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Reflection-Induced Perspectivities Among Triangles

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Abstract. In the plane of a reference triangle ABC, let DEF be a triangle and P a point. A line U through P is defined as a successful reflector through P if the reflection $D'E'F'$ of DEF in U is perspective to ABC, in the sense that the lines AD', BE', CF' concur. The point of concurrence is the perspector, $\mathcal{P}(DEF, P, U)$. The main theorem is that for given DEF and P, there are either infinitely many successful reflectors through P, or else there are at most four. Examples include choices of DEF and P for which the Steiner axes are the successful reflectors, and also choices for which there are infinitely many successful reflectors, in which case P is called a *pivot* for DEF. In three of the examples, the locus of $\mathcal{P}(DEF, P, U)$ as U rotates about P, called the *pivotal curve*, is the isogonal conjugate of a circle.

Key Words: barycentric coordinates, centroid, circumcircle, Euler triangle, extouch triangle, Gossard triangle, hexyl triangle, incircle, intouch triangle, isogonal conjugate, Kosnita triangle, medial triangle, pivot, pivotal curve, reflection, successful reflector, tangential triangle

MSC 2000: 51N20, 14H50, 15A15

1. Introduction

With the help of computers, many new relationships in the plane of a triangle have been discovered during the past few years. At first, a newcomer might assume that these discoveries depend on graphics software and that the discovery-making occurs visually. However, much of the new geometry is actually of a much more algebraic character, stemming from representations of points, lines, circles, and other curves using homogeneous coordinates. For

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example, almost every special point in the plane of a triangle has simple algebraic representations, and these lend themselves to algebraic criteria – most notably zero determinants for properties such as collinearity of three points, concyclicity of four points, concurrence of three lines, and so on. In that vein, this article introduces what appears to be a new kind of reflection theorem and shows that the axes of the Steiner ellipse play a prominent role.

We begin with a few terms and notation. In homogeneous barycentric coordinates (henceforth *barycentrics*), a point will be written as $\alpha : \beta : \gamma$. Points include the vertices and centroid of a reference triangle ABC, given by

$$A = 1 : 0 : 0, \quad B = 0 : 1 : 0, \quad C = 0 : 0 : 1, \quad G = 1 : 1 : 1.$$

The points

$$D = AG \cap BC, \quad E = BG \cap CA, \quad F = CG \cap AB$$

are the vertices of the medial triangle. Their anticomplements,

$$0 : 1 : -1, \quad -1 : 0 : 1, \quad 1 : -1 : 0,$$

are the vertices of the anticomplementary triangle. The line at infinity, $\mathcal{L}^\infty$, is given by the equation $\alpha + \beta + \gamma = 0$. These and many other objects in triangle geometry (and triangle algebra, in which the symbols a, b, c represent indeterminates or variables rather than sidelengths of a triangle) are developed elsewhere (e.g., [3], [4], [12], [13]).

2. Main theorem

Suppose that U, given by the barycentric equation $u\alpha + v\beta + w\gamma = 0$, is a line other than $\mathcal{L}^\infty$. The reflection $R(X)$ of a point $X = x : y : z$ in U is the point given by

$$\begin{aligned} R(X) \;&=\; f(a,b,c,u,v,w)x + g(a,b,c,u,v,w)(vy + wz) \;: \\ &\quad\; f(b,c,a,v,w,u)y + g(b,c,a,v,w,u)(wz + ux) \;: \\ &\quad\; f(c,a,b,w,u,v)z + g(c,a,b,w,u,w)(ux + vy), \end{aligned}$$

where

$$\begin{aligned} f(a,b,c,u,v,w) \;&=\; a^2u^2 - b^2v^2 - c^2w^2 + vw(b^2 + c^2 - a^2) \\ g(a,b,c,u,v,w) \;&=\; 2a^2u - v(a^2 + b^2 - c^2) - w(c^2 + a^2 - b^2). \end{aligned}$$

The procedure for obtaining this result is straightforward, following these steps: write a barycentric equation for the line U' through X perpendicular ([3, p. 29], [12], [13]) to U, find barycentrics for the point $Q = U \cap U'$, and then find $R(X)$ as the translation of Q by the vector $\overrightarrow{XQ}$.

