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Conic Solution of Euler's Triangle Determination Problem

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Abstract. We study Euler's problem of determination of a triangle from its circumcenter, orthocenter, and incenter as a problem of geometric construction. While it cannot be solved using ruler and compass, we construct the vertices by intersecting a circle with a rectangular hyperbola, both easily constructed from the given triangle centers.

Keywords: circumcenter, orthocenter, incenter, Feuerbach point, rectangular hyperbola

MSC 2007: 51M04, 51M15

1. Euler's triangle determination problem

Leonhard EULER studied in his famous paper [2] the problem of determining a triangle ABC from its circumcenter O, orthocenter H, and incenter I. Euler constructed a cubic polynomial whose roots are the lengths of the sides, and whose coefficients are rational functions of the distances among the three given triangle centers. He showed that when $OI = IH$, the cubic polynomial factors nontrivially, and gave the roots explicitly. With a numerical example, Euler showed that in the general case, the solution reduces to the trisection of an angle. In this note, we address Euler's determination problem as a construction problem. While the problem cannot be solved with the traditional restriction to ruler and compass, we shall nevertheless give the vertices as the intersections of two conics, one a circle and the other a rectangular hyperbola easily constructed from O, H, and I.

Let G and N be the points which divide OH in the ratio

$$OG : GN : NH = 2 : 1 : 3 .$$

These are the centroid and the nine-point center of the required triangle (if it exists). A necessary and sufficient condition for the existence of ABC is given by the following theorem. For details, see [4, 5, 6, 7].

Theorem 1 [A. GUINAND] *Let $\mathcal{D}$ be the open circular disk with diameter HG. A triangle ABC exists with circumcenter O, orthocenter H, and incenter I if and only if $I \in \mathcal{D} \setminus \{N\}$.*

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2. Ruler and compass construction of circumcircle and incircle

Let R and r denote respectively the circumradius and inradius of triangle ABC. In [2], EULER established, among other things, the famous relation

$$OI^2 = R(R - 2r). \tag{1}$$

With the help of the famous Feuerbach theorem [3], discovered half a century after Euler's paper [2], that the nine-point circle of a triangle is tangent internally to the incircle, we can easily construct the circumcircle, the incircle, the nine-point circle, and their point of tangency. According to the Feuerbach theorem,

$$NI = \frac{R}{2} - r. \tag{2}$$

Together with (1), this gives $2R \cdot NI = OI^2$, and suggests the following ruler-and-compass construction. See also [5], which studies the same problem from a paper-folding approach.

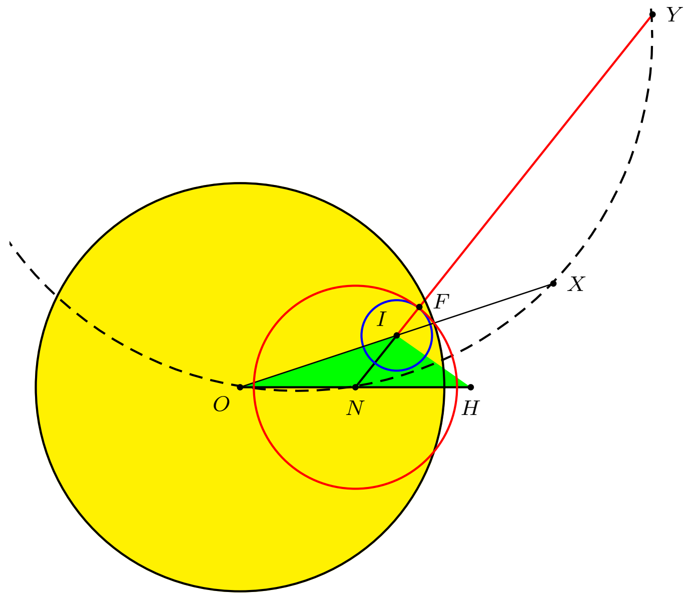

Figure 1: Incircle from O, H, I

Construction 2 Suppose O, H, I satisfy $I \in \mathcal{D} \setminus \{N\}$.

(1) Extend OI to X such that $OX = 2\,OI$; construct the circle ONX, and extend NI to intersect the circle again at Y. The length of IY *is twice the circumradius, and four times the radius of the nine-point circle.*

(2) Construct the circumcircle (O) and the nine-point circle (N).

