Journal for Geometry and Graphics
Volume 14 (2010), No. 1, 1–14.

Dilation-Induced Perspectivities among Triangles

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Abstract. In the plane of a triangle ABC, let DEF be a triangle and U a point. Except for special cases, there are at most two nontrivial U-dilations of DEF that are perspective to ABC. Of particular interest are configurations in which DEF is perspective to ABC, as when DEF is a cevian or anticevian triangle. Midcevian and mid-isotomic triangles are introduced. Many loci associated with perspectivities among these triangles are $\mathcal{Z}$ cubics.

Key Words: perspective triangles, cevian triangles, anticevian triangles, midcevian triangles, mid-isotomic triangles

MSC 2000: 51M04

1. Introduction

In the plane of a fixed triangle ABC, we shall use the notation DEF for a triangle and U and X for points. Certain assumptions and conventions will remain in effect throughout:

(A1) A, B, C, D, E, F, U are distinct points;

(A2) coordinates are homogeneous barycentric coordinates (henceforth simply *barycentrics*) relative to triangle ABC;

(A3) the phrase "triangles $D_tE_tF_t$ and ABC are perspective with perspector U" means that the lines AD_t, BE_t, CF_t concur in U.

Regarding (A3), five other possible perspectivities, such as the concurrence of lines AE_t, BF_t, CD_t, are not considered; accordingly, it is helpful to think of D_t, E_t, F_t as a labeled triangle; viz., D_t is the A-vertex, E_t the B-vertex, and F_t the C-vertex. For example, if DEF is the cevian triangle of a point $X = x : y : z$, then

$$D = 0 : y : z, \qquad E = x : 0 : z, \qquad F = x : y : 0;$$

ISSN 1433-8157/\$ 2.50

if DEF is the anticevian triangle of X, then

$$D = -x : y : z, \qquad E = x : -y : z, \qquad F = x : y : -z.$$

A point $X = x : y : z$ is a *finite point* if $x + y + z \neq 0$, and a *regular point* if $xyz \neq 0$; i.e., if X does not lie a sideline BC, CA, or AB of ABC. These and other introductory features of triangle geometry are presented in [6] and [11].

The notation $\sum$ will always mean a cyclic sum; e.g., $\sum a$ means $a+b+c$; and $\sum p(vq\beta - wr\gamma)\alpha^2$ means

$$p(vq\beta - wr\gamma)\alpha^2 + q(wr\gamma - up\alpha)\beta^2 + r(up\alpha - vq\beta)\gamma^2.$$

Notation of the form X_i refers to triangle centers as indexed in [7]. The first four X_i all appear in the sequel and are given by

$$\begin{aligned}
X_1 &= a : b : c = \text{ incenter} \\
X_2 &= 1 : 1 : 1 = \text{ centroid} \\
X_3 &= a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2) = \text{ circumcenter} \\
X_4 &= 1/(b^2 + c^2 - a^2) : 1/(c^2 + a^2 - b^2) : 1/(a^2 + b^2 - c^2) = \text{ orthocenter}
\end{aligned}$$

As a final introductory note, we mention that this article is a sequel to [10].

2. Dilation

For given $U = u : v : w$ and $X = x : y : z$, the point D_t given by

$$D_t = tkx + (1-t)hu \;:\; tky + (1-t)hv \;:\; tkz + (1-t)hw, \tag{1}$$

where

$$h = x + y + z \quad \text{and} \quad k = u + v + w,$$

is the *t-dilation of X from U*. Note that $D_0 = U$ and $D_1 = X$. For numerical values of a, b, c, the parameter t in (1) is merely a real number, but when a, b, c are regarded as indeterminates or variables, the parameter t is a 0-degree homogeneous symmetric function in a, b, c; that is, for q an indeterminate, we have

$$\begin{aligned}
t(qa, qb, qc) &= t(a, b, c); \\
t(a, b, c) &= t(b, c, a) = t(a, c, b).
\end{aligned}$$

Taking D to be successively the points D, E, F, we obtain from (1) points D_t, E_t, F_t. The triangle $D_tE_tF_t$ is a *U-dilation triangle of DEF*, as typified by Fig. 1. Note that $D_tE_tF_t$ is homothetic to DEF with homothetic ratio $|UD_t|/|UD| = t$.

3. General result

What dilations of a triangle DEF from a point U are perspective to ABC? We shall see in Theorem 1 that much can be said even if the triangle DEF is not assumed to be a member of any particular class of triangles, such as cevian or anticevian. Those cases are considered in Sections 5 and 6.