Suppose that ABC and DEF are triangles, and $P = p : q : r$ is a point. A line U through P is a *successful reflector through* P if the reflection $D'E'F'$ of DEF in U is perspective to ABC, in the sense that the lines AD', BE', CF' concur. We shall sometimes say merely that a line is a successful reflector of a triangle DEF; in that case P can be any point on the line.

A standard criterion for three lines L_1, L_2, L_3 given by

$$\begin{aligned} l_1\alpha + l_2\beta + l_3\gamma \;&=\; 0, &(1)\\ l_4\alpha + l_5\beta + l_6\gamma \;&=\; 0, &(2)\\ l_7\alpha + l_8\beta + l_9\gamma \;&=\; 0, &(3) \end{aligned}$$

to concur ([3, p. 28-29], [12], [13]) is that $\Delta = 0$, where

$$\Delta = \begin{vmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{vmatrix}. \tag{4}$$

We shall refer to Δ as a *concurrence determinant* of the three lines, and to the equation $\Delta = 0$ as a *concurrence equation.*

Theorem 1 *Suppose ABC and DEF are triangles, and $P = p : q : r$ is a point. Either there are infinitely many successful reflectors through P, or else there are at most four successful reflectors through P.*

Proof: The line at infinity is given parametrically by

$$x_t = 1 + t \quad y_t = -t \quad z_t = -1,$$

where $t = t(a, b, c)$ is homogeneous of degree 0 in a, b, c. Let U_t be the line $u\alpha + v\beta + w\gamma = 0$ of the points P and $x_t : y_t : z_t$, so that

$$u = -q + rt \quad v = p + r + rt \quad w = -q - pt - qt.$$

Represent the vertices of DEF and $D'E'F' = D_tE_tF_t$ as (the rows of) matrices

$$\begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{pmatrix},$$

respectively, so that

$$\begin{aligned}
u_1 &= f(a, b, c, u, v, w)d_1 + g(a, b, c, u, v, w)(vd_2 + wd_3) \\
u_2 &= f(b, c, a, v, w, u)d_2 + g(b, c, a, v, w, u)(wd_3 + ud_1) \\
u_3 &= f(c, a, b, w, u, v)d_3 + g(c, a, b, w, u, v)(ud_1 + vd_2) \\
u_4 &= f(a, b, c, u, v, w)d_4 + g(a, b, c, u, v, w)(vd_5 + wd_6) \\
u_5 &= f(b, c, a, v, w, u)d_5 + g(b, c, a, v, w, u)(wd_6 + ud_4) \\
u_6 &= f(c, a, b, w, u, v)d_6 + g(c, a, b, w, u, v)(ud_4 + vd_5) \\
u_7 &= f(a, b, c, u, v, w)d_7 + g(a, b, c, u, v, w)(vd_8 + wd_9) \\
u_8 &= f(b, c, a, v, w, u)d_8 + g(b, c, a, v, w, u)(wd_9 + ud_7) \\
u_9 &= f(c, a, b, w, u, v)d_9 + g(c, a, b, w, u, v)(ud_7 + vd_8).
\end{aligned}$$

We are interested in values of t for which the lines AD_t, BE_t, CF_t concur. Write these lines, respectively, as in (1). The concurrence equation $\Delta = 0$, with Δ as in (4), can now be written as

$$u_2u_6u_7 - u_3u_4u_8 = 0. \tag{5}$$

The left-hand side is a polynomial $\mathcal{P}(t)$ formally of degree 6 in t. If for a particular choice of P all the coefficients of powers of t are zero, then for all t, the line joining the points P and $x_t : y_t : z_t$ is a successful reflector through P. For the rest of this proof, assume that not all the coefficients are zero. Then, in spite of its hundreds of terms, the polynomial $\mathcal{P}(t)$ is found by computer to have

$$c^2t^2 + (b^2 + c^2 - a^2)t + b^2$$

as a factor. The discriminant is $-16\sigma^2$, where σ denotes the area of ABC. Thus, two of the roots of $\mathcal{P}(t)$ are nonreal, so that $\mathcal{P}(t)$ has at most four real roots. Consequently, there are at most four successful reflectors through P. □

