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A Geometrical Approach to the Numerical Stability Analysis of Some Projective Collinear Mapping Methods

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Abstract. This paper exposes the geometrical approach to the numerical stability analysis of the projective collinear mapping method based on Laguerre's points of the involution mapped. The ill-conditioned and corresponding unstable zones for this mapping method are defined and some alternative procedures for their corrections are proposed. As a result of the analysis, the stable and well conditioned collinear mapping technique, which can be used in the design of some 3D imaging software algorithms, are created and explained. The exposed analysis is a contribution to the theory of projective and computational geometry. Moreover, it proposes a numerical stability criterion for the practical mapping procedures in software applications which are based on the principles of the projective collineation.

Keywords: involution, Laguerre's points, numerical stability, projective collineation

MSC 2007: 51M04, 51N15

1. Introduction

This work is derived from an educational project in which the object oriented modeling of projective collinear transformations is accomplished. As a result of this process, a full featured Windows application that performs the collinear projective mapping and illustrates the main invariants of projective geometry is designed. One of the most interesting and most significant parts of this project was solving the practical problems of projective mapping numerical stability. This paper not just discloses the solution but also offers a possible approach in finding the answer to this important question.

At the beginning, it is not worthless to recall and emphasize that every mapping method can be characterized by the accuracy and errors caused by the limited precision of the technical instruments by which it is practically performed. This problem was considered thoroughly in literature, especially in the field of robust geometric computing for Computer-Aided Design.

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Very interesting and meticulous work has been done by S.-M. Hu and J. Wallner [2], who analyze *"the propagation of errors through geometric transformations, such as reflections, rotations, similarity transformations, and projections, and also the scalar product of vectors."*

J. Wallner, R. Krasauskas and H. Pottmann [8] *"... show how to treat some problems of error propagation in geometric constructions in a geometric way."*

And H. Pottmann, B. Odehnal, M. Peternell, J. Wallner, R. Ait Haddou [6] explain: *"A geometric approach to the computation of precise or well approximated tolerance zones for CAD constructions ..."* They emphasize that *"Almost always one assumes precise input or one is not so much concerned about the effect of input errors or tolerances on the output. However, for practical purposes, this is a fundamental question."*

All projective collinear mapping methods are based on the line intersections and, without regard whether the realization of these operations is graphical or numerical, it is possible to discuss about their stability and precision and analyze the alternative procedures which will make these projective transformations more stable and accurate.

Despite the fact that algebraic criteria for the stability determination are numerous, numerical stability analysis is obtained in this paper by the geometrical procedures. This geometrical approach is simple and it can be easily understood and practically accomplished. The essential assumption for this consideration is that the instability of the mapping realization is induced by the extremely small intersection angle of the point radius vectors.

2. Basic terminology and definitions

For further theoretical considerations, the following terminology and definitions are necessary to be defined:

Definition 1 A method of the projective collinear mapping from the field F_1 to the field F_2 is called *stable* in the particular zone Ω_2 of the field F_2, if small changes of its parameters correspond to small shifts of the points mapped. Otherwise the method of the collinear mapping is called *unstable* in the particular zone Ω_2 of the field F_2.

Definition 2 A method of the projective collinear mapping from the field F_1 to the field F_2 is called *well-conditioned* in a particular area Ω_1 of the field F_1, if the mapping method is stable in the associated area Ω_2 of the field F_2. The method of the projective collinear mapping from the field F_1 to the field F_2 is called *ill-conditioned* in the particular area Ω_1 of the field F_1, if the mapping method is unstable in the associated area Ω_2 of the field F_2.

Definition 3 The area within the projective collinear fields in which one particular mapping method is stable is called the *stability zone* of this mapping method. The area within the projective collinear fields associated with the corresponding stability zone is the *well-conditioned zone* of this mapping method. The *ill-conditioned zones* are the areas of the projective collinear fields located outside of the well-conditioned zones.

If the collinear mapping method is interpreted by the algebraic equations, as is shown clearly in [5], [1] and [7], the stability conditions of that mapping method can be expressed in algebraic language, i.e., in the well known matrix form. In that sense, the mathematical concepts of matrix stability, as well as the concept of ill-conditioned and well-conditioned matrices and systems of linear equations are defined. These definitions are exposed in [3]:

"**Definicija 1.** *Inverzna matrica A^{-1} matrice A je stabilna ako malim promenama elemenata matrice A odgovaraju male promene elemenata matrice A^{-1}. U protivnom matrica A^{-1} je nestabilna.*"
["**Definition 1.** *The inverse matrix A^{-1} of matrix A is stable if small changes of matrix A entries correspond to little changes of matrix A^{-1} entries. In the opposite case matrix A^{-1} is unstable.*"]

