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Two Kinds of Golden Triangles, Generalized to Match Continued Fractions

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Abstract. Two kinds of partitioning of a triangle ABC are considered: side-partitioning and angle-partitioning. Let $a = |BC|$ and $b = |AC|$, and assume that $0 < b \leq a$. Side-partitioning occurs in stages. At each stage, a certain maximal number q_n of subtriangles of ABC are removed. The sequence (q_n) is the continued fraction of a/b, and if $q_n = 1$ for all n, then ABC is called a side-golden triangle. In a similar way, angle-partitioning matches the continued fraction of the ratio C/B of angles, and if $q_n = 1$ for all n, then ABC is called a angle-golden triangle. It is proved that there is a unique triangle that is both side-golden and angle-golden.

Key Words: golden triangle, golden ratio, continued fraction

MSC 2000: 51M04

1. Introduction: rectangles and triangles

One of the fondest of all mathematical shapes is the golden rectangle, special because of its shape and "golden" because of its connection with the golden ratio. Here's the story:

> *Every rectangle has a length L and width W, and the shape of the rectangle is given by the single number L/W. There is only one shape of rectangle such that if a square of sidelength W is removed from an end of the rectangle, then the remaining rectangle has the same shape as the original. That is, $W/(L-W) = L/W$. Putting $x = L/W$ and simplifying gives $x^2 - x - 1 = 0$, so that $x = (1+\sqrt{5})/2$, the golden ratio, often denoted by φ.*

The square-removal process can be repeated indefinitely, always leaving a smaller rectangle having the same shape as the original. Only an original shape of φ ensures that there is always just one square available for removal at each stage. However, anticipating the removal of more than one square, it is natural to ask about other shapes of rectangles. As an example of mathematical folklore, it is known that if the original shape is a positive number r, then the number of removable squares at stage n is the n-th number in the continued fraction, $[a_1, a_2, \ldots, a_n, \ldots]$ of r, as proved below. In the case of the golden ratio, $r = [1, 1, 1, \ldots]$.

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As another example, you can easily construct, using a Euclidean ruler and compass, a rectangle having $W = 1$ and $L = 1 + \sqrt{2}$, and then convince yourself that there are 2 squares of length 1 that can be removed from an end of the rectangle, leaving a rectangle that offers 2 more squares, and so on forever, thus matching the continued fraction $[2, 2, 2, \ldots]$ of L. Or, for a nonconstructible venture, start with $W = 1$ and $L = e$, and find the numbers of removable squares to be given by $[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$.

Is there anything like this for triangles?

Indeed, in certain popular accounts, there is only one shape of triangle called *"golden"*. However, Bicknell and Hoggatt [1] showed that there are really infinitely many such triangles, and elsewhere [2] quite a different kind of golden triangle was introduced. Both kinds (which we call *side-golden* and *angle-golden*) match the continued fraction $[1, 1, 1, \ldots]$, and both kinds generalize to match arbitrary continued fractions. The purpose of this article is to present both generalizations, and to show, as a special case, that there is a unique triangle that is both side-golden and angle-golden.

Weiss [5] uses projective methods to study golden hexagons, including geometrically defined iteration processes and involving the golden ratio.

2. Segment ratios

We begin with a geometric version of the division algorithm ($a = bq + r$, where $0 \le r < b$), except that here, a and b may be any positive real numbers, not necessarily integers. Suppose that $0 < b \le a$, and let points A, B, C satisfy $a = |BC|$ and $b = |AC|$ as in Fig. 1:

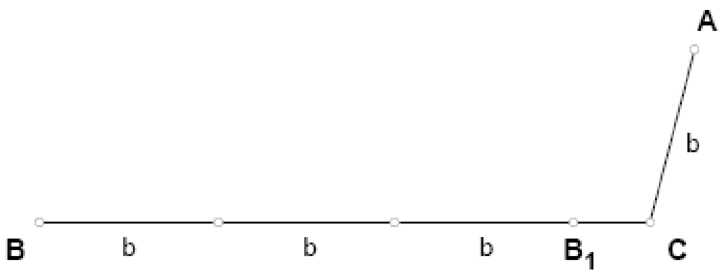

Figure 1: $|BC| = 3\,|AC| + |B_1C|$

Let B_1 be the point on segment BC satisfying

$$|BB_1| = \left\lfloor \frac{a}{b} \right\rfloor b\,;$$

that is, if we start at B and progress toward C with as many end-to-end links of length b as possible, without passing C, then B_1 is the final point reached. The remaining distance from B_1 to C is strictly less than b, so that we have the numbers q and r given by the division algorithm. This visual division algorithm extends to a visual euclidean algorithm by iteration: let C_1 be the point on segment AC satisfying