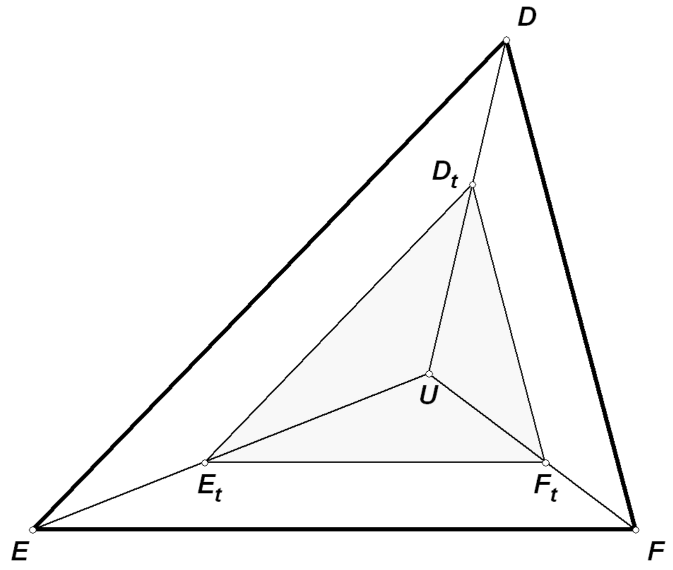

Figure 1: $D_tE_tF_t$ is a U-dilation triangle of DEF

Theorem 1. *Suppose that A, B, C, D, E, F, U are distinct finite points, and let n be the number of U-dilation triangles $D_tE_tF_t$ that are perspective to ABC. If n is finite then $n \leq 2$. More precisely, there exist conics $\mathbb{C}_0$, $\mathbb{C}_1$, $\mathbb{C}_2$ such that*
(B1) *if $U \notin \mathbb{C}_2$, then $n \leq 2$;*
(B2) *if $U \in \mathbb{C}_2$ and $U \notin \mathbb{C}_1$, then $n = 1$;*
(B3) *if $U \in \mathbb{C}_1 \cap \mathbb{C}_2$ and $U \notin \mathbb{C}_0$, then $n = 0$.*

Proof: Write $U = u : v : w$, and, to facilitate application of a computer algebra system, assign symbols as follows:

$$\begin{array}{lll} D = d_1 : d_2 : d_3 & E = d_4 : d_5 : d_6 & F = d_7 : d_8 : d_9 \\ D_t = u_1 : u_2 : u_3 & E_t = u_4 : u_5 : u_6 & F_t = u_7 : u_8 : u_9. \end{array}$$

The line AD_t is given by an equation $l_1\alpha + l_2\beta + l_3\gamma = 0$, where $\alpha : \beta : \gamma$ is a variable point. We represent the coefficients for this line and two others by

$$AD_t = l_1 : l_2 : l_3 \qquad BE_t = l_4 : l_5 : l_6 \qquad CF_t = l_7 : l_8 : l_9.$$

Let

$$h = u + v + w \quad k_1 = d_1 + d_2 + d_3 \quad k_2 = d_4 + d_5 + d_6 \quad k_3 = d_7 + d_8 + d_9,$$

so that the vertices of $\triangle D_tE_tF_t$ are given by

$$\begin{array}{lll} u_1 = td_1h + (1-t)uk_1 & u_2 = td_2h + (1-t)vk_1 & u_3 = td_3h + (1-t)wk_1 \\ u_4 = td_4h + (1-t)uk_2 & u_5 = td_5h + (1-t)vk_2 & u_6 = td_6h + (1-t)wk_2 \\ u_7 = td_7h + (1-t)uk_3 & u_8 = td_8h + (1-t)vk_3 & u_9 = td_9h + (1-t)wk_3, \end{array}$$

and coefficients for lines AD_t, BE_t, CF_t by

$$\begin{array}{lll} l_1 = 0 & l_2 = -u_3 & l_3 = u_2 \\ l_4 = u_6 & l_5 = 0 & l_6 = -u_4 \\ l_7 = -u_8 & l_8 = u_7 & l_9 = 0. \end{array}$$

The determinant

$$\delta = \begin{vmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{vmatrix}$$

has the form

$$t\,(u+v+w)\left[r_0(u,v,w) + tr_1(u,v,w) + t^2 r_2(u,v,w)\right],$$

where $r_i(u,v,w)$ has degree 2 in u,v,w, for $i = 0,1,2$. Let $\mathbb{C}_i$ be the conic $r_i(\alpha,\beta,\gamma) = 0$ for $i = 0,1,2$. The hypothesis that U is a finite point implies that $t\,(u+v+w) \neq 0$, so that the equation $\delta = 0$ is equivalent to

$$r_0(u,v,w) + tr_1(u,v,w) + t^2 r_2(u,v,w) = 0, \tag{2}$$

which is equivalent to the concurrence of the lines AD_t, BE_t, CF_t, which is equivalent to the conditions (B1)–(B3). Specifically, if $U \notin \mathbb{C}_2$, then (2) has at most two solutions t, so that $n \leq 2$. If $U \in \mathbb{C}_2$ and $U \notin \mathbb{C}_1$, then (2) linear, so that $n = 1$. The only remaining possibility is (B3). (Of course, if $n > 2$, then $D_tE_tF_t$ is perspective to ABC for all t, contrary to the hypothesis of the theorem.) □

The conic sections given by $r_i(\alpha,\beta,\gamma) = 0$ can be written out by substituting (α,β,γ) for (u,v,w) and using the following conveniences. Let

$$\begin{aligned}
c_0 &= h^2(d_2d_6d_7 - d_3d_4d_8) \\
c_1 &= h(u(d_2d_6k_3 - d_3d_8k_2) + v(d_6d_7k_1 - d_3d_4k_3) + w(d_2d_7k_2 - d_4d_8k_1)) \\
c_2 &= vwk_1(d_7k_2 - d_4k_3) + wuk_2(d_2k_3 - d_8k_1) + uvk_3(d_6k_1 - d_3k_2) \\
Q &= t^2c_0 + t(1-t)c_1 + (1-t)^2c_2.
\end{aligned}$$