3. Steiner axes

The isotomic conjugate of $\mathcal{L}^\infty$ is the Steiner ellipse, $\mathcal{E}$, also called the Steiner circumellipse in order to distinguish it from the Steiner inellipse; see [12] for details. A barycentric equation for $\mathcal{E}$ is

$$\alpha/a + \beta/b + \gamma/c = 0.$$

This ellipse meets the circumcircle (which is the *isogonal* conjugate of $\mathcal{L}^\infty$) in these four points: A, B, C, and the Steiner point, indexed as X_{99} in the *Encyclopedia of Triangle Centers* [4]. Barycentric equations for the major and minor axes of the Steiner ellipse are given in terms of the roots $\pm W$ of

$$W^2 = a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2 \tag{6}$$

and the functions

$$\begin{aligned} s_a &= s(a,b,c) = (b^2-c^2)(b^4+c^4-a^2b^2-a^2c^2) \\ t_a &= t(a,b,c) = (b^2-c^2)(a^2-b^2-c^2) \end{aligned}$$

as follows:

$$(s_a + t_aW)\alpha + (s_b + t_bW)\beta + (s_c + t_cW)\gamma = 0, \tag{7}$$
$$(s_a - t_aW)\alpha + (s_b - t_bW)\beta + (s_c - t_cW)\gamma = 0, \tag{8}$$

respectively.

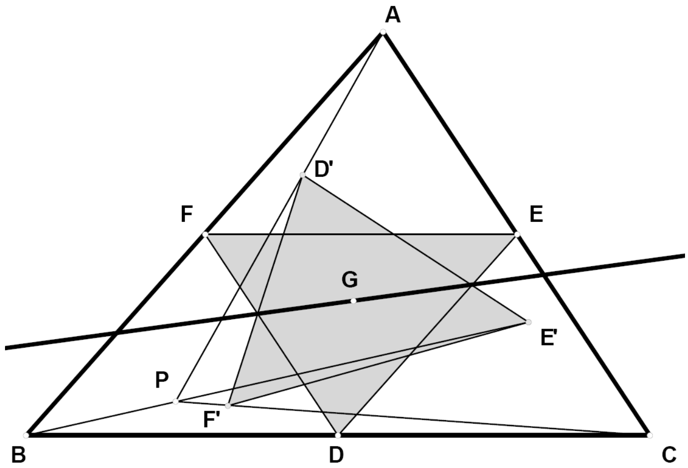

Figure 1: Steiner major axis as a successful reflector

Our colleague Paul Yiu, having seen a preliminary draft of Theorem 1 and an example involving the medial and anticomplementary triangles, noted that the reflectors are the Steiner axes, as proved in Theorems 2 and 3.

Theorem 2 *The medial triangle has exactly two successful reflectors through G. They are the Steiner axes of ABC.*

Proof: A computer-assisted proof is straightforward using barycentrics. Let U be an arbitrary line through G, and let $D'E'F'$ be the reflection in U of the medial triangle DEF. Let $X = x : y : z$ be a point on U, so that U is given by

$$u\alpha + v\beta + w\gamma = 0,$$

where $u = y - z$, $v = z - x$, $w = x - y$. The concurrence determinant factors as $\mathcal{C}_1\mathcal{C}_2^2$, where

$$\begin{aligned}\mathcal{C}_1 &= (b^2-c^2)(x^2+2yz)+(c^2-a^2)(y^2+2zx)+(a^2-b^2)(z^2+2xy), \qquad (9)\\ \mathcal{C}_2 &= f(a,b,c)x^2+f(b,c,a)y^2+f(c,a,b)z^2\\ &\quad +g(a,b,c)yz+g(b,c,a)zx+g(c,a,b)xy,\end{aligned}$$

where

$$\begin{aligned}f(a,b,c) &= a^2-2b^2-2c^2\\ g(a,b,c) &= 5a^2-b^2-c^2.\end{aligned}$$