$$|AC_1| = \left\lfloor \frac{|AC|}{|B_1C|} \right\rfloor |BC|\,;$$

then, in like manner, determine B_2 on B_1C and C_2 on C_1C, and inductively obtain sequences (B_i) and (C_i), both converging to C. In order to see a connection with a certain continued fraction, write

$$\begin{aligned}
a &= q_1 b + r_1, &&\text{where } q_1 = \left\lfloor \tfrac{a}{b} \right\rfloor && \text{and } 0 \le r_1 = a - q_1 b < b;\\
b &= q_2 r_1 + r_2, &&\text{where } q_2 = \left\lfloor \tfrac{b}{r_1} \right\rfloor && \text{and } 0 \le r_2 = b - q_2 r_1 < r_1;\\
r_1 &= q_3 r_2 + r_3, &&\text{where } q_3 = \left\lfloor \tfrac{r_1}{r_2} \right\rfloor && \text{and } 0 \le r_3 = r_1 - q_3 r_2 < r_2;\\
& && && \vdots\\
r_n &= q_{n+2} r_{n+1} + r_{n+3}, &&\text{where } q_{n+2} = \left\lfloor \tfrac{r_n}{r_{n+1}} \right\rfloor && \text{and } 0 \le r_{n+2} = r_n - q_{n+2} r_{n+1} < r_{n+1},
\end{aligned}$$

for all $n \ge 1$. These equations imply

$$\begin{aligned}
\frac{a}{b} &= q_1 + \frac{r_1}{b};\\
&= q_1 + \cfrac{1}{q_2 + \cfrac{r_2}{b}}\\
&\ \ \vdots\\
&= q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4 + \ddots}}}.
\end{aligned}$$

In other words, the continued fraction of a/b is $[q_1, q_2, q_3, \ldots]$.

Concerning the generalization of the golden rectangle mentioned in Section 1, make segment AC perpendicular to BC and complete an arbitrary rectangle by placing the fourth vertex in the obvious place. Then the points B_i and C_i indicate where the maximal number of squares can be removed at each stage. Specifically, if $a/b = [q_1, q_2, q_3, \ldots]$, then the number of removable squares at stage n is q_n.

3. Side-partitioning of a triangle

The sequences (B_i) and (C_i) of the preceding section determine a sequence of triangles:

$$ABC,\ AB_1C,\ C_1B_1C,\ C_1B_2C,\ C_2B_2C,\ C_2B_3C,\ C_3B_3C,\ \ldots\ .$$

If a/b is the golden ratio, then the triangles are all (side-) golden triangles as defined in [1], and in this case, all the triangles are similar to ABC.

If a/b is an arbitrary positive real number, then the connection that the sequence of triangles has to the continued fraction of a/b is as stated in Section 1. That is, at each stage, one of two things happens: either a triangle whose side on line BC is a maximal integer multiple of the preceding triangle's side on AC is removed, or else a triangle whose side on AC is a maximal integer multiple of the preceding triangle's side on BC is removed. At each stage, we may speak of the removed triangle and the remaining triangle. The remaining triangles are nested.

The removal process terminates in a finite number of stages if and only if a/b is a rational number. In this case, triangle ABC is partitioned into a finite number of triangles, and the nested remaining triangles do not have a limit point. On the other hand, if a/b is irrational, then the limit of the nest is the point C. Fig. 2 shows the first two triangles in a nest of similar triangles for the case $|BC| / |AC| = [2, 2, 2, \ldots]$.

