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# More on the Steiner-Lehmus Theorem*

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**Abstract.** For any point $P$ in the plane of the triangle $ABC$, we let $BB_P$, $CC_P$ be the cevians through $P$. Then the Steiner-Lehmus theorem states that if $I$ is the incenter of $ABC$ and if $BB_I = CC_I$ then $AB = AC$. Letting the internal angle bisector of $A$ meet $BC$ at $J$, it is stated in [13] that the same holds if $I$ is replaced by any point on the ray $AJ$. However, the proof there is valid for points on segment $AJ$ and for points on the extension of $AJ$ that are not very far away from side $BC$. In this paper, we consider all points $P$ on the line $AJ$ and we answer the question whether $BB_P = CC_P$ implies $AB = AC$, or equivalently whether $AB \neq AC$ implies $BB_P \neq CC_P$. For a triangle $ABC$ with $AB \neq AC$, we describe a line segment $XY$ on the line $AJ$ inside of which there exists $P$ with $BB_P = CC_P$ and ouside of which there are no such points.

*Key Words:* Steiner-Lehmus theorem, cevian, Ceva's Theorem

*MSC 2010:* 51M04

## 1. Introduction

The Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the corresponding sides are equal. This challenging statement has attracted a lot of attention since 1840 when Professor C. L. Lehmus, Berlin, wrote to C. Sturm asking for a purely geometrical proof. Publications related to this problem are surveyed in [8] and papers that appeared later than [8] include [1], [9], [13], and [14], where the last reference is devoted to trigonometric proofs of the theorem. Much of the huge amount of literature on this problem is devoted to new proofs, variations on the theme, and foundational issues, and very few results are true generalizations of the theorem. These generalizations are summarized in the following:

(a) *If $BB'$, $CC'$ divide angles $B$, $C$ of triangle $ABC$ into the same ratio $r$, i.e.,*

$$\angle ABB' : \angle B'BC = \angle ACC' : \angle C'CB = r, \tag{1}$$

*and if $BB' = CC'$, then $AB = AC$* (see [16], [11, X, p. 311], [17], and more recently [5], [4], [10], and [15]).

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(b) *If $BB'$, $CC'$ are the internal angle bisectors of angles $B$, $C$ of the triangle $ABC$ and if $AB > AC$, then $BB' + B'C > CC' + C'B$* (see [9]).

(c) *Statement* (a) *above is a neutral theorem.*

In view of this, the generalization given in [13] comes as a welcome addition to the literature on this theorem. This generalization proves that if $P$ is any point on the internal angle bisector $AJ$ of the angle $A$ in triangle $ABC$ and if $BB_P$, $CC_P$ are the cevians through this point, then

$$BB_P = CC_P \implies AB = AC. \tag{2}$$

The same holds if $P$ lies on the extension of $AJ$ as long as $B_P$, $C_P$ lie on the extensions of $AC$, $AB$, respectively. These are but two cases of a total of five cases that are fully treated in Theorem 1 below. These two cases correspond to Cases (A-ii), (B) in the proof of Theorem 1, and they are illustrated in Figs. 3 and 4.

We should also mention that, in our proofs, we consider the stronger implication

$$AB > AC \implies BB_P > CC_P \tag{3}$$

(or $AB > AC \implies BB_P < CC_P$) instead of (2). Clearly (3) implies (2). For an elaboration on whether (2) implies (3) (see Remark 7).

**Notation.** Throughout this paper, fix a triangle $ABC$ in which $AB > AC$, and let $a$, $b$, $c$, $A$, $B$, $C$ denote its side lengths and angles in the standard order. Let $AJ$ be the internal angle-bisector of $A$ and let $X$, $Y$ be the points on the extension of $AJ$ such that $CX$, $BY$ are parallel to $AB$, $AC$, respectively (see Fig. 1). Let $U$, $V$ be points on rays $AJ$, $JA$ that are infinitely far. For any point $P$ on the line $AJ$, we let $BB_P$, $CC_P$ be the cevians through $P$. The points $C_X$, $B_Y$ can be thought of as undefined or lie at infinity. Note that we have found it more convenient not to keep the shape and size of $ABC$ fixed in all the figures.

