),\, i(Y),\, A') = \frac{\overrightarrow{BY}}{\overrightarrow{CY}}, \quad \operatorname{cr}(i(C),\, i(A),\, i(Z),\, B') = \frac{\overrightarrow{CZ}}{\overrightarrow{AZ}},$$
$$\operatorname{cr}(i(A),\, i(B),\, i(X),\, C'') = \frac{\overrightarrow{AX}}{\overrightarrow{BX}},$$

we get that CX, BZ, and AY are concurrent if and only if $\overrightarrow{BY}/\overrightarrow{CY} \cdot \overrightarrow{CZ}/\overrightarrow{AZ} \cdot \overrightarrow{AX}/\overrightarrow{BX} = -1$. This finishes the proof. □

Corollary 3.2 (Menelaus' Theorem) *Let ABC be a triangle and let three points X, Y, and Z lie respectively on the lines AB, BC, and CA. Then X, Y, and Z are collinear if and only if*

$$\frac{\overrightarrow{AX}}{\overrightarrow{XB}} \cdot \frac{\overrightarrow{BY}}{\overrightarrow{YC}} \cdot \frac{\overrightarrow{CZ}}{\overrightarrow{ZA}} = -1.$$

Proof: Pick any point R that does not belong to the lines AB, BC, and CA. Let A'' be the intersection of the lines BC and AR; let B'' be the intersection of the lines CA and BR; and let C'' be the intersection of the lines AB and CR. By Ceva's Theorem, we get

$$\frac{\overrightarrow{B''C}}{\overrightarrow{AB''}} \cdot \frac{\overrightarrow{A''B}}{\overrightarrow{CA''}} \cdot \frac{\overrightarrow{C''A}}{\overrightarrow{BC''}} = 1,$$

Apply Theorem 2.2 in order to obtain that X, Y, and Z are collinear if and only if

$$\operatorname{cr}(i(B), i(C), i(A''), i(Y)) \cdot \operatorname{cr}(i(C), i(A), i(B''), i(Z)) \cdot \operatorname{cr}(i(A), i(B), i(C''), i(X)) = -1.$$

Since, one has

$$\begin{aligned}
&\operatorname{cr}(i(B), i(C), i(A''), i(Y)) \cdot \operatorname{cr}(i(C), i(A), i(B''), i(Z)) \cdot \operatorname{cr}(i(A), i(B), i(C''), i(X)) \\
&= \frac{\overrightarrow{BA''}/\overrightarrow{CA''}}{\overrightarrow{BY}/\overrightarrow{CY}} \cdot \frac{\overrightarrow{CB''}/\overrightarrow{AB''}}{\overrightarrow{CZ}/\overrightarrow{AZ}} \cdot \frac{\overrightarrow{AC''}/\overrightarrow{BC''}}{\overrightarrow{AX}/\overrightarrow{BX}} \\
&= -\frac{\overrightarrow{CY}}{\overrightarrow{BY}} \cdot \frac{\overrightarrow{AZ}}{\overrightarrow{CZ}} \cdot \frac{\overrightarrow{BX}}{\overrightarrow{AX}} \\
&= \left(\frac{\overrightarrow{BY}}{\overrightarrow{YC}} \cdot \frac{\overrightarrow{CZ}}{\overrightarrow{ZA}} \cdot \frac{\overrightarrow{AX}}{\overrightarrow{XB}} \right)^{-1},
\end{aligned}$$

this completes the proof. □

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