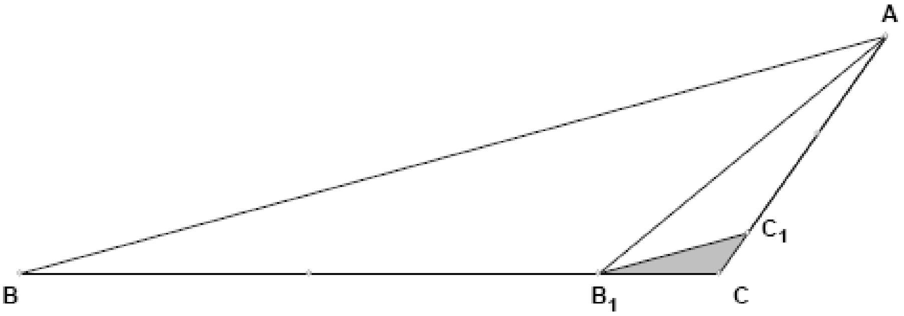

Figure 2: Nested similar triangles

4. Angle-partitioning of a triangle

We turn now to angle-partitioning of triangle ABC, which, unlike side-partitioning, involves the placement of points on all three sides of ABC. To begin, suppose that the angles satisfy $A \le B \le C$, so that $a \le b \le c$. Let X be the point on segment AB that satisfies $|XC| = |BC|$, as in Fig. 3, so that

$$\triangle ABC = (\text{isosceles } \triangle XBC) \cup (\text{remaining } \triangle AXC).$$

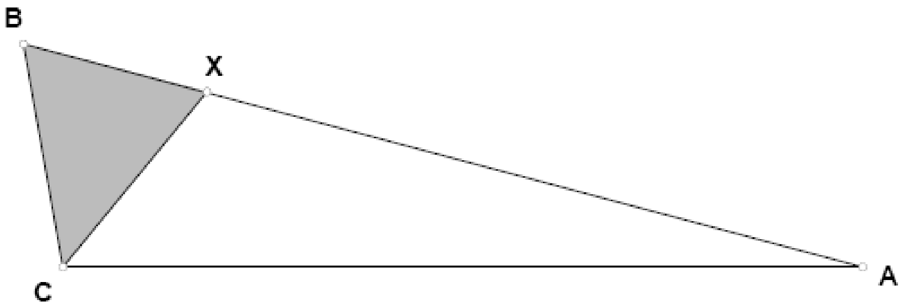

Figure 3: First step in angle-partitioning

An *isosceles removal* is the removal of $\triangle XBC$ from $\triangle ABC$, leaving $\triangle AXC$. Angle-partitioning simply consists of repeating the isosceles removals, either indefinitely, or else until no triangle remains.

A *maximal removal* consists of as many isosceles removals as possible for a given least vertex angle. In Fig. 4, the least vertex angle, A, stays fixed during a maximal removal consisting of 4 isosceles removals.

It is known [3] that if $B \le C$ in a given triangle ABC, and if the successive numbers of maximal removals are $q_1, q_2, \ldots$, then the ratio C/B of angles is the number whose continued fraction is $[q_1, q_2, \ldots]$. An *angle-golden triangle* is one for which only one isosceles removal is possible at each stage. Such a triangle is given by the condition $C = \varphi B$, and $C/B = [1, 1, 1, \ldots]$.

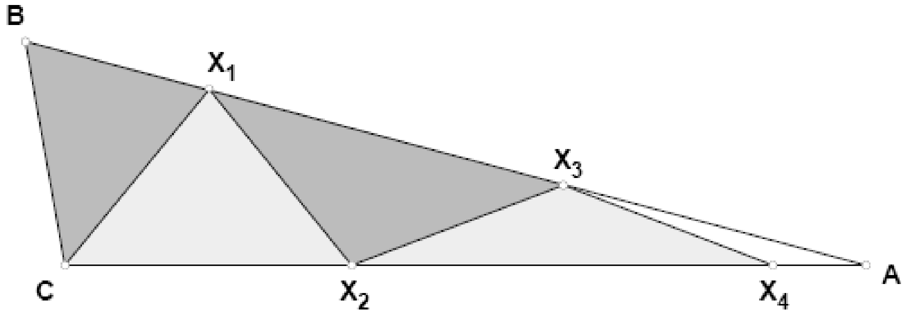

Figure 4: A maximal removal of four isosceles triangles

5. Loci and intersections

For given $a/b \geq 1$, the locus of a point A for which $\triangle ABC$ has side-ratio $|BC|\,/\,|AC| = a/b$ is the circle represented in Fig. 5. In cartesian coordinates centered at B with positive x-axis in the direction from B to C, the circle is given by