Now, multiplying together the left-hand sides of (7) and (8), and substituting for W^2 as in (6), we find that the product, after cancellations, is $\mathcal{C}_1$. That is, the conic $\mathcal{C}_1 = 0$ is the union of the Steiner axes. Thus, regarding (4), for any X on either Steiner axis, we have $\Delta = 0$. The only other way to have $\Delta = 0$ is for X to lie on the conic $\mathcal{C}_2 = 0$; however, this conic has center G and also passes through G, so that it is entirely imaginary except for the single point G. □

The next theorem concerns the anticomplementary triangle. We digress for a moment to clarify just which triangle this is, because in recent literature, it is sometimes called the *antimedial triangle* or the *precevian triangle of the centroid*. It is the triangle whose medial triangle is the reference triangle ABC. For example, its A-sideline is the line through A parallel to line BC. The traditional name refers to the fact that the vertices are the anticomplements of A, B, C. Correspondingly, the medial triangle is sometimes called the *complementary triangle*.

Theorem 3 *The anticomplementary triangle has exactly two successful reflectors through G. They are the Steiner axes of ABC.*

Proof: A proof is nearly identical to that of Theorem 2. □

The next theorem reveals a distinctive property of the 1st Brocard triangle [8]. The proof depends on the fact that two lines are parallel if and only if they meet in the same point on $\mathcal{L}^\infty$.

Theorem 4 *Every line parallel to the Steiner major axis or the Steiner minor axis is a successful reflector of the 1st Brocard triangle.*

Proof: The vertices of the 1st Brocard triangle are given by

$$D = a^2 : c^2 : b^2, \quad E = b^2 : a^2 : c^2, \quad F = c^2 : b^2 : a^2.$$

Let $u\alpha + v\beta + w\gamma = 0$ be an arbitrary line U and D', E', F' the reflections of D, E, F in U. The concurrence determinant for the lines AD', BE', CF' factors as $\widehat{\mathcal{C}_1}\widehat{\mathcal{C}_2}\widehat{\mathcal{C}_3}$, where the conics $\widehat{\mathcal{C}_2} = 0$ and $\widehat{\mathcal{C}_3} = 0$ are imaginary except for the point G. Regarding the first factor, however, we have

$$\widehat{\mathcal{C}_1} = (b^2-c^2)(u^2+2vw)+(c^2-a^2)(v^2+2wu)+(a^2-b^2)(w^2+2uv),$$

which is simply (9) with u, v, w substituted for x, y, z. Thus, as in the proof of Theorem 2, the equation $\widehat{\mathcal{C}_1} = 0$ holds if and only if the *point* $u : v : w$ satisfies (7) or (8). One point

