

# Double Tangent Circles and Focal Properties of Sphero-Conics

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**Abstract.** We give two proofs for the characterization of a sphero-conic as locus of points such that the absolute value of the sum or difference of tangent distances to two fixed circles is constant. The first proof is based on methods of descriptive and projective geometry, the second is purely algebraic in nature. In contrast to earlier results, our proofs remain valid in case of purely imaginary tangent distances (when the sphero-conic is enclosed by both circles). Minor modifications make the algebraic proof work in the hyperbolic plane as well.

*Key Words:* spherical geometry, sphero-conic, double tangent circle, focal property, elliptic geometry, hyperbolic geometry

*MSC 2000:* 51M04, 51N05, 51N25

## 1. Introduction

In the usual focal definition of conic sections the distance to a focal point can be replaced by the *tangent distance* to a circle, that is, the distance between a point  $x$  and the point of tangency on either of the two circle tangents through  $x$ . The locus of points such that the absolute value of either sum or difference of tangent distances to two circles  $S_1, S_2$  is constant is a conic section  $C$ . The circles are tangent to  $C$  in two points (they are *double tangent circles* of  $C$ ) and their centers span an axis of  $C$ . Conversely, for every conic  $C$  the absolute sum or difference to two double tangent circles, centered on the same axis of  $C$ , is constant. The characterization of conics via sum or difference of tangent distances to double tangent circles applies to the three affine types of conics (ellipse, parabola, and hyperbola) and a similar extension is possible for the characterization of a conic via focal point and director line.

These results are known at least since the 19th century, see [1, 2, 5, 8, 10]. Among the various proofs, only that of [8] remains valid if  $S_1$  and  $S_2$  enclose the conic  $C$ . In this case the tangent distances are purely imaginary numbers.

Since the focal properties of sphero-conics are analogous to that of planar conics, it is natural to consider the same characterization on the sphere. Indeed, this is the topic of [3]. However, only double-tangent circles of equal radius that *inscribe* the sphero-conic are considered. (Although the extension to circles of different radius is straightforward.) We will

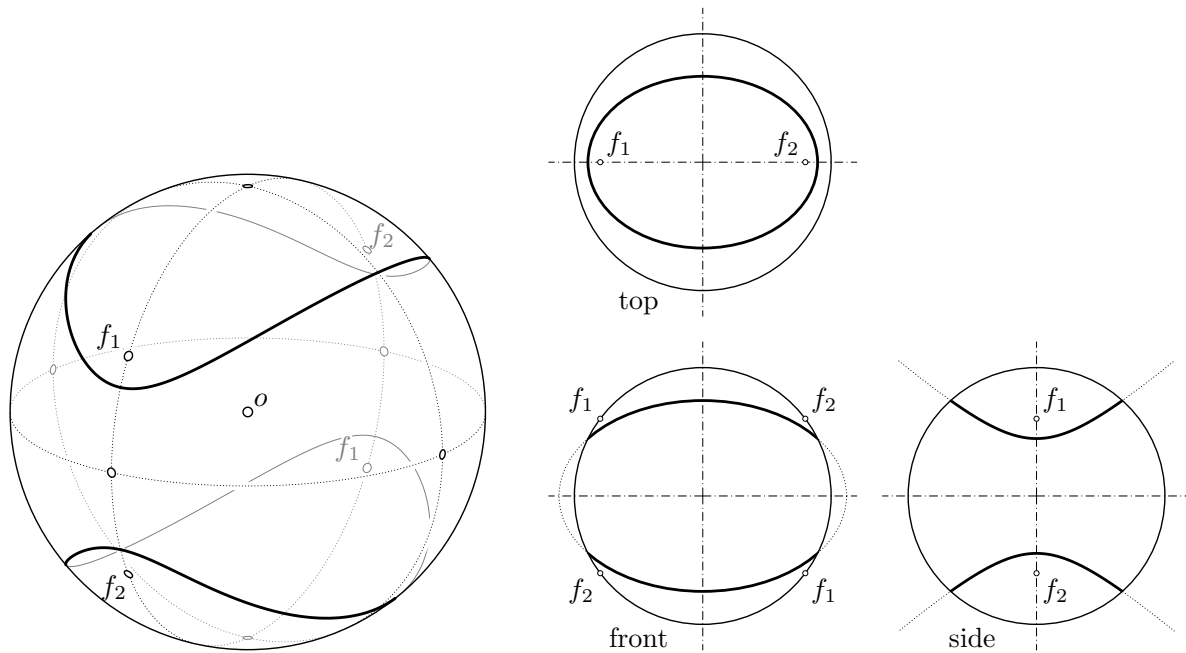


Figure 1: A sphero-conic and its focal points  $f_1, f_2$  in orthogonal projection and in top, front and side view

present two different proofs that remain valid in the most general cases, including *enclosing* double tangent circles and circles of different radius.