"**Definicija 2.** *Matrica A je slabo uslovljena ako je njena inverzna matrica A^{-1} nestabilna.*"
["**Definition 2.** *Matrix A is ill-conditioned if its inverse matrix A^{-1} is unstable.*"]

3. Measure of the local absolute and local relative mapping error

It is well known that fields of projective collineation possess two pencils of vanishing lines and the absolute involutions mapped, whose supports they are, represent elliptical involuted ranges. A projective collinear mapping can be accomplished directly by using Laguerre's points, pairs of projectively associated circular pencils of rays and especially defined polar coordinate systems. This fact are exposed in [4]:
"U posebno znač ajne invarijante kolokalnih kolinearnih polja ubrajaju se i njihove žiže. One omogućavaju da se, uvodenjem polarnih koordinata, primene jednostavni i grafič ki precizni postupci direktnog preslikavanja jednog polja u drugo."
["*Especially important invariants of collocal collinear fields are their foci. By them, polar coordinate systems can be defined and simple and graphically precise methods of direct mapping from one to the another field can be applied.*"]
and [5]:
"Lagerovim tač kama preslikanih apsolutnih involucija definisani su polarni koordinatni sistemi u poljima opšte kolineacije, tako da se preslikavanja tač aka vrše sa konstantnim ugaonim koordinatama, variranjem dužina njihovih potega. Lagerove tač ke preslikanih apsolutnih involucija, odnosno žiže kolokalnih kolinearnih polja, predstavljaju centre, a glavne normale su pridružene ose tih polarnih koordinatnih sistema."
["*By Laguerre's points of the involution mapped, the polar coordinate systems are defined in general collinear fields in such a way that the point mapping is accomplished with the constant angular coordinates by varying the radius vectors' length. Laguerre's points of the involution mapped, i.e., the foci of the collocal collinear fields, are the centers and the principal normal lines are the associated axes of those polar coordinate systems.*"]

This work considers the numerical stability of the above-mentioned collinear mapping method based on Laguerre's points of the involution mapped.

The principal parameters of the projective collinear mapping method based on Laguerre's points of the involution mapped are: homogenous coordinates of Laguerre's points and angular coordinates of the field points. Since the positions of Laguerre's points must be defined by coordinates or determined by an alternative mapping method, errors of their location do not have the influence on the mapping method stability considered in this paper. Consequently to this conclusion, full attention will be concentrated only to the relation between the mapping stability and the point angular coordinates.

Let us presume that angular coordinates of the points in the field F_1 are determined with some errors and that these errors cause the errors of the angular coordinate φ and ψ of the points mapped in the field F_2. Also, let us presume that the absolute value of all angular errors is not greater than some particular number ε. It is necessary to evaluate the

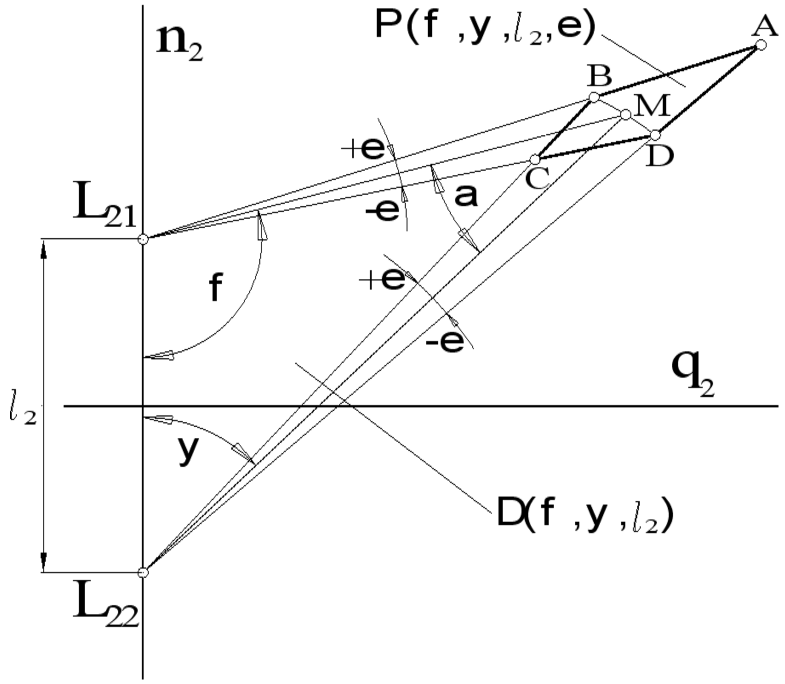

Figure 1: The measure of the local absolute and local relative mapping error

alteration of the point mapped positions in the function of this number ε. Acting similarly, J. Wallner, R. Krasauskas and H. Pottmann [8] write:

"We consider a very simple geometry construction: the intersection of two lines in the Euclidean plane. The input consists of two lines, and the output is a point. ... If we speak of error propagation, we mean the following: Suppose each item of the input data can vary independently in some domain (for instance, a point varies in a disk). We can think of input data given imprecisely or of tolerance zones for the input data. We ask for the set of all possible outputs. If this is not possible, we would at least want to know some tolerance zone which contains all possible outputs."