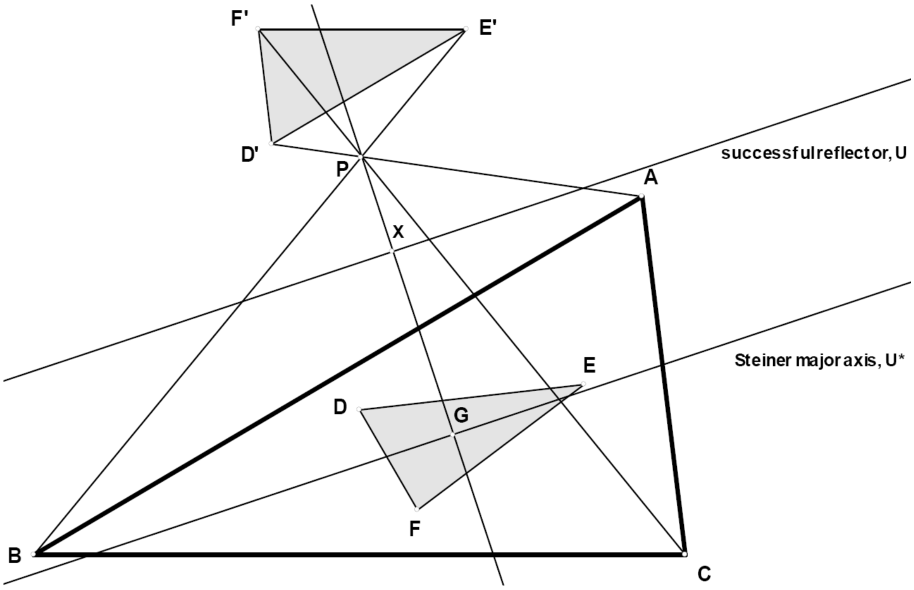

Figure 2: 1st Brocard triangle and successful reflector U

that satisfies (7) is the isogonal conjugate, X_{1379}^{-1}, of the point X_{1379} of intersection of the Brocard axis and the circumcircle. Since X_{1379}^{-1}, is on $\mathcal{L}^{\infty}$, every line meeting $\mathcal{L}^{\infty}$ in X_{1379}^{-1} is a successful reflector of DEF. One member of this family of parallel lines is the Steiner major axis.

Likewise, a point that satisfies (8) is the isogonal conjugate, X_{1380}^{-1}, of the other point of intersection of the Brocard axis and the circumcircle. Since X_{1380}^{-1}, is on $\mathcal{L}^{\infty}$, every line meeting $\mathcal{L}^{\infty}$ in X_{1380}^{-1} is a successful reflector of DEF. One member of this family of parallel lines is the Steiner minor axis. □

4. Pivots

Suppose that for a given triangle DEF there is a point P such that *every* line through P is a successful reflector. Then P is called a *pivot* for DEF. If U is a line through such a pivot and $D'E'F'$ is the reflection of DEF in U, then the perspector of $D'E'F'$ and ABC can be regarded as a function of the line U. As U rotates about P, the perspector traces the *pivotal curve* of P. We shall see that several well known triangles have pivots. (Indeed, by Theorem 4, we may view X_{1379}^{-1} as a pivot for the 1st Brocard triangle, with the Steiner minor axis as pivotal curve. Likewise, X_{1380}^{-1} is a pivot for the 1st Brocard triangle, with the Steiner major axis as pivotal curve.)

Theorem 5 *The incenter is a pivot for the intouch triangle. The pivotal curve is the isogonal conjugate of the incircle.*

Proof. The intouch triangle is defined by its vertices D, E, F given by the following homogeneous trilinear coordinates (or simply *trilinears*):

$$\begin{aligned} D &= 0 : \sec^2(B/2) : \sec^2(C/2) \\ E &= \sec^2(A/2) : 0 : \sec^2(C/2) \\ F &= \sec^2(A/2) : \sec^2(B/2) : 0. \end{aligned}$$

(Geometrically, D is the orthogonal projection of the incenter on sideline BC, and likewise for E and F.) Suppose that U is an arbitrary line through the incenter I. It can be easily checked by computer that the reflection $D'E'F'$ about U is perspective to DEF and that the perspector is given by trilinears

$$\frac{\csc^2(A/2)}{(bz-cy)^2} : \frac{\csc^2(B/2)}{(cx-az)^2} : \frac{\csc^2(C/2)}{(ay-bx)^2}, \tag{10}$$

where X is any point on U except I. The pivotal curve of I is a quartic curve whose isogonal conjugate, remarkably, is the incircle. Specifically, the point

$$(bz-cy)^2\sin^2(A/2) : (cx-az)^2\sin^2(B/2) : (ay-bx)^2\sin^2(C/2)$$

obtained from reciprocals of trilinears in (10) lies on the incircle, given by the equation

$$a_1^4\alpha^4 + b_1^4\beta^4 + c_1^4\gamma^4 - 2b_1^2c_1^2\beta\gamma - 2c_1^2a_1^2\gamma\alpha - 2ab_1^2b_1^2\alpha\beta = 0,$$

where

$$a_1 = \cos(A/2), \quad b_1 = \cos(B/2), \quad c_1 = \cos(C/2)\,. \qquad \square$$