The first proof is based on elementary concepts of descriptive and projective geometry. In case of circumscribing circles we resort to descriptive geometry with imaginary elements. The second proof is inspired by the elegant proof of [8, p. 263] by means of the “method of abridged notation”. It is purely algebraic in nature and can easily be extended to the hyperbolic plane.

We believe that both methods of proof have their merits and the underlying techniques deserve attention, especially in the era of computer proofs with straightforward but often obscure calculations.

## 2. Preliminaries

### 2.1. Spherical and elliptic geometry

The locus of spherical geometry is the unit sphere  $S$  in Euclidean three space. The lines in this geometry are the great circles on  $S$ . By identifying antipodal points of  $S$  one obtains the spherical model of the elliptic plane. Its projection into the bundle about the sphere center  $o$  yields the bundle model, where the “points” are straight lines and the “lines” are planes through the center of  $S$ . The intersection of these lines and planes with a plane not incident with the sphere center transforms the bundle model into the projective Cayley-Klein model of planar elliptic geometry. We will formulate our results within the context of elliptic rather than spherical geometry. This is not really necessary but occasionally it is convenient to have at hand concepts of projective geometry.

The distance between two points  $a, b$  in the spherical model of elliptic geometry is defined as the length of the great arc segment connecting  $a$  and  $b$ . The distance function is two-valued. The two distances  $\delta, \delta' \in [0, \pi]$  add up to  $\pi$ .

## 2.2. Sphero-conics

A sphero-conic  $C$  is the intersection of  $S$  with a cone of second degree  $\Gamma$  whose vertex is the center  $o$  of  $S$ , see [6, pp. 154–161], [7, Chapter IX], or [9, pp. 259–260]. Hence,  $C$  is a spherical curve of degree four (Fig. 1, left-hand side).

The cone  $\Gamma$  has three planes of symmetry, pairwise perpendicular and incident with the sphere-center  $o$ . The orthogonal projections of  $C$  onto two of them are ellipses, the projection onto the third is a hyperbola. Projected points in the outside of  $S$  are the images of conjugate complex points of  $C$  (Fig. 1, right-hand side). Conversely, any spherical curve whose orthogonal projection is a conic that is concentric with the projection of the unit sphere center is (part of) a sphero-conic. The great circles in the planes of symmetry are called the axes of the sphero-conic.

It is well-known that a sphero-conic is the locus of points such that the sum of distances to two fixed points  $f_1, f_2$  is constant. At the same time, it is the locus of points such that the absolute difference of distances to  $f_1$  and  $f_2$  is constant. This is a consequence of the ambiguity of the elliptic distance. Thus, a sphero-conic comprises the focal properties of ellipse and hyperbola and, indeed, elliptic geometry only distinguishes between regular conics in general and circles.

## 3. Double tangent circles of sphero-conics

The center of a spherical circle  $S$  in double contact with a sphero-conic  $C$  lies on an axis of  $C$ . Accordingly, we can distinguish three families of double tangent circles (Fig. 2). The members of the first family are contained in the interior of  $C$ , the members of the second family enclose  $C$ , the members of the third family lie in the exterior of  $C$  but do not contain interior points of  $C$ . In an orthogonal projection onto any of the three planes of symmetry of  $C$  the members of one family of double tangent circles are in an edge-view. Except for members of the third family, the points of tangency may be complex. The focal points of  $C$  can be considered as double tangent circles of the first family with zero radius.

In [3], KACZMAREK and PRETKI prove that for every point of a sphero-conic the absolute sum or difference of tangent distances to two double tangent circles is constant. Their proof is, however, only valid for double tangent circles of the first and third family. Furthermore, KACZMAREK and PRETKI restrict their attention to circles of equal radius, probably because they also aim at defining a “generalized semi-axis length”. The extension of their proof to circles of different radius is straightforward but members of the second family are still excluded. We will give two new proofs that remain valid also in the most general situation.