Then $r_0(u,v,w) = c_2$, $r_1(u,v,w) = c_1 - 2c_2$, $r_2(u,v,w) = c_2 - c_1 + c_0$, and (2) is equivalent to $Q = 0$. Consequently, if $c_0 = 0$, then $d_2d_6d_7 = d_3d_4d_8$ or $h = 0$; either the triangles ABC and DEF are perspective, so that n is infinite, contrary to the hypothesis of Theorem 1, or else, if $h = 0$, then U lies on the line at infinity, but this, too, is contrary to the hypothesis. Therefore $c_0 \neq 0$. If $c_1 = 0$, then U lies on the line

$$(d_2d_6k_3 - d_3d_8k_2)\alpha + (d_6d_7k_1 - d_3d_4k_3)\beta + (d_2d_7k_2 - d_4d_8k_1)\gamma = 0,$$

and if $c_2 = 0$, then U lies on the conic

$$k_1(d_7k_2 - d_4k_3)\beta\gamma + k_2(d_2k_3 - d_8k_1)\gamma\alpha + k_3(d_6k_1 - d_3k_2)\alpha\beta = 0,$$

which passes through the vertices A, B, C. If all the barycentrics in Theorem 1 are normalized (so that each point $x : y : z$ satisfies $x+y = z = 1$), then the expressions simplify considerably because $h = k_1 = k_2 = k_3 = 1$.

4. Perspective images of ABC

Suppose that DEF is perspective to ABC, and let $X = x : y : z$ denote the perspector, so that

$$D = t_1 : y : z, \qquad E = x : t_2 : z, \qquad F = x : y : t_3,$$

for some triple (t_1, t_2, t_3). If a, b, c are numerical, then so are t_1, t_2, t_3, whereas if a, b, c are indeterminates or variables, then t_1, t_2, t_3 are functions of (a, b, c) having the same degree of homogeneity as x, y, z. We assume that $t_1 \neq x$, $t_2 \neq y$, $t_3 \neq z$ and refer to DEF an *X-perspective image of ABC*. Much studied examples of such DEF are cevian, anticevian, circumcevian, and circum-anticevian triangles. In the sequel, we shall also present examples which we call midcevian and mid-isotomic triangles.

Theorem 2. *Suppose that DEF is an X-perspective image of ABC and that U is a finite point other than X. Let $D_tE_tF_t$ be the t-dilation of DEF from U. Let*

$$\begin{aligned} M &= \sum vw\left[xt_1(z-y) + yt_2(z+y) - zt_3(z+y) - xt_1(t_2+t_3) + x(z^2-y^2)\right]; \\ N &= \sum \left[u^2\left(yz(t_3-t_2-z+y\right) + vwx\left(yt_3 - zt_2 - t_1t_2 + t_1t_3 + xy - xz\right)\right]. \end{aligned}$$

If $MN \neq 0$, then $D_tE_tF_t$ is perspective to ABC if and only if $t \in \{0, 1, -M/N\}$. If $t = -M/N$, the perspector is the point $P(X,U) = p : q : r$ given by

$$\begin{aligned} p &= \frac{y - z - (t_2 - t_3)}{vz(x+y) - wy(x+z) - (wyt_2 - vzt_3)} \\ q &= \frac{z - x - (t_3 - t_1)}{wx(y+z) - uz(y+x) - (uzt_3 - wxt_1)} \\ r &= \frac{x - y - (t_1 - t_2)}{uy(z+x) - vx(z+y) - (vxt_1 - uyt_2)}. \end{aligned}$$

Proof: Referring to the proof of Theorem 1, we find that (2) has a nontrivial linear factor $M + Nt$:

$$\delta = t(1-t)\,(u+v+w)\,(M+Nt),$$

Putting $t = -M/N$, we find the perspector

$$l_5l_9 - l_6l_8 \;:\; l_6l_7 - l_4l_9 \;:\; l_4l_8 - l_5l_7$$

to be, after simplifications, as asserted. □

In Fig 2, DEF is an X-perspective image of ABC, as in Theorem 2. As D_t traces line DU, the U-dilation $D_tE_tF_t$ takes three positions for which it is perspective to ABC — that is, for which def is a single point. Of the three, two are trivial (when $d = e = f = X$ and when $d = e = f = U$). The existence of the other position is established by Theorem 2. As D_t traces line DU, each of the points d, e, f traces a conic (not shown), and the three conics meet in the three perspectors.

It will be convenient in the sequel to refer to $D_tE_tF_t$, for $t = -M/N \neq 0$, as the *special triangle*, special in the sense that it is the only triangle that is both homothetic to DEF and also perspective to ABC.

We turn now to certain loci related to special triangles. A subclass of these loci are "self-isogonal circumcubics" — elegantly described in a context of linear algebra by H.S.M. Coxeter [1] — "self-isogonal" in the sense that if J is any regular point on such a cubic $\mathcal{Z}$, then the isogonal conjugate of J is on $\mathcal{Z}$ also; and "circum-" because $\mathcal{Z}$ passes through the vertices A, B, C. The full class, in which "self-isogonal" is replaced by "self-P-isoconjugate," consists of cubics each the locus of a point $\alpha : \beta : \gamma$ satisfying an equation of the form