It frequently happens in geometry that a family of lines forms an envelope called a deltoid (as at [6], an interactive Java applet). For example, the Steiner deltoid [11] is the envelope of the Simson lines of a triangle. As a supplement to Theorem 5, we note that if Q is the isogonal conjugate of the perspector and X goes around the incircle, then the envelope of the lines XQ is a deltoid. This *intouch deltoid*, as shown in Fig. 3, is homothetic to the Steiner

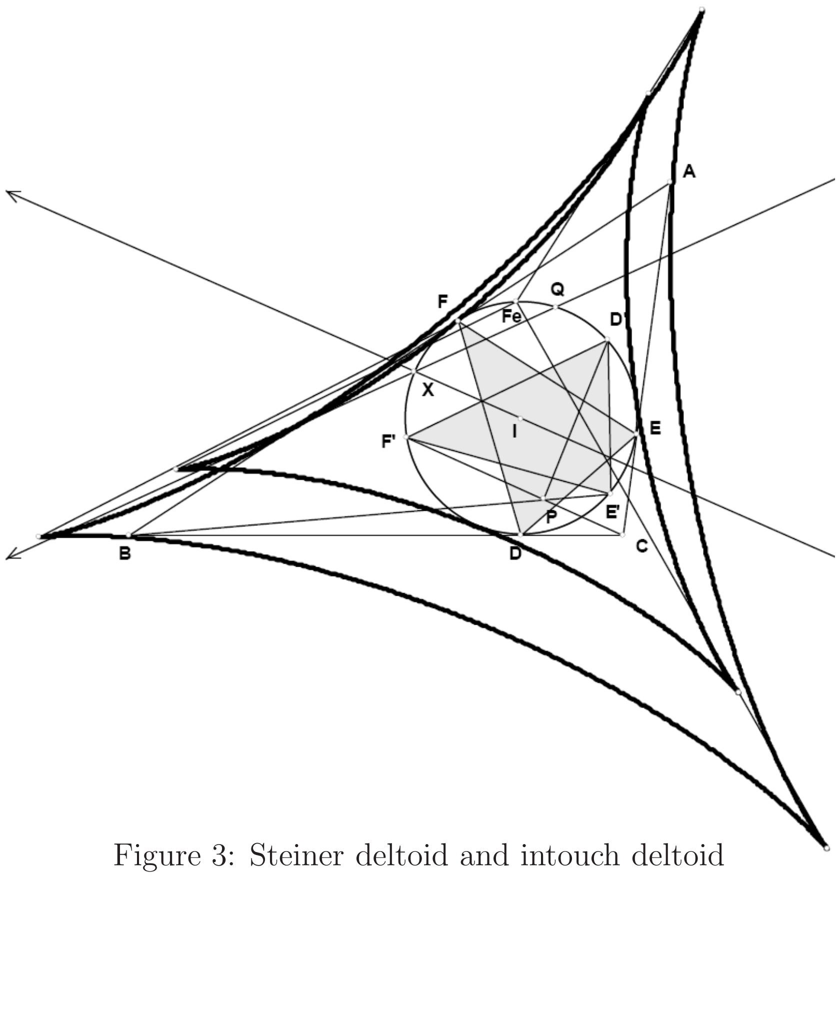

Figure 3: Steiner deltoid and intouch deltoid

deltoid. The homothetic center is the Feuerbach point, labeled Fe, and the homothetic ratio is $2r/R$, where r and R are the radii of the incircle and the circumcircle, respectively.

The method of proof of Theorem 5 enables the discovery and confirmation of other pivots, as in the following examples. Barycentrics for perspectors are omitted because of their lengths.

Example 1. Let DEF be the tangential triangle [7]. The circumcenter O is a pivot, and the pivotal curve is the isogonal conjugate of the circle with center H and radius $2R$, where H denotes the orthocenter and R the circumradius of ABC. (The isogonal conjugate of a circle is a curve of degree at most 4.)