As shown in Fig. 1, the Laguerre's points L_{21} and L_{22} are defined in the field F_2, as well as the mapped point $M(\varphi,\ \psi)$ whose radius vectors intersect at the angle α. From the presumption that angular coordinates are determined with error limit $\pm\varepsilon$, it can be concluded that the real positions of the point mapped can be altered inside the square ABCD whose sides are formed by the radius vectors with angular coordinates $(\varphi \pm \varepsilon,\ \psi \pm \varepsilon)$. The square ABCD actually represents the "Fat Point" and this consideration is in accordance with its definition [8]:

"In Euclidian geometry, points are the most elementary geometric objects. A fat point is a set of points. Usually when computing with fat points, one restricts attention to sets with certain properties, like closed convex ones, or even balls."

For this reason, the square area $\Pi(\varphi, \psi, l_2, \varepsilon)$ can represent the measure of the local absolute mapping error, in function of angular coordinates φ and ψ, distance l_2 between Laguerre's points, and error limit ε.

Since the point position and the measure of local absolute mapping error are determined by the pair of radius vectors, the measure of the local relative error of the mapping method must be determined by the function of both radius vectors' length. Therefore, the measure of the local relative error of the mapping method $\rho(\varphi, \psi, l_2, \varepsilon)$ can be represented by the ratio

$$\rho(\varphi, \psi, l_2, \varepsilon) = \Pi(\varphi, \psi, l_2, \varepsilon)/\Delta(\varphi, \psi, l_2). \tag{1}$$

In this formula, $\Delta(\varphi, \psi, l_2)$ is the area of the triangle $L_{21}L_{22}M$ which algebraically comprises the product of two points radius vectors length. Since the angular coordinate ψ is a function of α and φ, the measure of the local relative error ρ can also be expressed by the ratio

$$\rho(\varphi, \alpha, l_2, \varepsilon) = \Pi(\varphi, \alpha, l_2, \varepsilon)/\Delta(\varphi, \alpha, l_2). \tag{2}$$

4. Geometrical loci of constant relative mapping error

Now, let us assume that the radius vectors' intersection angle α is constant. Under this assumption, the geometrical loci of constant relative error of the mapping method based on Laguerre's points will be considered and determined. As shown in Fig. 2, it is clear that the locus of mapped points, whose radius vectors cut at a constant angle α, belongs to the pair of equal radii circles that intersect at Laguerre's points L_{21} and L_{22}.