## 2. The main result

In this section, we give, in Theorem 1 below, a full treatment of all the possible positions of $P$ on the line $AJ$. As mentioned earlier, Cases (A-ii) and (B) in the proof are the two cases treated in [13]. Compared to the proof given in [13] which is purely geometric, our proof of (A-ii) is trigonometric in that it makes use of the law of sines and simple properties of trigonometric functions. It is instructive to compare our proof to the proof of the Steiner-Lehmus theorem given in [6] and to compare the proof in [13] to that given in [18].

We shall freely use the simple fact that if $M, N > 0$ and if $M + N < 180°$, then

$$M < N \iff \sin M < \sin N. \tag{4}$$

This is seen by constructing a triangle $LMN$ having $M$, $N$ as two angles, using the law of sines, and then using the fact that $M < N \iff LN < LM$.

**Theorem 1.** *Let $ABC$ be a triangle in which $c > b$. Let $J$, $X$, $Y$, $U$, $V$, $B_P$, $C_P$ be as defined above.*

(A) *If $P$ lies on ray $JV$, then $BB_P > CC_P$.*

(B) *If $P$ lies on line segment $JX$, then $BB_P < CC_P$.*

(C) *If $P$ lies on ray $YU$, then $BB_P > CC_P$.*

(D) *There exists a point $P$ on line segment $XY$ such that $BB_P = CC_P$.*

*Proof:* **(A)** We split the proof of (A) into two cases.

**Case (A-i)** Suppose that $P$ lies on the extension of $JA$.
Let $BJ = q$, $JC = p$, $PB_P = s$, $PC_P = t$, $B_PB = u$, $C_PC = v$, as shown in Fig. 2, and suppose that $v \geq u$. We shall reach a contradiction.

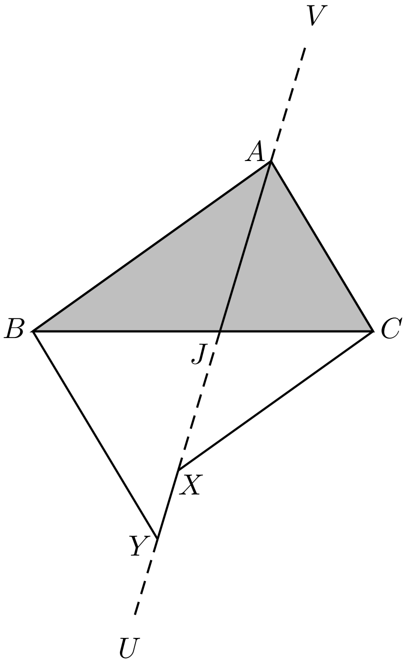


Figure 1: Notation

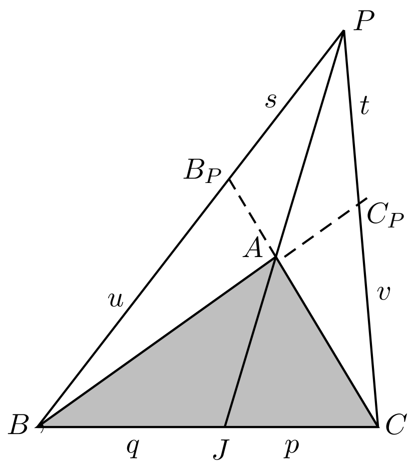


Figure 2: Case (A-i)

The angle bisector theorem implies that $q/p = c/b$ and therefore $q > p$. Ceva's theorem implies that $qvs = ptu$. Since $q > p$, $v \geq u$, it follows that $t > s$. Adding this to $v \geq u$, we obtain $PC > PB$.
On the other hand, $\angle PAB$, $\angle PAC$ are equal and obtuse, and $b < c$. Applying the law of cosines to triangles $PAB$, $PAC$, we obtain the contradiction $PC < PB$.