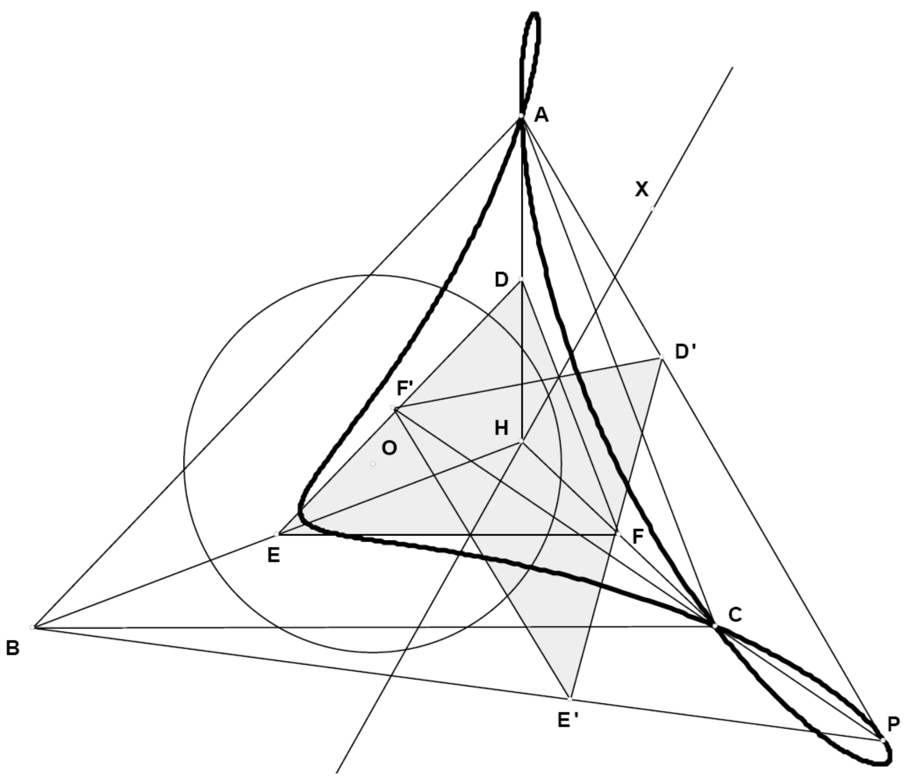

Figure 4: Euler triangle and pivotal curve

Example 2. Let DEF be the Euler triangle [9]. The orthocenter H is a pivot, and the pivotal curve is the isogonal conjugate of the circle with center O and radius $R/2$, as shown in Fig. 4. In this figure, as X moves around any circle with center H, the line HX is a successful reflector. The perspector P of triangles ABC and $D'E'F'$ traces the pivotal curve.

Example 3. Let DEF be the hexyl triangle [10]. The incenter I is a pivot, and the pivotal curve is the isogonal conjugate of the circle $\ominus$ with center I and radius $2R$ (so that $\ominus$ is the circumcircle of the hexyl triangle).

Example 4. Let DEF be the Gossard triangle [5]. The Zeeman-Gossard point X_{402} is a pivot, and the perspector lies on $\mathcal{L}^{\infty}$.

Example 5. Let DEF be the Kosnita triangle ([1], [2]), whose vertices are the circumcenters of the triangles OBC, OCA, OAB. The circumcenter O is a pivot, and the pivotal curve is the isogonal conjugate of the circle having center H and radius R. This circle passes through the point X_{265}, so that the isogonal conjugate of this point, namely X_{186}, lies on the pivotal curve. In [4], X_{182} is identified as the inverse-in-circumcircle of H.