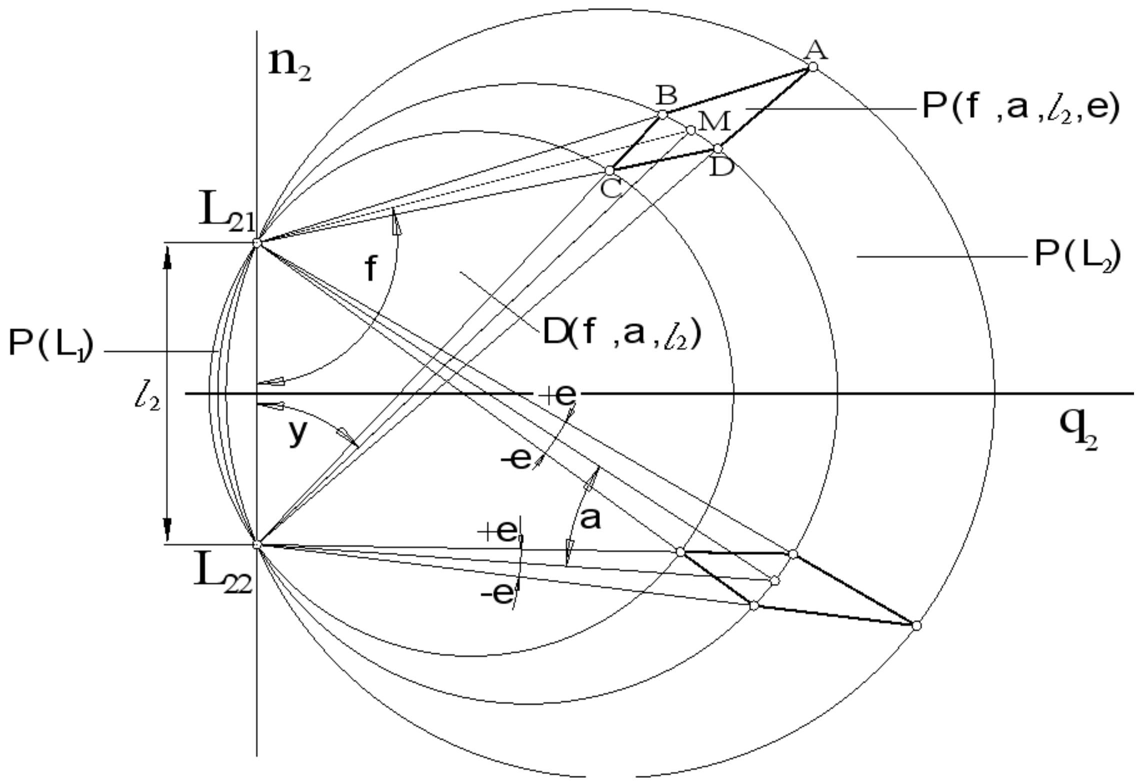

Figure 2: The geometrical loci of constant relative mapping error

The function $\rho = \rho(\varphi, \alpha, l_2, \varepsilon)$ of the local relative mapping error can be effectively determined from the following equations:

$$P_0 = \frac{l_2 \cdot (\sin\varphi \cdot \sin(\varphi + \alpha))}{2\sin\alpha}; \quad P_1 = \frac{l_2 \cdot (\sin(\varphi + \varepsilon) \cdot \sin(\varphi + \alpha - \varepsilon))}{2\sin(\alpha - 2\varepsilon)}$$

$$P_2 = \frac{l_2 \cdot (\sin(\varphi + \varepsilon) \cdot \sin(\varphi + \alpha + \varepsilon))}{2\sin\alpha}; \quad P_3 = \frac{l_2 \cdot (\sin(\varphi - \varepsilon) \cdot \sin(\varphi + \alpha - \varepsilon))}{2\sin\alpha}$$

$$P_4 = \frac{l_2 \cdot (\sin(\varphi - \varepsilon) \cdot \sin(\varphi + \alpha + \varepsilon))}{2\sin(\alpha + 2\varepsilon)}; \quad \rho = \frac{P_1 - P_2 - P_3 + P_4}{P_0}. \tag{3}$$

This function is computed numerically for different values of angle α and the constant value of ε. The result is shown in the following tables:

Table 1: Numerical values of the function $\rho = \rho(\varphi, \alpha, l_2, \varepsilon)$

$\alpha = 0.1;\ \varepsilon = 10^{-5}$		$\alpha = 0.01;\ \varepsilon = 10^{-5}$	
φ	ρ	φ	ρ
0.01	8.02672113179	0.01	8.00030267188
0.1	8.02672044451	0.1	8.00029874459
1.0	8.02672040531	1.0	8.00029867272
2.0	8.02672040527	2.0	8.00029867257
3.0	8.02672054031	3.0	8.00029871479
4.0	8.02672040555	4.0	8.00029867298
5.0	8.02672040514	5.0	8.00029867247
6.0	8.02672041992	6.0	8.00029868221
6.1	8.02672045689	6.1	8.00029869708
6.2	8.02671983168	6.2	8.00029880329
6.28	8.02671781117	6.28	8.00026181718

Table 2: Numerical values of the function $\rho = \rho(\varphi, \alpha, l_2, \varepsilon)$

$\alpha = 2.1\,10^{-5};\ \varepsilon = 10^{-5}$		$\alpha = 2.01\,10^{-5};\ \varepsilon = 10^{-5}$	
φ	ρ	φ	ρ
0.01	19.5122145929	0.01	199.501445977
0.1	19.5121953166	0.1	199.501248871
1.0	19.5121951237	1.0	199.501246898
2.0	19.5121951233	2.0	199.501246894
3.0	19.5121952189	3.0	199.501247872
4.0	19.5121951243	4.0	199.501246905
5.0	19.5121951230	5.0	199.501246891
6.0	19.5121951459	6.0	199.501247125
6.1	19.5121951797	6.1	199.501247471
6.2	19.5121954036	6.2	199.501249760
6.28	19.5123887086	6.28	199.503225632

The following conclusions can be drawn from equations (3) and numerical data exposed in Tables 1 and 2:

1. For the constant value of the error limit ε, the function $\rho = \rho(\varphi, l_2, \varepsilon)$, which measures the local relative mapping error, increases as the value of the radius vectors intersection angle decreases. This is the most important characteristic of this function.
2. For one particular and constant error limit ε and any value of angular coordinate φ, the function $\rho = \rho(\varphi, \alpha, l_2, \varepsilon)$ can become infinitely large if it satisfies the following condition:
$$|\alpha| \leq 2 \cdot \varepsilon \tag{4}$$
This condition will be reached in the numerical example exposed above if
$$|\alpha| \leq 2 \cdot 10^{-5} \tag{5}$$
3. For one particular value of angles α and ε, the function $\rho = \rho(\varphi, \alpha, l_2, \varepsilon)$ is very nearly constant and practically independent from the angular coordinate φ. Therefore, it can be concluded that the elliptical pencil of circles whose base points are L_{21} and L_{22} represents the geometrical locus of approximately constant relative errors ρ in the field F_2 of the mapping method based on Laguerre's points of the absolute involution mapped. The analogue conclusion can be formulated for the geometrical locus of approximately constant relative errors ρ in the field F_1.

5. Measure of the integral absolute and integral relative mapping error

The vertices A, B, C and D of the squares whose area $\Pi(\varphi, \psi, l_2, \varepsilon)$ represents the measure of the mapping method local absolute error belongs to the same elliptical pencil of circles whose base points are Laguere's points L_{21} and L_{22}. As shown in Fig. 1, all mapped points belong to the circular lunettes Λ_1 and Λ_2 whose area

$$\Pi(\Lambda_1, \Lambda_2) = \Pi(\Lambda_1) + \Pi(\Lambda_2) \tag{6}$$

decreases if the intersection angle α increases, and vice versa. It is essential to note that lunettes area $\Pi(\Lambda_1, \Lambda_2)$ can become infinitely large if the value of intersection angle α satisfies condition (4). This means that the error of the practical realization of the collinear mapping can be immeasurably large if the value of intersection angle α becomes less or equal to the doubled value of the error limit ε. From this considerations and the fact that all mapped points belong to the circular lunettes Λ_1 and Λ_2, it can be concluded that the area $\Pi(\Lambda_1, \Lambda_2)$ represents a measure of integral absolute error of the projective collinear mapping method based on the Laguerre's points.

According to the definitions of local relative and integral absolute mapping errors, the measure of the integral relative mapping error can be expressed by the ratio

$$R(\Lambda_1, \Lambda_2) = \Pi(\Lambda_1, \Lambda_2)/\Pi(\Lambda_{01}, \Lambda_{02}) \tag{7}$$

where $\Pi(\Lambda_{01}, \Lambda_{02})$ is the area of circular lunettes Λ_{01} and Λ_{02} shown in Fig. 2.

From these considerations, one can draw the conclusion that the reciprocal value $R^{-1}(\Lambda_1, \Lambda_2)$ of the measure of the integral relative mapping error represents a measure for the constructive graphical and numerical stability of the collinear mapping method based on Laguerre's points of the absolute involution mapped. The angle $\alpha_0 = 2\varepsilon$ is the critical intersection angle of the mapped points' radius vectors, and its numerical value depends on the technical instruments quality by which the collinear mapping is effectively performed.

6. The analyze deduction

The following deductions referring to the collinear mapping from the field F_1 to the field F_2 can be drawn from the exposed stability analyzes:

1. Every circle from the elliptical pencil of circles whose base points are L_{21} and L_{22} represents the geometrical locus of constant local relative errors ρ in the field F_2 of the mapping method based on Laguerre's points of the absolute involution mapped.
2. There are exactly two circles in the elliptical pencil whose base points are L_{21} and L_{22} which correspond to the critical value of the radius vectors intersection angle α_0. These circles represent the stability zone limits in the field F_2 of the mapping method based on Laguerre's points, and they are called circles of critical stability for those mapping method.
3. Circles of critical stability in the collinear fields F_1 and F_2 are associated by the projective collineation to the pair of hyperbolas in the fields F_2 and F_1 respectively. These hyperbolas represent the hyperbolas of critically conditioned mapping method based on Laguerre's points of the absolute involution mapped.