$$\sum p(vq\beta - wr\gamma)\alpha^2 = 0$$

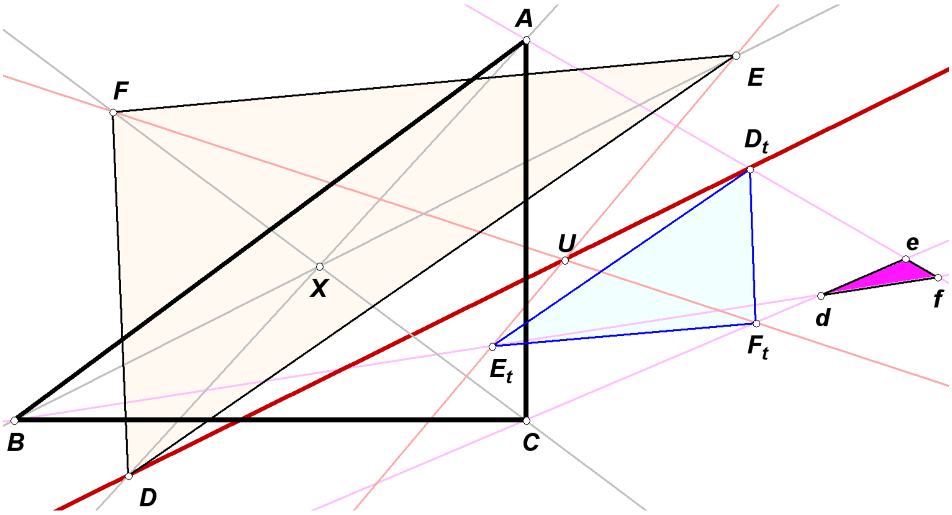

Figure 2: DEF is an X-perspective image of ABC (Theorem 2), D_t traces line DU

for some points $P = p : q : r$ and $U = u : v : w$. This equation is used to define the cubic $\mathcal{Z}(U, P)$ in [8] and [9], where the coordinates are homogeneous trilinears, and the same class of cubics is defined when the coordinates are barycentrics, as will be the case in the sequel. Indeed, when working with barycentrics, it is common to use the $p\mathcal{K}$ classification scheme, presented in [5]. It will be helpful to be able to cross back and forth between the two kinds of classification. First suppose $\mathcal{Z}(U, P)$ is given, where U and P are in trilinears. Then

$$\mathcal{Z}(U, P) = p\mathcal{K}(P, U'),$$

where U' is the *trilinear* product given by

$$U' = X_2 \cdot X = bcu : cav : abw.$$

For the reverse, we start with $p\mathcal{K}(P, U)$ where U and P are in barycentrics. Then

$$p\mathcal{K}(P, U) = \mathcal{Z}(U, P'),$$

where P' is the *barycentric* product given by

$$P' = X_{31} * P = a^3p : b^3q : c^3r.$$

For example, the Thomson cubic is $\mathcal{Z}(X_{30}, X_1) = p\mathcal{K}(X_6, X_{30})$; the Lucas cubic is $\mathcal{Z}(X_{69}, X_{31}) = p\mathcal{K}(X_2, X_{69})$. In [9] and elsewhere, hundreds of $\mathcal{Z}$ cubics are discussed in families and individually.

Theorem 3. *Suppose that DEF is an X-perspective image of ABC and that U is a finite point other than X such that $MN \neq 0$ (as in Theorem 2). Let $D_tE_tF_t$ be the special triangle. The locus of a point $\alpha : \beta : \gamma$ whose cevian triangle $A'B'C'$ is perspective to $D_tE_tF_t$ is a $\mathcal{Z}$ cubic. Likewise, the locus of a point $\alpha : \beta : \gamma$ whose anticevian triangle $A'B'C'$ is perspective to $D_tE_tF_t$ is a $\mathcal{Z}$ cubic.*

Proof: Let $u_1, u_2, \ldots, u_9$ be as in the proof of Theorem 1. Then the lines $A'D_t$, $B'E_t$, $C'F_t$ are given by the equations

$$h_i\alpha + h_{i+1}\beta + h_{i+2}\gamma = 0,$$

for $i = 1, 4, 7$, respectively, where

$$\begin{array}{lll} h_1 = \beta u_3 - \gamma u_2 & h_2 = \gamma u_1 & h_3 = -\beta u_1 \\ h_4 = -\gamma u_5 & h_5 = \gamma u_4 - \alpha u_6 & h_6 = \alpha u_5 \\ h_7 = \beta u_9 & h_8 = -\alpha u_9 & h_9 = \alpha u_8 - \beta u_7. \end{array}$$

Let

$$\delta = \begin{vmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{vmatrix}.$$

The three lines concur if and only if $\delta = 0$. Since

$$\delta = \begin{pmatrix} (u_3u_4u_8 - u_2u_6u_7)\alpha\beta\gamma \\ +[(u_3u_5u_9 - u_3u_6u_8)\beta - (u_2u_5u_9 - u_2u_6u_8)\gamma]\alpha^2 \\ +[(u_1u_4u_9 - u_3u_4u_7)\gamma - (u_3u_6u_7 - u_1u_6u_9)\alpha]\beta^2 \\ +[(u_1u_5u_8 - u_2u_4u_8)\alpha - (u_1u_5u_7 - u_2u_4u_7)\beta]\gamma^2 \end{pmatrix},$$

the locus, clearly cubic, is a $\mathcal{Z}$ cubic if and only if the first two terms cancel, and a computer calculation shows that we do indeed have $u_3u_4u_8 = u_2u_6u_7$.