(3) Construct the intersection F of the circle (N) with the half line NI, and the circle, center I, passing through F. *This is the incircle and F is the (Feuerbach) point of tangency with the nine-point circle* (see Fig. 1).

Starting with an arbitrary point A on (O), by drawing tangents, we can complete a triangle ABC with incircle (I) and circumcircle (O). The locus of the orthocenter of the variable triangle ABC is the circle, center P, passing through H. We determine the specific triangle with H as orthocenter. First we consider a special case where the constructibility with ruler and compass is evident from EULER's calculations.

3. The case $OI = IH$

If $OI = IH$, consider the intersection A of the half line NI with the circumcircle (see Fig. 2). We complete a triangle ABC with (O) as circumcircle and (I) as incircle. The orthocenter of triangle ABC lies on the reflection of AO in the line AI. This is the line AH. Since angle AHN is a right angle, for the midpoint M of AH, we have $NM = \frac{1}{2}AH = \frac{1}{2}AO = \frac{R}{2}$. This means that M is a point on the nine-point circle. It is the midpoint of the segment joining A to the orthocenter. It follows that the orthocenter must be H.

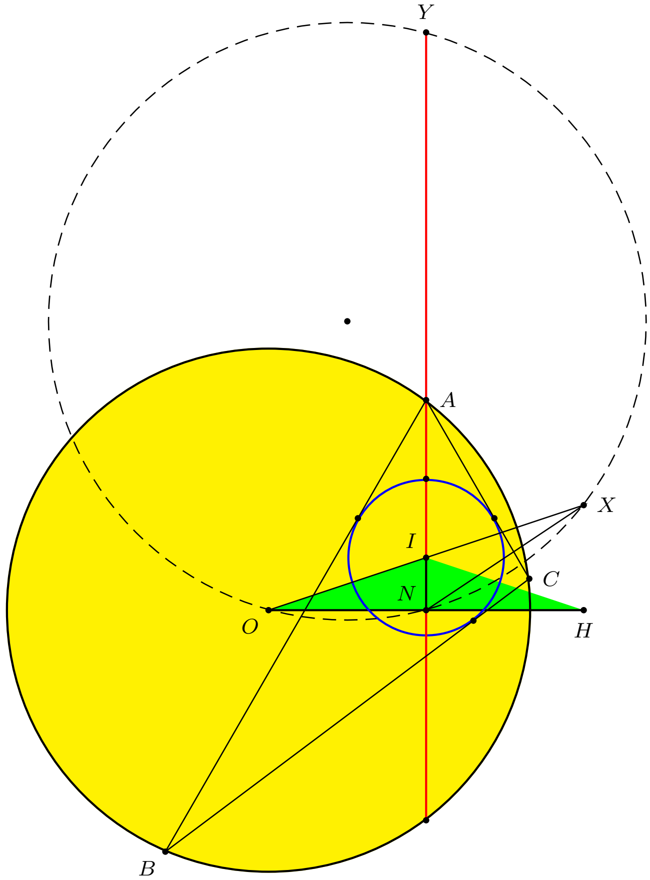

Figure 2: Triangle from O, H, I with $OI = IH$

Remark: One referee has kindly pointed us to [1], which shows that in this case the angle BAC must be $\frac{\pi}{3}$, and the Euler line cuts off an equilateral triangle with the sides AB and AC.

4. The general case: construction with the aid of a conic

Set up a cartesian coordinate system with origin at O. Assume $H = (k, 0)$ and $I = (p, q)$. Since $\angle HAI = \angle OAI$, and likewise for B and C, the vertices A, B, C are on the locus of the point P for which $\angle HPI = \angle OPI$. If $P = (x, y)$, a routine calculation shows that the locus of P is a curve $\mathcal{K}$: $K(x, y) = 0$, where

$$\begin{aligned} K(x,y) \;:=\; & 2qx^3 - (2p-k)x^2y + 2qxy^2 - (2p-k)y^3 \\ & - 2(p+k)qx^2 + 2(p^2-q^2)xy + 2(p-k)qy^2 \\ & + 2kpqx - k(p^2-q^2)y. \end{aligned}$$

Note that $K(k,0)=0$, i.e., $\mathcal{K}$ contains the point H.