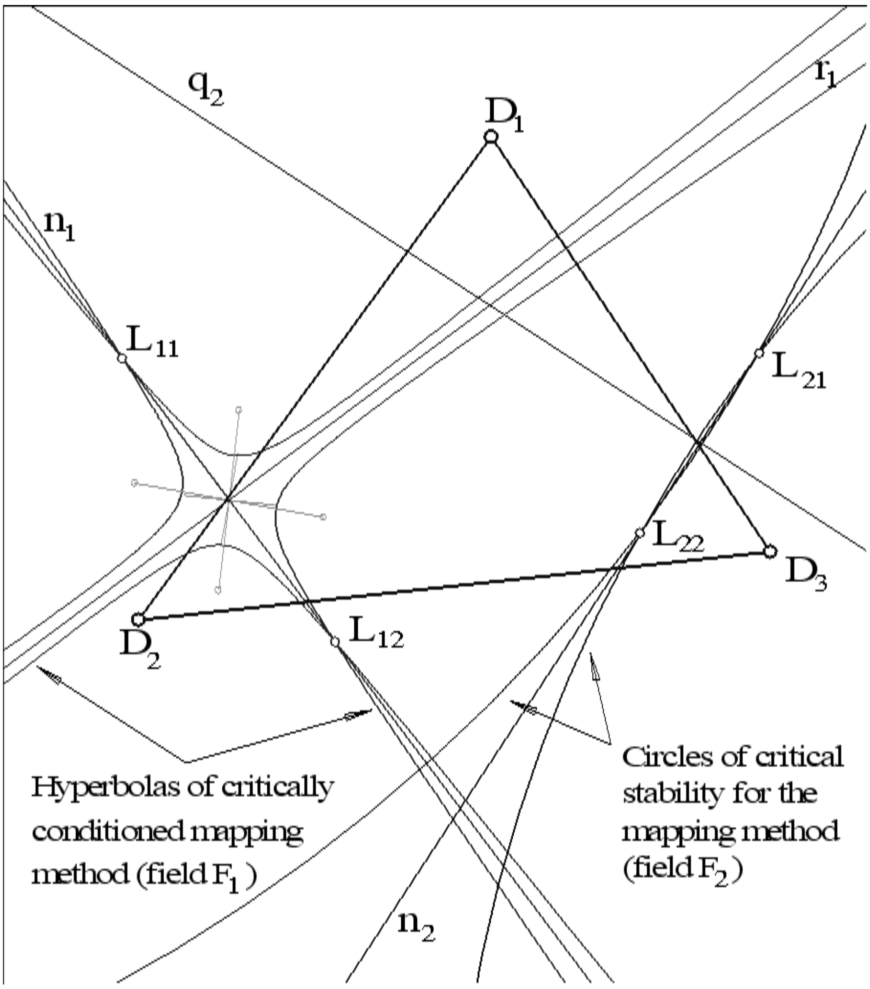

Figure 3: Hyperbolas of critically conditioned mapping method and circles of critical stability for the mapping method based on Laguerre's points of the absolute involution mapped

A pair of projective collinear fields F_1 and F_2 and the hyperbolas of critically conditioned mapping method are shown in Fig. 3, as well as the instability zones and associated ill-conditioned areas. For one particular value of the error limit ε by which the polar coordinates are determined, those hyperbolas correspond to one particular value of the critical angle $\alpha_0 = 2\varepsilon$. As clearly shown in Fig. 3, two ill-conditioned zones of these mapping method exist in the field F_1, the first of which comprises the vanishing line r_1 and the second one the principal normal line n_1. The pair of ill-conditioned zones of the same mapping method exists in the second field too, but they comprise the vanishing line q_2 and the principal normal line n_2. The most important consequence of these considerations is that the collinear mapping method based on Laguerre's points can not be applied in those ill-conditioned areas, and the alternative, well-conditioned mapping methods must be accomplished and performed in above mentioned zones.

7. The alternative mapping methods

In Fig. 4 fields of projective collineation, their vanishing lines r_1, q_2, and Laguerre's points L_{11}, L_{12}, L_{21}, L_{22} on the corresponding principal normal lines n_1 and n_2 are shown. It is evident that the collinearly associated points A_1 and A_2 belong to the straight lines s_1 and s_2, which are parallel to the vanishing line r_1, and q_2 in the collinear fields F_1 and F_2, respectively. From this facts and the theorem of the projectively associated points of the corresponding principal normal lines, the following equation can be formulated:

$$\eta \cdot \xi = \frac{l_1 \cdot l_2}{4} \tag{8}$$

where

- η, ξ is the distance between projectively associated points from the corresponding collinear field vanishing lines,
- l_1 is the distance between F_{11} and F_{12},
- l_2 is the distance between F_{21} and F_{22}.