The same method shows that starting with the anticevian triangle of $\alpha : \beta : \gamma$ also yields a $\mathcal{Z}$ cubic. □

5. Cevian triangles

The simplest perspective image of the reference triangle ABC is, for given $X = x : y : z$, the cevian triangle of X. Its vertices are the points

$$AX \cap BC = 0 : y : z \qquad BX \cap CA = x : 0 : z \qquad CX \cap AB = x : y : 0.$$

Corollary 1. *Suppose $X = x : y : z$ is a regular point and DEF is the cevian triangle of X. Suppose U and $D_tE_tF_t$ are as in Theorem 2. Let*

$$\begin{aligned} M &= \sum u\,(wy - vz)\,x^2 \\ N &= \sum \left[vw(y-z) + w^2y - v^2z\right] x^2. \end{aligned}$$

If $MN \neq 0$, then $D_tE_tF_t$ is perspective to ABC if and only if $t \in \{0, 1, -M/N\}$. If $t = -M/N$, then the perspector P is the point

$$\frac{y-z}{vz(x+y) - wy(x+z)} : \frac{z-x}{wx(y+z) - uz(y+x)} : \frac{x-y}{uy(z+x) - vx(z+y)},$$

and the line UP, given by

$$\sum [vx(x+y) - wy(x+z)]\alpha = 0,$$

passes through the point

$$X^* = x(y+z) \,:\, y(z+x) \,:\, z(x+y).$$

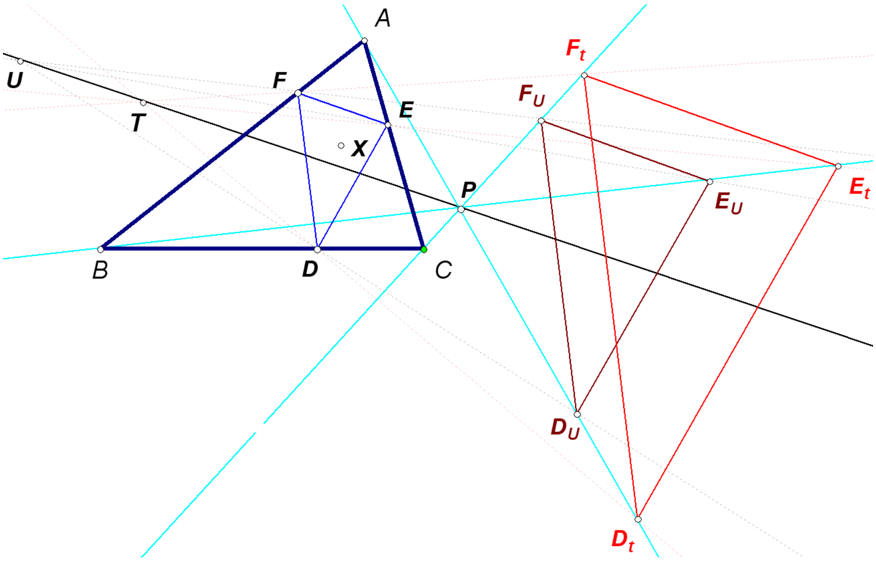

Figure 3: The perspector P of $D_rE_rF_r$ and ABC remains fixed as R varies on UP

Proof: In the proof of Theorem 2, take $t_1 = 0$, $t_2 = 0$, $t_3 = 0$, leading to the solution $t = -M/N$ of the equation $M + Nt = 0$. It is easy to check that UP is as asserted and that $X^* \in UP$. □

Regarding Fig. 3, starting with A, B, C, X, U, the following are calculated as in Corollary 1: $M, N, t = -M/N$, the special triangle $D_tE_tF_t$, the perspector P, and the line UP. Then a variable point R is placed on line UP, the corresponding t calculated, and the t-dilation of DEF from R is shown as $D_rE_rF_r$. The main point of the figure is this: that the perspector P of $D_rE_rF_r$ and ABC remains fixed as R varies on UP.

A geometric rendering of Corollary 1 is that there is exactly one nontrivial $D_tE_tF_t$ perspective to ABC if, for given U, the point X lies on neither of the cubic curves

$$\sum u\,(w\beta - v\gamma)\,\alpha^2 = 0 \quad \text{and} \quad \sum [vw(\beta - \gamma) + w^2\beta - v^2\gamma]\alpha^2 = 0,$$

or equivalently, if, for given X, the point U lies on neither of two conic curves

$$\sum x\left(y^2 - z^2\right)\beta\gamma = 0 \quad \text{and} \quad \sum\,(y - z)\left(x^2\beta\gamma + yz\alpha^2\right) = 0.$$

Example 1. If U in Corollary 1 is the centroid, $1 : 1 : 1$, then $t = -1/2$, and the special triangle $D_tE_tF_t$ is especially simple:

$$D_t = y + z : z : y, \qquad E_t = z : z + x : x, \qquad F_t = y : x : x + y.$$

We call $D_tE_tF_t$ the *mid-isotomic triangle of* X. Note that the perspector of $D_tE_tF_t$ and ABC is the isotomic conjugate of X.

In Section 7, we present a construction and properties of mid-isotomic triangles.