**Case (A-ii)** Suppose that $P$ lies on line segment $AJ$, and let $B_1$, $B_2$, $C_1$, $C_2$, $B'$, $C'$ be as shown in Fig. 3.
Since $c > b$, it follows from the angle bisector theorem that $BJ > JC$. Thus the midpoint $M$ of $BC$ lies between $B$ and $J$. Since

$$\angle AJC = \frac{A}{2} + B < \frac{A}{2} + C = \angle AJB,$$

it follows that $\angle AJC < 90°$. By the exterior angle theorem, $\angle PMC < \angle PJC < 90°$. Applying the law of cosines (or Proposition 24 of Book I of Euclid's *Elements*, sometimes referred to as *the open mouth theorem*, as in [12],) to triangles $PMB$, $PMC$, we conclude that $PB > PC$ and therefore

$$C_1 > B_1 \ , \quad \sin C_1 > \sin B_1. \tag{5}$$

Next, it follows from the trigonometric version of Ceva's theorem ([3, Theorem 1.15.3, p. 56]) that

$$\frac{\sin B_2}{\sin C_2} = \frac{\sin B_1}{\sin C_1}\,.$$

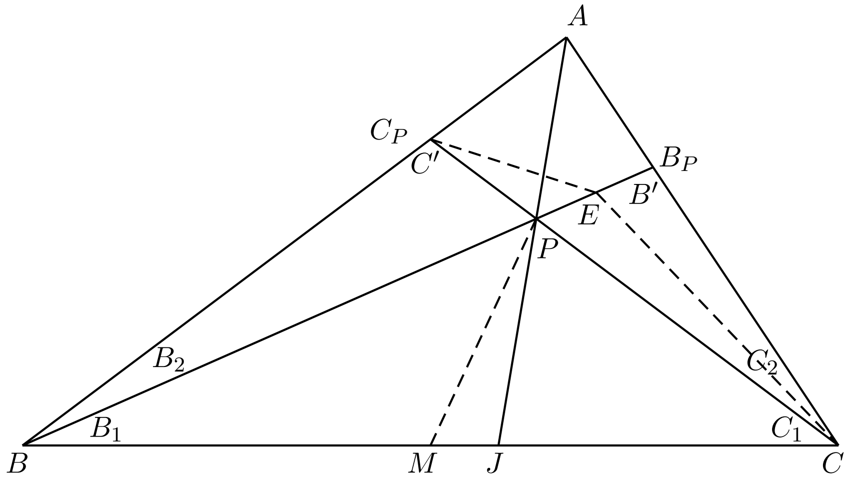


Figure 3: Case (A-ii)

From this and (5), we conclude that

$$\sin C_2 > \sin B_2 \ , \quad C_2 > B_2. \tag{6}$$

It also follows from this and the exterior angle theorem that

$$C' > B'. \tag{7}$$

To prove that $BB_P > CC_P$, we suppose now that $BB_P \leq CC_P$ (and $AB > AC$) and reach a contradiction.

Using the law of sines and the assumptions that $\sin B < \sin C$ and $CC_P \geq BB_P$, we have

$$\frac{\sin C'}{a} = \frac{\sin B}{CC_P} \ < \ \frac{\sin C}{BB_P} = \frac{\sin B'}{a} \, .$$

Therefore $\sin C' < \sin B'$. But $C' > B'$ by (7). Therefore

$$C' \text{ is obtuse.} \tag{8}$$

By (6), there is a point $E$ on line segment $PB_P$ such that

$$\angle ECC_P = \angle EBC_P. \tag{9}$$

Then $ECBC_P$ is cyclic. By (8), $\angle CC_PB$, and hence $\angle CEB$, are obtuse. Therefore $\angle ECB$, $\angle C_PBC$ are acute. Also, $\angle ECB > \angle C_PBC$ by (5) and (9). Therefore the chord of $\angle ECB$ (in circle $BCEC_P$) is greater than that of $\angle C_PBC$, i.e., $EB > C_PC$, contradicting the assumption that $BB_P \leq CC_P$.

**(B)** Suppose that $P$ lies on line segment $JX$.
Let $BJ = q$, $JC = p$, $BC_P = s$, $CB_P = t$, as shown in Fig. 4. Applying Ceva's theorem to the cevians $AJ$, $BB_P$, $CC_P$ in triangle $ABC$, we obtain $qt(c+s) = p(b+t)s$. By the angle bisector theorem, we have $q/p = c/b$. Therefore $ct(c + s) = b(b + t)s$ and hence $ts(c - b) = b^2s - c^2t$. Since $c > b$, it follows that $s > t$. Therefore $s + c > t + b$, i.e., $AC_P > AB_P$. Now apply Case (A-ii) to triangle $AB_PC_P$ to conclude that $B_PB < C_PC$, as desired.