This hyperbolical function of coordinates η and ξ can be applied directly in the ill-conditioned zones of the projective collinear mapping method based on Laguerre's points of the absolute involution mapped. Point A_1 on the straight line s_1 that is parallel to the principal normal line n_1 is shown on Fig. 4, as well as the radius vector p_{11} of the point A_1, its distance η from the line n_1, and angular coordinate φ_1. The distance ξ between mapped point A_2 and vanishing line q_2 is determined in the field F_2 by the above mentioned hyperbolical function. The position of the radius vector p_{21}, associated to the radius vector p_{11}, is found from the fact that the angular coordinate φ_2 is equal to the angular coordinate φ_1. The mapped point A_2 represents the intersection point between line s_2 and radius vector p_{21}. Since the intersection angle between lines s_2 and p_{21} is extremely close to the right angle for the points located very close to the principal normal line n_2, this mapping method shows a great stability in the mentioned area. This collinear transformation becomes unstable for the points which are extremely fare from the principal normal lines, and very close to the vanishing lines. From the above, it can be concluded that this alternative mapping method possesses stability precisely in those zones in which the classical procedure, based on Laguerre's points, is unstable.

Another collinear transformation, which can stabilize the classical mapping method, is represented in Fig. 5. It is well known that the pencil of straight lines, whose vertex is point O_1, is projectively transformed, from the field F_1 to the field F_2, into the pencil of parallel lines that are orthogonal to the vanishing line q_2. As is shown in Fig. 5, the position of each line, which belongs to the pencil (O_1), is determined in the field F_1 by the angular coordinate

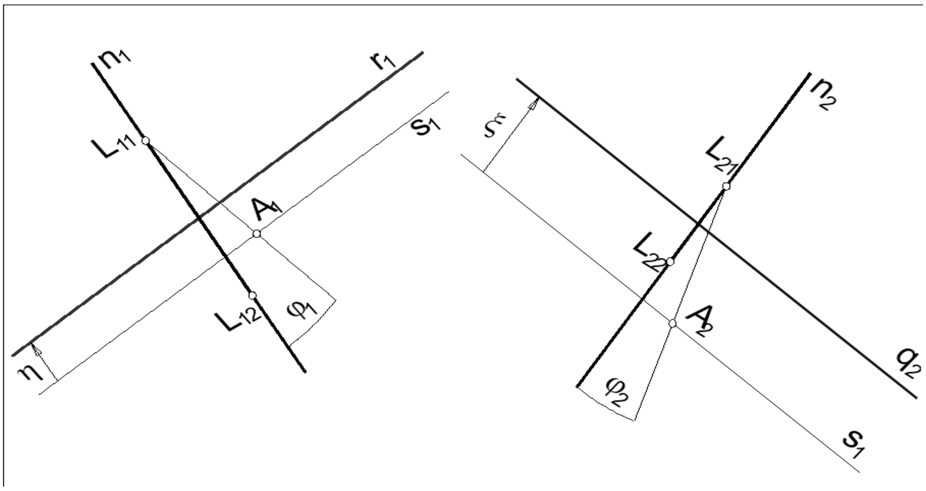

Figure 4: The alternative mapping method

φ, and the position of the corresponding line, which belongs to the projectively associated orthogonal pencil, is determined in the field F_2 by the linear coordinate ν. The following equation describes the relation between these two coordinates:

$$\nu = \frac{l_2 \cdot \tan\varphi}{2} \tag{9}$$

From the relations (8) and (9), the theorem of the coordinate orthogonal net can be formulated:

The collinear transformation of the angular coordinate φ and linear coordinate η of each point in the field F_1 into the pair of linear coordinates ν and ξ of the corresponding point in the field F_2, is described by following relations:

$$\nu = \frac{l_2 \cdot \tan\varphi}{2}, \quad \xi = \frac{l_1 \cdot l_2}{4 \cdot \eta}. \tag{10}$$

The linear coordinates ν and ξ of the mapped point represent the coordinate orthogonal netting in the field F_2.

The relations describing the collinear coordinate transformation from the field F_2 to the field F_1, as well as the coordinate orthogonal netting in the field F_1 can be formulated similarly to the relations (10).

It is important to emphasize that all finite mapped points are determined in the exposed mapping procedure by the intersection of orthogonal straight lines, which means that this collinear procedure is well-conditioned and stabile for all finite points in the pair of projectively associated collinear fields. This mapping method is ill-conditioned in the field F_1 and F_2 only for points whose coordinates have the following properties:

$$\begin{aligned} \varphi_1 &\rightarrow 90^0 \wedge \eta \rightarrow 0 \text{ (Field } \mathrm{F}_1) \\ \varphi_1 &\rightarrow 90^0 \wedge \eta \rightarrow 0 \text{ (Field } \mathrm{F}_2) \end{aligned}$$

and consequently unstable for points in the field F_2 and F_1 (respectively) which are extremely fare from the principal normal lines vanishing lines. From the above, it can be concluded that this mapping method can compensate the ill-conditioned and unstable collinear mapping transformations based on Laguerre's points of absolute involutions mapped.