By computing the circumradius R, we easily obtain the equation of the circumcircle $(O):\ G(x,y)=0$, where

$$G(x,y) := x^2+y^2-\frac{(p^2+q^2)^2}{(2p-k)^2+4q^2}.$$

It is possible to find a linear function $L(x,y)$ such that

$$Q(x,y) := K(x,y)-L(x,y)G(x,y)$$

does not contain third degree terms. For example, by choosing

$$L(x,y) := 2qx-(2p-k)y-2qk,$$

we have

$$\begin{aligned} Q(x,y) \;=\; & -2pqx^2+2(p^2-q^2)xy+2pqy^2 \\ & -\frac{k^2(k-4p)(p^2-q^2)+k(3p^2-5q^2)(p^2+q^2)+2p(p^2+q^2)^2}{(2p-k)^2+4q^2}\cdot x \\ & +\frac{2q(kp((2p-k)^2+4q^2)+(p^2+q^2)^2)}{(2p-k)^2+4q^2}\cdot y-\frac{2k(p^2+q^2)^2q}{(2p-k)^2+4q^2}. \end{aligned}$$

The finite intersections of (O) with $\mathcal{K}$ are precisely the same with the conic $\mathcal{C}$ defined by $Q(x,y)=0$. Note that the coefficients of x and y in L are dictated by the elimination of the third degree terms in $K-L\cdot G$. We have chosen the constant term such that $L(k,0)=0$, so that $L(x,y)=0$ represents the line HP parallel to NI. It follows that $Q(k,0)=0$, and the conic $\mathcal{C}$ contains the vertices and the orthocenter of the required triangle ABC. It is necessarily a rectangular hyperbola. This fact also follows from the factorization of the quadratic part of Q, namely, $-2(px+qy)(qx-py)$. This means that $\mathcal{C}$ is a rectangular hyperbola whose asymptotes have slopes q/p and $-p/q$. These are parallel and perpendicular to the segment OI.

To construct the rectangular hyperbola $\mathcal{C}$, we identify its center O'. This is the point with coordinates (u,v) for which the quadratic polynomial $Q(x-u,y-v)$ has no first degree terms in x and y. A routine calculation gives

$$\begin{aligned} O' &= \left(\frac{k((2p-k)^2-p^2+5q^2)+2p(p^2+q^2)}{2((2p-k)^2+4q^2)},\ \frac{(p^2+q^2-kp)q}{(2p-k)^2+4q^2}\right) \\ &= \left(\frac{k}{2}+\frac{(2p-k)(p^2+q^2)+2kq^2}{2((2p-k)^2+4q^2)},\ \frac{(p^2+q^2)q-kpq}{(2p-k)^2+4q^2}\right). \end{aligned} \tag{3}$$

Since $\mathcal{C}$ is a rectangular hyperbola, its center O' lies on the nine-point circle (N). The following observation leads to a very simple construction of the center.

Proposition 3 *The Feuerbach point F lies on the asymptote perpendicular to OI.*

Proof: The nine-point circle has the equation

$$\left(x-\frac{k}{2}\right)^2+y^2-\frac{(p^2+q^2)^2}{4((2p-k)^2+4q^2)}=0. \tag{4}$$