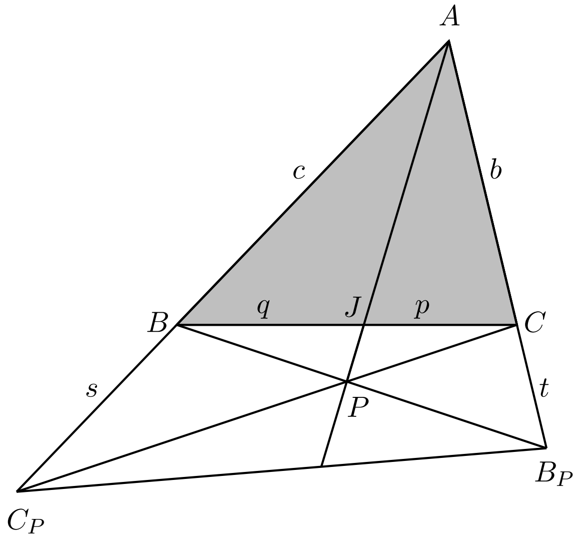


Figure 4: Case (B)

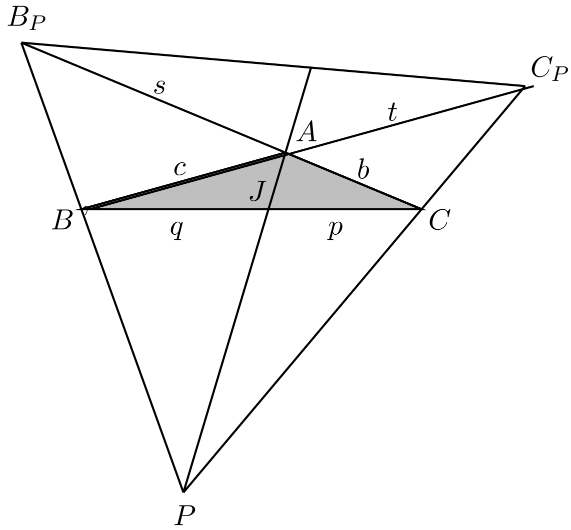


Figure 5: Case (C)

**(C)** Suppose that $P$ lies on ray $YU$.
Let $BJ = q$, $JC = p$, $AB_P = s$, $AC_P = t$, as shown in Fig. 5. Applying Ceva's theorem to the cevians $AJ$, $BB_P$, $CC_P$ in the triangle $ABC$, we obtain $q(b + s)t = ps(t + c)$. By the angle bisector theorem, we have $q/p = c/b$. Therefore $c(b + s)t = bs(t + c)$ and hence $(c - b)st = cb(s - t)$. Since $c > b$, it follows that $s > t$. Now apply Case (A-i) to triangle $AB_PC_P$ to conclude that $B_PB > C_PC$, as desired.

**(D)** Let $Q$ move contiuously from $X$ to $Y$ along the line segment $XY$. Then the ratio $BB_Q/CC_Q$ moves from 0 to $\infty$. By continuity, there is a point $P$ on line segment $XY$ for which $BB_P/CC_P = 1$, as desired.
This completes the proof. □

*Remark* 2. In view of Theorem 1, Theorem 2 of [13] is incorrect. It claims that

$$BB_P = CC_P \implies AB = AC$$

for points $P$ lying on ray $AJ$, but proves it only for points lying on line segment $AX$. According to Part (D) of Theorem 1, the statement is not true if $P$ lies on line segment $XY$.

*Remark* 3. According to Part (D) of Theorem 1, there exists a point $P$ on the line segment $XY$ for which the cevians $BB_P$, $CC_P$ are equal. One wonders whether $P$ may coincide with the $A$-excenter of $ABC$, i.e., whether $PB$, $PC$ bisect $B$, $C$ externally. It follows from Part (b) of Theorem 4 of [7] that this happens if and only if