### 3.1. Tangent distance to a circle

The tangent distance  $\delta$  from a point  $p$  to a spherical circle  $S$  of radius  $\varrho$  can be constructed in an elementary way. Assume at first that  $p$  is not contained in the interior of  $S$ . Clearly,  $\delta$  is the same for all points on the circle  $P$  through  $p$  and concentric with  $S$ . Consider now an edge-view of  $S$  (Fig. 3, left-hand side). Among all tangent circles of  $S$  we find two whose supporting plane is in edge-view as well. Rotating one of them into the drawing plane immediately yields the tangent distance  $\delta$ .

By the spherical version of Pythagoras’ Theorem we have

$$\cos \mu = \cos \varrho \cos \delta \tag{1}$$

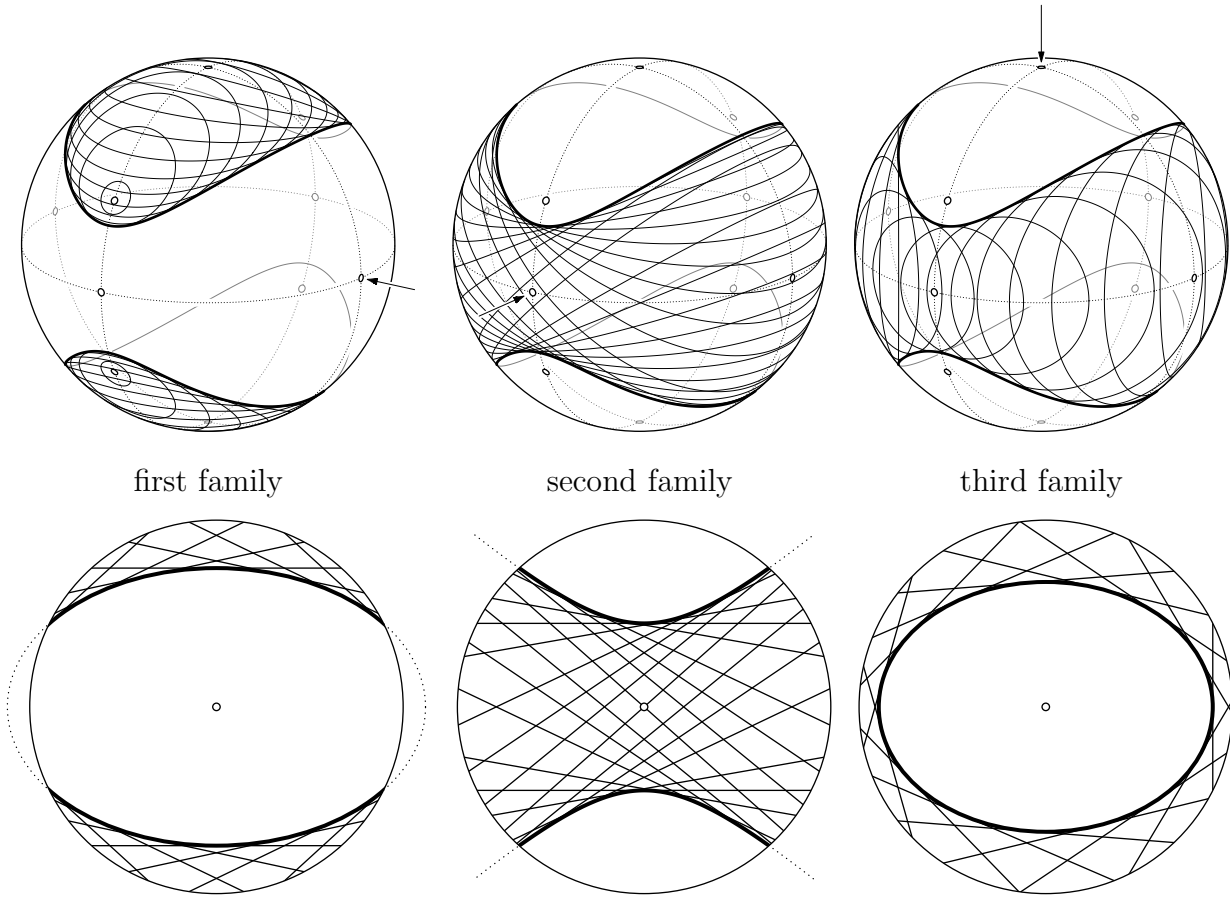


Figure 2: Double tangent circles of a sphero-conic

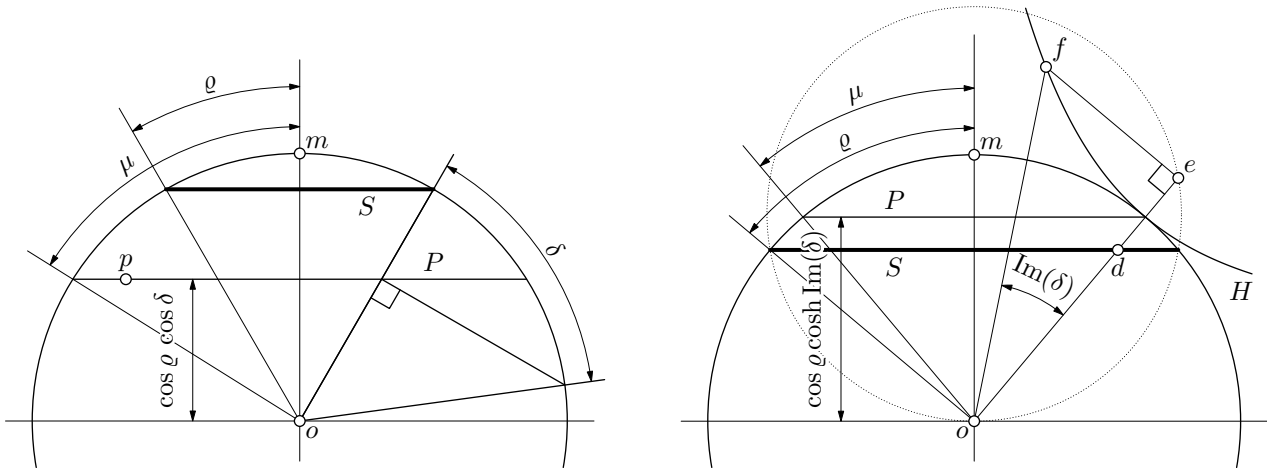


Figure 3: Real and purely imaginary tangent distance  $\delta$  to a spherical circle

where  $\mu$  is the distance of  $p$  and the center of  $S$  (Fig. 3 can be used to proof this theorem). If  $m$  is the center of  $S$ , the equation of the supporting plane of  $P$  is

$$\langle m, x \rangle = \cos \varrho \cos \delta \tag{2}$$

with the inner product  $\langle m, x \rangle = m_1x_1 + m_2x_2 + m_3x_3$ .

If  $p$  is contained in the interior of  $S$  the tangent distance can be defined by (1). Because of  $\mu < \varrho$  it is purely imaginary and its imaginary part is

$$\operatorname{Im}(\delta) = \operatorname{arccosh} \frac{\cos \mu}{\cos \varrho}. \tag{3}$$