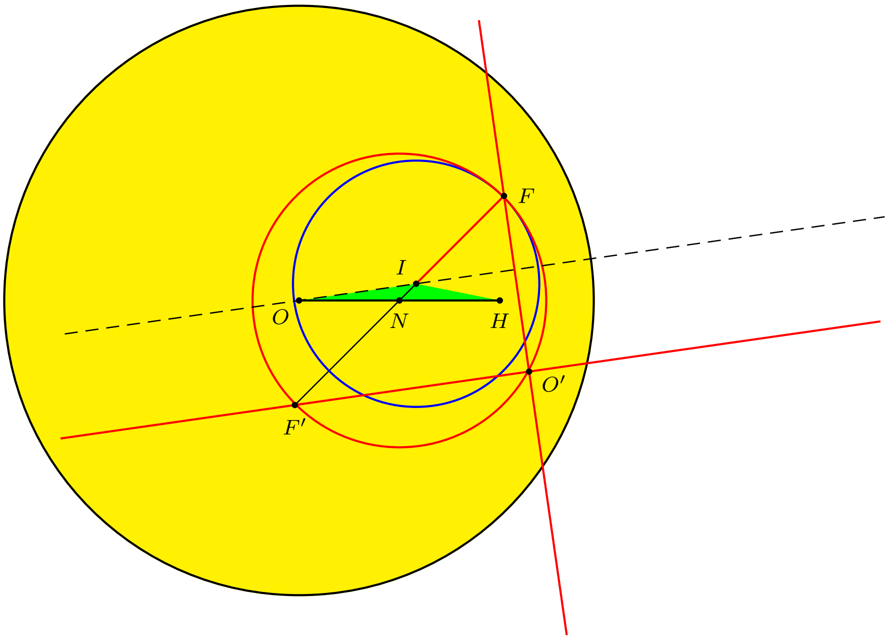

Figure 3: Center O' of rectangular hyperbola $\mathcal{C}$

This intersects the line NI at two points, the Feuerbach point and its antipode (on the nine-point circle). The Feuerbach point is the point

$$F = \left(\frac{k}{2} + \frac{(2p-k)(p^2+q^2)}{2((2p-k)^2+4q^2)}, \frac{(p^2+q^2)q}{(2p-k)^2+4q^2}\right).$$

It is easy to see that the line $O'F$ has slope $-\frac{p}{q}$, and is perpendicular to the line OI. □

Corollary 4 *The center O' of the rectangular hyperbola $\mathcal{C}$ is the second intersection of the nine-point circle (N) with the perpendicular from F to OI (Fig. 3).*

It is well known that if $\mathcal{C}$ is a rectangular hyperbola passes through the vertices of a triangle ABC, its fourth intersection with the circumcircle at the reflection of the orthocenter of the triangle in the center of the hyperbola. Therefore, one of the intersections of $\mathcal{C}$ with the circle (O) is the reflection of H in O'. The other three are the vertices of the required triangle (see Fig. 4).

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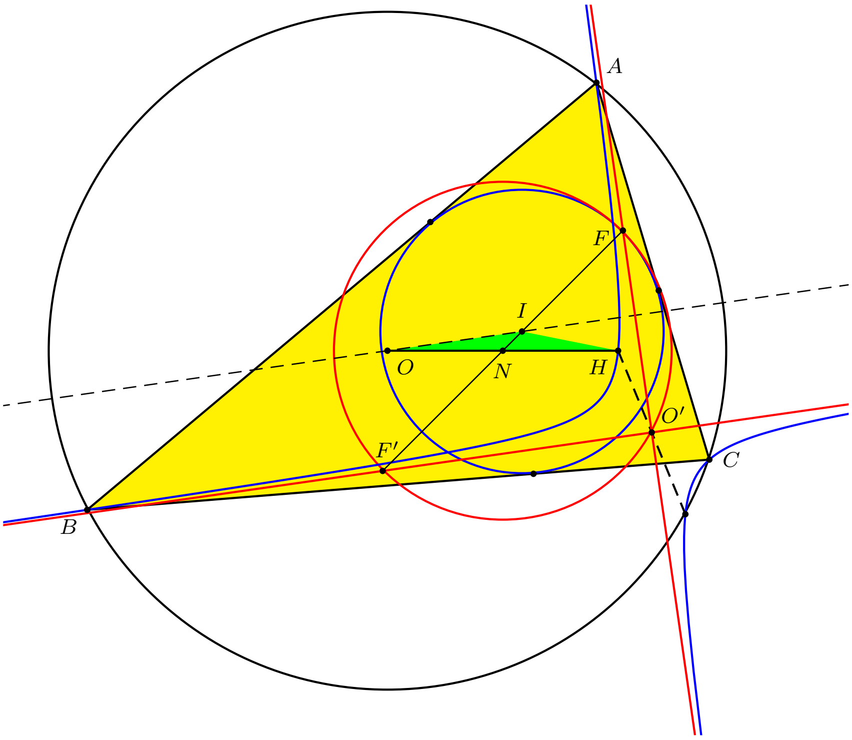

Figure 4: Triangle ABC with given O, H, I

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