$$(x-a)^2 + y^2 = b^2. \tag{1}$$

For $r > 0$, to find the locus of A for which $\triangle ABC$ has angle-ratio $C/B = r$, we write θ for the angle CBA. Then A is the point (x,y) of intersection of the lines $y = x\tan\theta$ and $y = (a-x)\tan r\theta$, so that

$$x = \frac{a\tan r\theta}{\tan\theta + \tan r\theta}\,. \tag{2}$$

If r is irrational, then (x,y) is nonperiodic in θ, and the locus consists of infinitely many branches as suggested by Fig. 5.

The branch that passes through segment BC meets the circle at a single point for $\theta \in [0,\pi]$. Because the branch is unbounded and the circle is bounded, a necessary and sufficient condition that the two loci meet is that the circle comes closer to B than the branch does. That is, there is a triangle that has side-ratio $|BC|\,/\,|AC| = a/b$ and angle-ratio r if and only if

$$a - b < \frac{ar}{1+r}\,, \tag{3}$$

where

$$\frac{ar}{1+r} = \lim_{\theta\to 0}\frac{a\tan r\theta}{\tan\theta + \tan r\theta}\,.$$

In order to prove (3), we rewrite it using C/B for r as

$$a/b < 1 + C/B = (\pi - A)/B,$$

so that we wish to prove that

$$A + (a/b)B < \pi, \tag{4}$$

an unusual mix of sides and angles. To prove (4), note that the altitude from A to BC has length $b\sin C$, so that