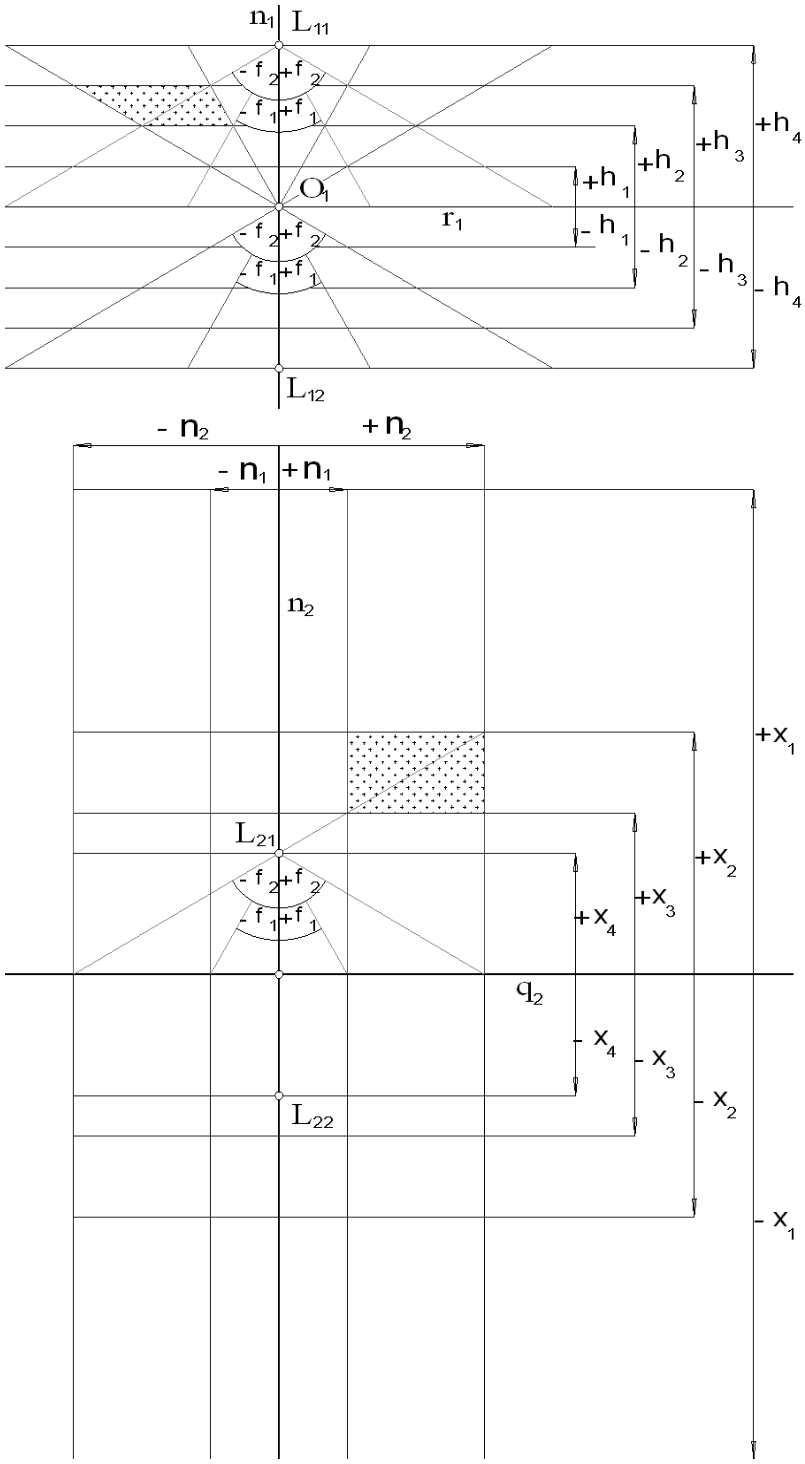

Figure 5: The associated coordinate nettings in the fields F_1 and F_2

8. Conclusion

This paper analyzes the stability of the projective collinear mapping method based on Laguerre's points of the involution mapped. The ill-conditioned and unstable zones for this collinear transformation are defined and some alternative procedures for its correction are proposed. As the result of this analysis, the stabile and well-conditioned general collinear fields mapping methods, which can be used in computer graphics and object model design of the collinear projective transformations, are created and explained. The exposed analysis is a contribution to the theory of computational and projective geometry; moreover, it makes the mapping procedures in software models of projective collineation more accurate and effective.

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