Example 2. $X = X_4$, $U = X_1$. Here, DEF is the orthic triangle, determined by

$$x : y : z = \tan A : \tan B : \tan C,$$

and $U = a : b : c$. The perspector P is given by

$$X_{72} = a\,(b + c)\left(b^2 + c^2 - a^2\right) \;:\; b\,(a + c)\left(a^2 - b^2 + c^2\right) \;:\; c\,(a + b)\left(a^2 + b^2 - c^2\right).$$

Example 3. $X = X_{63}$, $U = X_3$. Here, we obtain $P = a : b : c$, the incenter. The special triangle has vertices

$$\begin{aligned} D_t &= f(a,b,c) : b : c \\ E_t &= a : f(b,c,a) : c \\ F_t &= a : b : f(c,a,b), \end{aligned}$$

where

$$f(a,b,c) = \frac{2a^2(b+c)\,(b^2+c^2-a^2)}{(a^2-b^2+c^2)\,(a^2+b^2-c^2)}.$$

Example 4. $X = X_{69}$, and U and arbitrary finite point on the Euler line, other than X_3 (for which $M = N = 0$). In this case, the perspector P is the orthocenter, X_4, so that the line PU is the Euler line. Representing the Euler line by a variable point $U = U(\tau)$, given by

$$u : v : w = a(bc + \tau a_1)\ :\ b(ca + \tau b_1)\ :\ c(ab + \tau c_1),$$

where

$$a_1 = a(b^2 + c^2 - a^2),\ b_1 = b(c^2 + a^2 - b^2),\ c_1 = c(a^2 + b^2 - c^2))$$

(so that $a_1 : b_1 : c_1 = \cos A : \cos B : \cos C$), we find that the special triangle has vertices

$$\begin{aligned} D_t &= f(a,b,c) : a^2 + b^2 - c^2 : a^2 - b^2 + c^2 \\ E_t &= b^2 - c^2 + a^2 : f(b,c,a) : b^2 + c^2 - a^2 \\ F_t &= c^2 + a^2 - b^2 : c^2 - a^2 + b^2 : f(c,a,b) \end{aligned}$$

where

$$f(a,b,c) = \frac{-4a^4\,(b^2\tau + c^2\tau - a^2\tau + 2b^2c^2)}{b^4\tau + c^4\tau - a^4\tau - 2b^2c^2\tau - 4a^2b^2c^2}.$$

Example 5. $X = X_7$ (Gergonne point) and $U = X_6$ (symmedian point). Here, we obtain

$$P = X_9 = a(b - a + c)\ :\ b(a - b + c)\ :\ c(a + b - c),$$

and the special triangle has vertices

$$\begin{aligned} D_t &= f(a,b,c)\ :\ b(a-b+c)\ :\ c(a+b-c) \\ E_t &= a(b+c-a)\ :\ f(b,c,a)\ :\ c(a+b-c) \\ F_t &= a(b+c-a)\ :\ b(c+a-b)\ :\ f(c,a,b) \end{aligned}$$

where

$$f(a,b,c) = \frac{a^2\,(2bc + 2ca + 2ab - a^2 - b^2 - c^2)}{b^2 + c^2 - ab - ac}.$$

6. Anticevian triangles

Another simple perspective image of the reference triangle ABC is, for given $X = x : y : z$, the anticevian triangle of X, with vertices

$$D = -x : y : z \qquad E = x : -y : z \qquad F = x : y : -z.$$

Writing $A' = AX \cap BC = 0 : y : z$, the point D is the $\{A, A'\}$-harmonic conjugate of X, and likewise for E and F.

Corollary 2. *Suppose $X = x : y : z$ is a regular point and DEF is the anticevian triangle of X. Suppose U and $D_tE_tF_t$ are as in Theorem 2. Let*

$$\begin{aligned} M &= \sum x(y-z)(x-y-z)vw; \\ N &= \sum yz(y-z)u^2. \end{aligned}$$

If $MN \neq 0$, then $D_tE_tF_t$ is perspective to ABC if and only if $t \in \{0, 1, -M/N\}$. If $t = -M/N$, then the perspector P is the point

$$\begin{aligned} &\frac{y-z}{wy\,(x-y+z) - vz\,(x+y-z)} \\ :\;&\frac{z-x}{uz\,(y-z+x) - wx\,(y+z-x)} \\ :\;&\frac{x-y}{vx\,(z-x+y) - uy\,(z+x-y)}, \end{aligned}$$

and the line UP is given by

$$\sum[vz(x+y-z) - wy(x+z-y)]\alpha = 0.$$

Proof: In the proof of Theorem 2, take $t_1 = -x$, $t_2 = -y$, $t_3 = -z$, leading to the solution $t = -M/N$ of the equation $M + Nt = 0$. □

Example 6. As a complement to Example 1, if $U = 1 : 1 : 1$, then $t = -2$, and the special triangle $D_tE_tF_t$ is given by

$$\begin{aligned} D_t &= -x-y-z : x+y-z : x-y+z \\ E_t &= y-z+x : -x-y-z : y+z-x \\ F_t &= z+x-y : z-x+y : -x-y-z. \end{aligned}$$

Starting with $G = 1 : 1 : 1$ and $G_A = -1 : 1 : 1$, we can construct D_t as follows: let

$$\begin{aligned} P_1 &= GX \cap BC = 0 : x-y : x-z \\ P_2 &= P_1G_A \cap AX = -x : y : z \\ X^* &= \text{(isotomic conjugate of } X\text{)} = 1/x : 1/y : 1/z \\ P_3 &= AX^* \cap BC = 0 : z : y \\ D_t &= P_2G \cap P_3G_A. \end{aligned}$$

The vertices E_t and F_t are likewise constructed. Note that $D_tE_tF_t$ is inscribed in the anticomplementary triangle, $G_AG_BG_C$.

The perspector of $D_tE_tF_t$ and ABC is the point

$$\frac{1}{y+z-x} : \frac{1}{z+x-y} : \frac{1}{x+y-z},$$

which, for example, lies on the Euler line if $X = a^2 : b^2 : c^2 = X_6$.