There exists a construction of  $\operatorname{Im}(\delta)$  similar to that of a real tangent distance (see Fig. 3, right). Exchanging the role of  $\mu$  and  $\varrho$  we construct a point  $d$  whose distance to the unit sphere center  $o$  equals  $\cos \varrho / \cos \mu$ . By inversion of  $d$  at the unit circle we find a point  $e$  of distance  $\cos \mu / \cos \varrho$  to  $o$ . Using a right hyperbola  $H$  we finally obtain the angle  $\operatorname{Im}(\delta) = \operatorname{arccosh}(\cos \mu / \cos \varrho)$ .

If  $m$  is the center of  $S$ , the equation of the supporting plane of the circle  $P$  (the locus of all points with purely imaginary tangent distance  $\delta$ ) is

$$\langle m, x \rangle = \cos \varrho \cosh \operatorname{Im}(\delta). \tag{4}$$

### 3.2. Geometric proof

Now we are ready for the first proof of our central result.

**Theorem 1** *The locus of all points on the unit sphere such that the absolute sum or difference of tangent distances to two fixed circles  $S_1, S_2$  is constant is a sphero-conic with double tangent circles  $S_1$  and  $S_2$ . The centers of  $S_1$  and  $S_2$  lie on the same axis of  $C$ .*

*Proof:* We consider the case of real tangent distances at first. In an edge view of  $S_1$  and  $S_2$  we have to show that the projection of the sought locus  $C$  is a conic that is concentric with the projection of the sphere center and tangent to  $S_1$  and  $S_2$  (Fig. 4, left-hand side). We start with two points  $c_1, c_2$  of spherical distance  $\delta$  on the great arc  $G$  through the centers of  $S_1$  and  $S_2$ . For  $i \in \{1, 2\}$  we denote one of the elliptic orthogonal projections of  $c_1$  on  $S_i$  by  $s_i$  and the distance from  $c_i$  to  $s_i$  by  $\delta_i$ .