5. Successful reflectors through the incenter

Theorem 6 *The cevian triangle DEF of a point X has four successful reflectors through the incenter, I. Three of them are the lines AI, BI, CI.*

Proof: Let $X = x : y : z$ be a point not on a sideline BC, CA, AB. The cevian triangle DEF of X has vertices

$$D = AX \cap BC, \quad E = BX \cap CA, \quad F = CX \cap AB.$$

The determinant $u_2u_6u_7 - u_3u_4u_8$ in (5) factors as

$$(a+b+c)^4\,(ct-b)\,(b+at+bt)\,(a+c+ct)\,(h-kt)\left(b^2-a^2t+b^2t+c^2t+c^2t^2\right),$$

where the linear factor $h - kt$ is too long to be shown here. The other three factors that are linear in t give equations of the lines AI, BI, CI. □

Example 6. In Theorem 6, we take $X = G$, so that DEF is the medial triangle. In this case, the successful reflector other than the lines AI, BI, CI is the line

$$u(a,b,c)\alpha + u(b,c,a)\beta + u(c,a,b)\gamma = 0,$$

where

$$u(a,b,c) = (b+c-a)\,(b^2+c^2-4bc+ab+ac),$$

and the perspector, $u_4u_7 : u_4u_8 : u_6u_7$, is the triangle center $h(a,b,c) : h(b,c,a) : h(c,a,b)$, where

$$h(a,b,c) = \frac{3a-b-c}{(2a-b-c)(b^2+c^2-4bc+ba+ca)}.$$

Example 7. As another illustration of Theorem 6, let X be the Nagel point, given in trilinears (as are all coordinates in this example) by

$$X_8 = bc(b+c-a) : ca(c+a-b) : ab(a+b-c).$$

The cevian triangle of X_8 is known as the extouch triangle — whereas the intouch triangle in Theorem 5 has as vertices the touchpoints of the incircle on the sidelines BC, CA, AB, the vertices of the extouch triangle are touchpoints of the excircles on the sidelines. Let

$$g(a,b,c) = b^3+c^3+2a^3-3a^2b-3a^2c-4ab^2-4ac^2-b^2c-bc^2+12abc.$$

The successful reflector other than the lines AI, BI, CI is the line

$$l(a,b,c)\alpha + l(b,c,a)\beta + l(c,a,b)\gamma = 0$$

given by

$$l(a,b,c) = (b+c-a)\,g(a,b,c),$$

and the perspector is the point $p(a,b,c) : p(b,c,a) : p(c,a,b)$ given by

$$p(a,b,c) = \frac{a(b+c-a)(a^2+b^2+c^2+2ab+2ac-6bc)}{(2a-b-c)g(a,b,c)}.$$

Figure 5 shows the extouch triangle DEF, its reflection $D'E'F'$ in the successful reflector IJ (which is the trilinear polar of the point Q), and the perspector P in which the lines AD', BE', CF' concur.

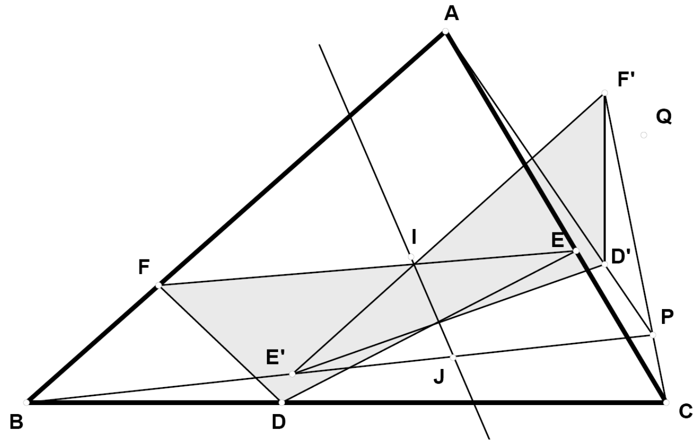

Figure 5: Extouch triangle and successful reflector IJ

6. Concluding remark

Unlike traditional geometry, the methods employed in this article are "symbolic" in two ways that bear comment: first, they depend largely on the factoring of determinants that are functions of the symbols a, b, c, and second, they are highly computer-dependent on expressions that are in some cases of great length. Consequently, one might feel that, although the results have geometric meaning, the methods leave something to be desired. The authors agree that traditional geometric proofs would be of interest. On the other hand, the proofs given here cover more ground, since a, b, c need not be sidelengths of a traditional triangle.

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