The $\mathcal{Z}$-cubic of Theorem 3, i.e., the locus of a point $\alpha : \beta : \gamma$ whose cevian triangle is perspective to $D_tE_tF_t$, is given by

$$\sum x(y+z)\left[(x-y+z)\beta - (x+y-z)\gamma\right]\alpha^2 = 0.$$

7. Mid-isotomic triangles

Mid-isotomic triangles $D_tE_tF_t$ are introduced in Example 1 as the special triangles when DEF is a cevian triangle and $U = 1 : 1 : 1$. (Recall that the special triangle is the one homothetic to DEF and perspective to ABC.) As these triangles may not have been considered as a class elsewhere, we present a construction (Fig. 4). Theorem 2 applies to these triangles, so that Theorem 3 also applies, and we give corresponding examples.

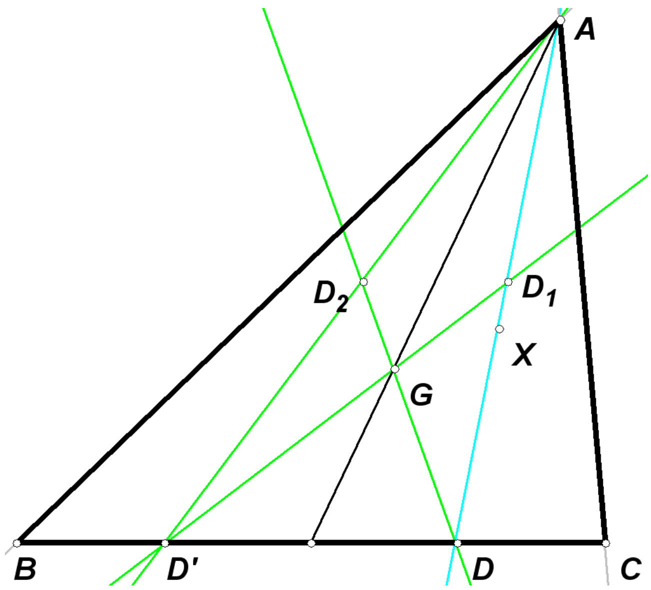

Figure 4: D_1 is the A-vertex of the midcevian triangle of X and D_2 the A-vertex of the mid-isotomic triangle of X

Figure 4 shows the A-vertex, labeled D_2, of the mid-isotomic triangle of a point X. Its construction depends on that of the A-vertex, labeled D_1, of the midcevian triangle of X. Steps for constructing D_1 and D_2 follow:

$$\begin{aligned} D &= AX \cap BC \\ D_1 &= \text{midpoint of segment } AD \\ D' &= \text{reflection of } D \text{ in the midpoint of } BC \\ D_2 &= \text{midpoint of segment } AD'. \end{aligned}$$

Theorem 2 applies to mid-isotomic triangles; to see that this is so, start by writing the A-vertex as $D = y + z : z : y$. (That is, we write DEF for the special triangle and will soon write $D_tE_tF_t$ for the special-of-special triangle.) Then $D = 1/y + 1/z : 1/y : 1/z$, and likewise for E and F, so that DEF is the X^*-perspective image of ABC, where X^* is the isotomic conjugate of X. Now, for any $U \neq 1 : 1 : 1$, the special triangle $D_tE_tF_t$ as given by Theorem 2 has A-vertex

$$\begin{aligned} D_t &= x(y+z)\left[(u-v)(ux-wz)z - (u-w)(ux-vy)y\right]/(vy-wz) \\ &: y(x-z)\left(v(y+z) - u(x+z)\right) \ : \ z(x-y)(w(y+z) - u(x+y)). \end{aligned}$$

By Theorem 2, $D_tE_tF_t$ is homothetic to the cevian triangle of X^* and perspective to ABC, with perspector P given by

$$\begin{aligned} p &= \frac{x(y-z)}{v(x+y)-w(x+z)} \\ q &= \frac{y(z-x)}{w(y+z)-u(y+x)} \\ r &= \frac{z(x-y)}{u(z+x)-v(z+y)}. \end{aligned}$$

We view the foregoing appearance of the mid-isotomic triangles in connection with Theorems 2 and 3 as sufficient reason for inquiring further into the properties of these triangles quite apart from the two theorems. Accordingly, continuing with the notation DEF for the mid-isotomic triangle, suppose that $X = x : y : z$ is a finite regular point. The locus of a point $\alpha : \beta : \gamma$ whose cevian triangle $A'B'C'$ is perspective to DEF is a $\mathcal{Z}$ cubic; to prove this, note that coefficients for the line $A'D$ are $\beta y - \gamma z : \gamma(y+z) : -\beta(y+z)$, and likewise for $B'E$ and $C'F$. Then the desired locus is given by $\delta = 0$, where

$$\begin{aligned} \delta &= \begin{vmatrix} \beta y - \gamma z & \gamma(y+z) & -\beta(y+z) \\ -\gamma(z+x) & \gamma z - \alpha x & \alpha(z+x) \\ \beta(x+y) & -\alpha(x+y) & \alpha x - \beta y \end{vmatrix} \\ &= (yz+zx+xy)\left[(y\beta - z\gamma)\alpha^2 + (z\gamma - x\alpha)\beta^2 + (x\alpha - y\beta)\gamma^2\right]. \end{aligned}$$

From this product we see that if $1/x + 1/y + 1/z \neq 0$, then the expected cubic is the self-isotomic cubic given by

$$(y\beta - z\gamma)\alpha^2 + (z\gamma - x\alpha)\beta^2 + (x\alpha - y\beta)\gamma^2 = 0. \tag{3}$$