According to Fig. 3, left, we construct a point  $p$  whose tangent distance to  $S_i$  equals  $\delta_i$ . Rotating the points  $c_1$  and  $c_2$  along  $G$  while preserving their constant distance  $\delta$  we can construct further positions of points  $p$ . The absolute sum or difference of tangent distances to  $S_1$  and  $S_2$  from these points equals  $\sigma - \delta$  where  $\sigma$  is the distance of  $s_1$  and  $s_2$ . Hence, the absolute sum or difference of tangent distances is constant and  $C$  is the sought locus.

If  $g(\varphi)$  is an arc-length parametrization of  $G$  we may think of  $c_1$  and  $c_2$  as depending on the parameter  $\varphi$ , for example  $c_1 = g(\varphi), c_2 = g(\varphi + \delta)$ . According to (2), the points  $p(\varphi)$  lie in the intersection of the planes

$$P_1(\varphi): \langle m_1, x \rangle - \cos \varrho_1 \cos \varphi = 0, \quad P_2(\varphi): \langle m_2, x \rangle - \cos \varrho_2 \cos(\varphi + \delta) = 0. \tag{5}$$

The substitution  $\varphi = 2 \arctan t$  yields rational parametrizations of degree two for the two pencils of parallel planes  $P_1(t), P_2(t)$  and a rational parametrizations  $p(t)$  of degree four for the projected intersection locus. However, the planes  $P_1(\pm i)$  and  $P_2(\pm i)$  coincide with the plane at infinity. Therefore, the factor  $1 + t^2$  can be split off from the parametrization  $p(t)$  and a rational parametrization of degree two remains. The locus  $C$  of points  $p(t)$  is a conic section.

The projection of  $C$  is an ellipse because it is bounded by the supporting planes of  $S_1$  and  $S_2$ . Furthermore, it is symmetric with respect to  $o$ . Therefore,  $C$  itself is a sphero-conic with axis  $G$ . For  $c_i(\varphi) = s_i$ , the projection of  $p(\varphi)$  lies on the projection of  $S_i$ . Hence,  $S_1$  and  $S_2$  are double tangent circles of  $C$ .

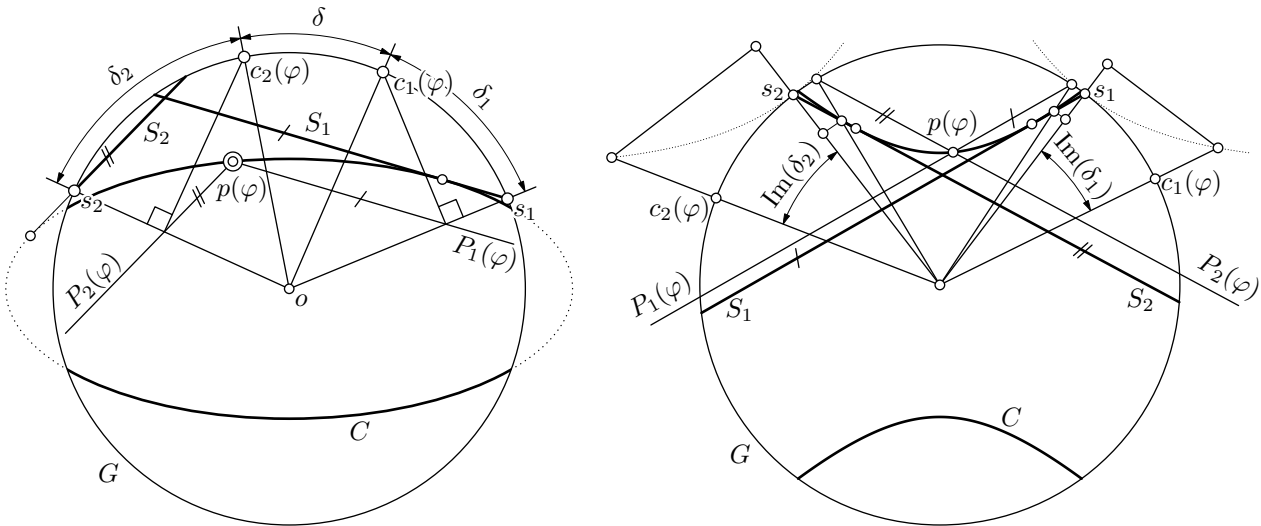


Figure 4: Locus  $C$  of constant absolute tangent distance sum or difference

The case of purely imaginary tangent distances can be treated in similar fashion. The according construction is shown in Fig. 4, right-hand side. It is derived from the construction of Fig. 3, right, but the right hyperbolas' vertices are drawn at the points  $s_1$  and  $s_2$ .