$$\sin^2 \frac{A}{2} = \sin \frac{B}{2} \sin \frac{C}{2}\,.$$

*Remark* 4. For every point $P$ on line $UV$ of Fig. 1, let $\rho(P) = BB_P/CC_P$. We have seen that $\rho(P)$ is never 1 outside segment $XY$ except of course when $P = J$. Also, $\rho(X) = 0$ and $\rho(Y) = \infty$, and therefore $\rho(P) = 1$ for at least one $P$ on segment $XY$. It would be interesting to have a more detailed sketch of the function $\rho(P)$ against the axis $UV$ that shows the increasing-decreasing behavior of $\rho$ and that shows the values of $\rho$ at the infinite points $U$

and $V$. In particular, one wonders whether there are triangles for which $\rho$ assumes the value 1 at more than one point on segment $XY$. To settle this question, we let $P$ be a point on line segment $XY$ for which $BB_P = CC_P$. Then $C_P$, $B_P$ lie on rays $BA$, $AC$, respectively. Let $AC_P = y$, $CB_P = x$. Using Maple and the theorems of Ceva and Stewart, we obtain the equations

$$\begin{aligned} f(y) &= (c^2 - cb)y^3 - (a^2c - b^2c + b^3 - b^2c + 2c^2b)y^2 \\ &\quad +(a^2bc - 2b^3c + b^2c^2 + c^3b)y - (c^3b^2 + c^2b^3) \;=\; 0, \qquad (10) \\ x &= \frac{b^2(y+c)}{bc - y(c-b)}. \qquad (11) \end{aligned}$$

These equations yielded a unique solution for every numerical choice of the triple $(a, b, c)$ that we tried. For example, taking $(a, b, c) = (3, 4, 6)$, we obtain $y \approx 20.36$, $x \approx 25.23$.

## 3. Other variants

We now record several variants of Statement (A) of Theorem 1 for points $P$ lying on line segment $AJ$.

**Theorem 5.** *Let $ABC$ be a triangle and let $P$ be a point on the internal angle bisector $AJ$ of $A$. Let $BB_P$, $CC_P$ be the cevians throuh $P$, and let $B_1$, $B_2$, $C_1$, $C_2$, $B'$, $C'$ be as in Fig. 3. If $c > b$, then*

*(i)* $C_1 > B_1$, *(ii)* $C_2 > B_2$, *(iii)* $C' > B'$, *(iv)* $AC_P > AB_P$, *(v)* $BC_P > CB_P$.

*Consequently, each of the equalities $C_1 = B_1$, $C_2 = B_2$, $C' = B'$, $AC_P = AB_P$, $BC_P = CB_P$ implies that $b = c$.*

*Proof:* The first three inequalities are nothing but (5), (6), and (7) in the proof of Theorem 1. To prove the last two, let $x = AC_P$, $y = AB_P$. From the angle bisector theorem and Ceva's theorem, it follows that

$$1 = \frac{x}{c-x}\,\frac{BJ}{JC}\,\frac{b-y}{y} = \frac{x}{c-x}\,\frac{c}{b}\,\frac{b-y}{y}. \qquad (12)$$

Therefore $xc(b-y) = yb(c-x)$, i.e., $(x-y)bc = xy(c-b)$. Since $c > b$, it follows that $x > y$. This proves (iv). Finally it follows from (12) that

$$\frac{BC_P}{CB_P} = \frac{c-x}{b-y} = \frac{c}{b}\,\frac{x}{y} > 1,$$

and hence (v). □

*Remark* 6. One may wonder whether one can add the inequality $PB_P > PC_P$ to the list appearing in the conclusion of Theorem 5. The answer is given in the problem proposal [2], where we are asked to prove that if $A$ is acute then there is a point $P$ on segment $AJ$ for which $PB_P = PC_P$. This is not so if $A$ is obtuse.

*Remark* 7. It is natural to feel that (3) is stronger than (2) in the sense that (3) implies (2) but (2) does not imply (3). We show now that (2) does imply (3) in the case when $P$ lies on line segment $AJ$. Other cases can be treated similarly. Thus assume that (2) holds and that $c > b$. Let $P$ be on line segment $AJ$. To show that $BB_P > CC_P$, we suppose that $BB_P \le CC_P$, and we reach a contradiction. Note that $B_A = C_A = A$. Thus $BB_A > CC_A$. From this and the assumption $BB_P \le CC_P$, it follows by continuity that there is a point $Q$ on the closed line segment $PA$ for which $BB_Q = CC_Q$. By (2), we obtain the contradiction $c = b$.

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