Example 7. If $X = X_{75} = bc : ca : ab$, then (3) is the Spieker perspector cubic $\mathcal{Z}(X_{75}, X_{31})$, indexed as **K034** in [4]. As $\alpha : \beta : \gamma$ traces $\mathcal{Z}(X_{75}, X_{31})$, the perspector $\alpha_1 : \beta_1 : \gamma_1$ of DEF and the cevian triangle $A'B'C'$, it can be proved, traces $\mathcal{Z}(X_2, X_{58})$, indexed as **K344** in [4]. It would be of interest to have a formula for a mapping $\alpha : \beta : \gamma \to \alpha_1 : \beta_1 : \gamma_1$ from $\mathcal{Z}(X_{75}, X_{31})$ to $\mathcal{Z}(X_2, X_{58})$. The points X_i on $\mathcal{Z}(X_{75}, X_{31})$, for $i = 1, 2, 7, 8, 63, 75, 92, 347$, are mapped respectively to the points X_i for $i = 1, 37, 226, 10, 9, 2, 281, 1214$, on $\mathcal{Z}(X_2, X_{58})$.

Example 8. For comparison with (3), for arbitrary finite regular X, the locus of $\alpha : \beta : \gamma$ whose *anticevian* triangle $A''B''C''$ is perspective to DEF is also a $\mathcal{Z}$ cubic:

$$\sum \left[(x+y)(yz - zx + xy)\beta - (x+z)(yz + zx - xy)\gamma\right]\alpha^2 = 0. \tag{4}$$

Taking $X = X_{75} = bc : ca : ab$, we obtain in (4) the Spieker central cubic $\mathcal{Z}(X_8, X_{58})$, indexed as **K033** in [4]. As $\alpha : \beta : \gamma$ traces $\mathcal{Z}(X_8, X_{58})$, the perspector $\alpha_1 : \beta_1 : \gamma_1$ of DEF and $A''B''C''$ traces $\mathcal{Z}(X_2, X_{58})$, already encountered in Example 7. Here, the points X_i on $\mathcal{Z}(X_8, X_{58})$, for $i = 1, 4, 8, 10, 40, 65, 72$, are mapped respectively to the points X_i, for $i = 1, 281, 2, 37, 9, 226, 10$, on $\mathcal{Z}(X_2, X_{58})$.

8. Midcevian triangles

As indicated by the construction in Fig. 3, a class of triangles closely related to mid-isotomic triangles are a class we call midcevian triangles, defined by vertices of the form

$$D = y+z : y : z, \qquad E = x : z+x : z, \qquad F = x : y : x+y.$$

As DEF is an X-perspective image of ABC, Theorem 2 applies. As usual, let $D_tE_tF_t$ denote the special triangle (homothetic to DEF and perspective to ABC) determined by X and a finite point $U \neq X$. The perspector of $D_tE_tF_t$ and ABC is given by Theorem 2:

$$\begin{aligned} p &= \frac{y-z}{vz(x+y)-wy(x+z)} \\ q &= \frac{z-x}{wx(y+z)-uz(y+x)} \\ r &= \frac{x-y}{uy(z+x)-vx(z+y)}. \end{aligned}$$

Next, in the manner of Section 7, we consider perspectivities of the midcevian triangle DEF with cevian and anticevian triangles. If $X = x : y : z$ is a regular point, then DEF is perspective to the cevian triangle of a point $\Upsilon = \alpha : \beta : \gamma$ if and only if

$$(x+y+z)\sum(\beta/y-\gamma/z)\alpha^2 = 0,$$

which is to say that Υ is on the line at infinity or on the $\mathcal{Z}$ cubic indicated. Finally, DEF is perspective to the anticevian triangle of a point Υ if and only if Υ lies on the $\mathcal{Z}$ cubic given by

$$\sum\left[z(x+y)\,(x-y+z)\,\beta - y(x+z)\,(x+y-z)\,\gamma\right]\alpha^2 = 0.$$

9. Concluding remarks

Here we summarize the foregoing results and mention problems for further study. Theorem 1 is very general in the sense the DEF is an arbitrary triangle. In Theorem 2 and all the rest of the article except an example just below, DEF is an X-perspective image of ABC. It is reasonable to expect that Theorem 1 has interesting implications for other other triangles DEF. Theorem 3 indicates that many loci associated with $D_tE_tF_t$ are $\mathcal{Z}$ cubics. Subsequent examples in Sections 5–8 suggest that the locus of the perspector P for $\alpha : \beta : \gamma$ on such a $\mathcal{Z}$ cubic is again a $\mathcal{Z}$ cubic, as in Examples 7 and 8. The associated mappings between the two cubics warrant further investigation.

We conclude with a choice of DEF that is not a perspective image of ABC, namely the pedal triangle of the centroid. In the case, (2) has only one nonzero root, $t = -1$, and the triangle $D_tE_tF_t$ homothetic to DEF and also perspective to ABC (whereas DEF is not perspective to ABC) is given by

$$\begin{aligned} D_{-1} &= 4a^2 \,:\, a^2-b^2+c^2 \,:\, a^2+b^2-c^2 \\ E_{-1} &= b^2+c^2-a^2 \,:\, 4b^2 \,:\, b^2-c^2+a^2 \\ F_{-1} &= c^2-a^2+b^2 \,:\, c^2+a^2-b^2 \,:\, 4c^2. \end{aligned}$$

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Received November 30, 2008; final form March 10, 2010