The distance between  $c_1$  and  $c_2$  is  $\text{Im}(\delta)$  where  $\delta \in i\mathbb{R}$ . We consider an arc-length parametrization  $g(\varphi)$  of the great circle  $G$  through the centers of  $S_1$  and  $S_2$  and let  $c_1(\varphi) = g(\varphi)$ ,  $c_2(\varphi) = g(\varphi + \text{Im}(\delta))$ . Re-parameterizing via  $\varphi = 2 \operatorname{arctanh} t$  we obtain, by virtue of (4), rational parametrizations of degree two of the pencils of parallel planes  $P_1(t)$  and  $P_2(t)$  and a rational parametrization  $p(t)$  of degree four of the sought locus  $C$ . The planes  $P_1(t)$  and  $P_2(t)$  become infinite for  $t = \pm 1$  and the factor  $1 - t^2$  can be split off from  $p(t)$ . The projection of  $C$  is a conic section. It is symmetric with respect to  $o$  and its real points lie on the side of the supporting plane of  $S_i$  that does not contain  $o$ . Therefore, its projection is a hyperbola. For  $c_i(\varphi) = s_i$  we obtain the point of tangency on  $S_i$ . This shows that  $C$  is indeed a sphero-conic with enclosing double tangent circles  $S_1, S_2$  and completes the proof.  $\square$

**Theorem 2** *If  $C$  is a sphero-conic and  $S_1, S_2$  are double tangent circles with centers on the same axis of  $C$ , the absolute sum or distance of tangent distances from points of  $C$  to  $S_1$  or  $S_2$  is constant.*

*Proof:* Two double-tangent circles  $S_1$  and  $S_2$  and a point  $c$  define two sphero-conics  $C, C'$ . Furthermore, the circles  $S_1, S_2$  and the point  $c$  define two loci  $D, D'$  of constant absolute sum or difference of tangent distances. If  $\gamma_1$  and  $\gamma_2$  are the tangent distances from  $c$  to  $S_1$  and  $S_2$  one locus corresponds to the absolute sum or difference  $\delta = |\gamma_1 + \gamma_2|$  and the other to  $\delta = |\gamma_1 - \gamma_2|$ . Hence we either have  $C = D, C' = D'$  or  $C = D', C' = D$  and the theorem is proved.  $\square$

### 3.3. Proof by abridged notation

In this section we give an algebraic proof of Theorems 1 and 2. It is inspired by the proof the planar version of these theorems by G. SALMON in [8, p. 263] that uses algebraic equations but with certain abbreviation terms that are never expanded. Instead, their geometric meaning is interpreted appropriately. SALMON calls this the “method of abridged notation”.

The equation of a spherical circle with normalized center  $m$  and radius  $\varrho$  reads

$$S: \langle x, x \rangle \cos^2 \varrho - \langle m, x \rangle^2 = 0. \quad (6)$$

where

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 \quad (7)$$

is the usual scalar product in  $\mathbb{R}^3$  [7, p. 176].

We now consider two circles  $S_1, S_2$  with equations of the shape (6). Any sphero-conic  $C$  having double contact with  $S_1$  and  $S_2$  can be written as

$$C: \lambda^2 E^2 - 2\lambda(\cos^2 \varrho_1 S_2 + \cos^2 \varrho_2 S_1) + F^2 = 0 \quad (8)$$

where

$$E = \cos \varrho_1 \langle m_2, x \rangle + \cos \varrho_2 \langle m_1, x \rangle, \quad F = \cos \varrho_1 \langle m_2, x \rangle - \cos \varrho_2 \langle m_1, x \rangle. \quad (9)$$

This can be seen from the fact that  $E$  and  $F$  are linear in  $x$  and the equation of  $C$  can be written as

$$(\lambda E + F)^2 - 4\lambda \cos^2 \varrho_2 S_1 = 0 \quad \text{or} \quad (\lambda E - F)^2 - 4\lambda \cos^2 \varrho_1 S_2 = 0. \quad (10)$$