$$\tan B = \frac{b\sin C}{a - b\cos C}\,.$$

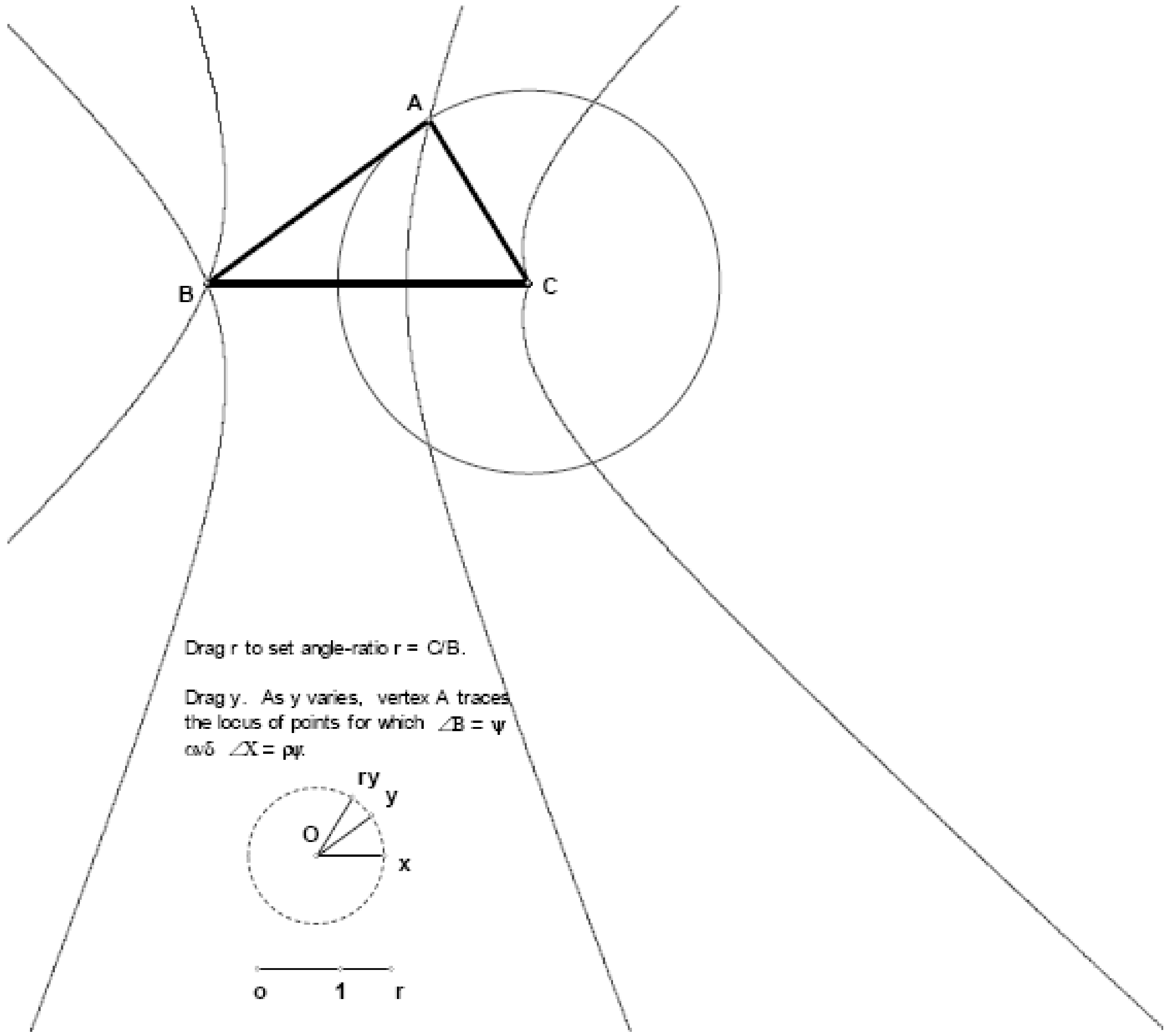

Figure 5: Two loci of vertex A: a circle and a multi-branch curve

We can thus view $A + (a/b)B$ as a function of angle C. Specifically, write a/b as t and A as $\pi - B - C$, so that

$$\begin{aligned} g(C) &= A + (a/b)B \\ &= \pi - C + (t-1)\arctan\frac{\sin C}{t - \cos C}\,; \\ g'(C) &= \frac{t(t+1)(1-\cos C)}{-t^2 - 1 + 2t\cos C}\,. \end{aligned}$$

If $a > b$, or if $a = b$ and $\cos C \neq 0$, then clearly, $g'(C) < 0$ for all C, so that $g(C)$ is strictly decreasing, and (4) follows because $\lim\limits_{C \to 0} g(C) = \pi$. In the remaining case, that $a = b$ and $\cos C = 0$, clearly (4) holds.

For $r > 1$, the equations (1), (2) and $y = x \tan\theta$ yield

$$\frac{\sin(r+1)\theta}{\sin\theta} = \pm\frac{a}{b}\,.$$

In order to find a triangle that is both side-golden and angle-golden, put $r = a/b = \varphi$. The resulting equation has solutions only if the right-hand side is positive, so that the equation can be written as $\sin(\varphi + 1)\theta = \varphi \sin\theta$. There is only one solution θ in $[0, \pi]$, hence only one angle B for which the resulting triangle is both side-golden and angle-golden. This doubly golden triangle has angles $C = \varphi B$, $A = \pi - B - C$, and B, approximately

$$0.6574054829783607699528810721252458396268099700424278723498608444862\,.$$

6. Summary theorem

In this section we summarize as a theorem the main result proved above, showing connections between partitioning and continued fractions:

Theorem. *Suppose ABC is an arbitrary triangle, labeled so that* $|AC| \le |BC|$ *and (angle B)* $\le$ *(angle C). For stage n of side-partitioning, let* q_n *be the number of triangles removed, and for stage n of angle-partitioning, let* p_n *be the number of triangles removed. Then* $[q_1, q_2, q_3, \ldots]$ *and* $[p_1, p_2, p_3, \ldots]$ *are the continued fractions of* a/b *and* C/A*, respectively.*

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