Solving (8) for  $\lambda$  we obtain

$$\lambda = \frac{(\cos \varrho_1 \sqrt{S_2} \pm \cos \varrho_2 \sqrt{S_1})^2}{(\cos \varrho_1 \langle m_2, x \rangle + \cos \varrho_2 \langle m_1, x \rangle)^2}. \quad (11)$$

One of these two equations is satisfied by every point  $x$  of  $C$ . The points of tangency between  $C$  and  $S_1$  or  $S_2$  satisfy both. If they are real and different, they separate segments on  $C$  where one or the other of the two equations (11) is valid.

Assume now that  $x$  is normalized as well, that is  $\langle x, x \rangle = 1$ . Then the expression  $\langle m_i, x \rangle$  equals  $\cos \mu_i$  where  $\mu_i$  is the distance between  $x$  and  $m_i$ , and the expression  $S_i$  evaluates to  $\cos^2 \varrho_i - \cos^2 \mu_i$ . Furthermore, the numbers  $\mu_i$  and  $\varrho_i$  are related to the tangent distance  $\delta_i$  from  $x$  to  $S_i$  via the spherical version of Pythagoras' Theorem:

$$\cos \mu_i = \cos \varrho_i \cos \delta_i. \quad (12)$$

We insert this into (11) and, using the second addition formulas

$$\begin{aligned} \sin \delta_1 \pm \sin \delta_2 &= 2 \sin\left(\frac{1}{2}(\delta_1 \pm \delta_2)\right) \cos\left(\frac{1}{2}(\delta_1 \mp \delta_2)\right), \\ \cos \delta_1 + \sin \delta_2 &= 2 \cos\left(\frac{1}{2}(\delta_1 + \delta_2)\right) \cos\left(\frac{1}{2}(\delta_1 - \delta_2)\right), \end{aligned} \quad (13)$$

we find an expression for the constant  $\lambda$  that depends only on the sum or difference of the tangent distances  $\delta_1$  and  $\delta_2$ :

$$\lambda = \frac{(\sin \delta_1 \pm \sin \delta_2)^2}{(\cos \delta_1 + \cos \delta_2)^2} = \tan^2\left(\frac{\delta_1 \pm \delta_2}{2}\right). \quad (14)$$

Hence, for every point of  $C$  either the absolute sum or the absolute difference of tangent distances  $\delta_i$  is constant. This finishes the second proof of Theorem 2. Its converse, Theorem 1 can be shown by reversing our arguments.

### 3.4. Generalized focal property in the hyperbolic plane

Because of the algebraic equivalence of complex elliptic and hyperbolic geometry [4, Kapitel VII, §3] the proof by abridged notation of Theorems 1 and 2 can easily be adapted to the hyperbolic plane. We have to use the hyperbolic scalar product

$$\langle x, y \rangle_h = x_1y_1 + x_2y_2 - x_3y_3 \quad (15)$$

instead of (7) and replace  $\varrho_i, \mu_i, \delta_i$  by  $i\varrho_i, i\mu_i, i\delta_i$ , respectively. The only difficulty arises if one of the two double tangent circles is a *horocycle*, that is, a circle with center  $m$  that cannot be normalized because of  $\langle m, m \rangle = 0$ . These cases can, however, be proved by an appropriate passage to the limit.

All in all, we have the following characterization of conics in the elliptic and hyperbolic plane:

**Theorem 3** *The locus of all points in the elliptic or hyperbolic plane such that the absolute sum or difference of tangent distances to two fixed circles  $S_1, S_2$  is constant is a conic with double tangent circles  $S_1$  and  $S_2$ . The centers of  $S_1$  and  $S_2$  lie on the same axis of  $C$ . Conversely, for any conic with these properties, the absolute sum or difference of tangent distances to  $S_1$  and  $S_2$  is constant.*

## 4. Conclusion

We presented two proofs for the characterization of sphero-conics as locus of points with constant absolute sum or difference of tangent distances to two fixed circles. This result is analogous to an old characterization of planar conics. Our proofs are based on concepts from descriptive and projective geometry or, in the second case, on the “method of abridged notation”. In contrast to an earlier result by KACZMAREK and PRETKI in [3], both proofs also cover the case of purely imaginary tangent distances. The algebraic proof can easily be extended to the hyperbolic plane.

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