Journal for Geometry and Graphics
Volume 14 (2010), No. 2, 171–180.

Discrete Gliding Along Principal Curves*

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Abstract. We consider n-dimensional discrete motions such that any two neighbouring positions correspond in a pure rotation ("rotating motions"). In the Study quadric model of Euclidean displacements these motions correspond to quadrilateral nets with edges contained in the Study quadric ("rotation nets"). The main focus of our investigation lies on the relation between rotation nets and discrete principal contact element nets. We show that every principal contact element net occurs in infinitely many ways as trajectory of a discrete rotating motion (a discrete gliding motion on the underlying surface). Moreover, we construct discrete rotating motions with two non-parallel principal contact element net trajectories. Rotation nets with this property can be consistently extended to higher dimensions.

Key Words: Discrete differential geometry, kinematics, rotating motion, curvature line discretization, principal contact element net, gliding motion.

MSC 2010: 53A17, 53A05

1. Introduction

Discrete differential geometry is an active field of geometrical research. Its aim is the development of discrete notions for well-known concepts from differential geometry. The resulting theories are often more elementary and concrete when compared to their smooth counterparts. While classic differential geometry is largely based on analysis, elementary geometric incidence or closure theorems are at the core of discrete differential geometry. It is a discipline that naturally lends itself to applications that require numeric simulation, visualization, or the building of real world objects. An excellent introduction to the current state of research is the monograph [3].

In this article we relate recent progress in the theory of discrete curvature line parametrizations to spatial kinematics in $\mathbb{R}^3$. We study discrete nets of proper (orientation-preserving) Euclidean displacements such that neighbouring positions correspond in a relative rotation

* Plenary lecture at the 14th International Conference on Geometry and Graphics, August 5–9, 2010, Kyoto/Japan.

ISSN 1433-8157/\$ 2.50

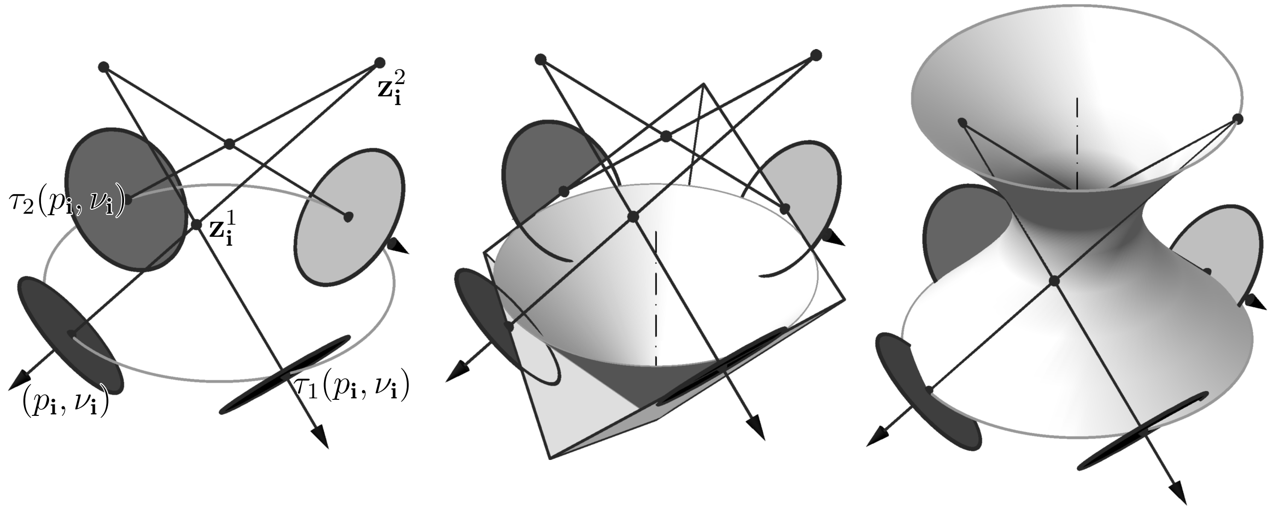

Figure 1: Elementary quadrilateral in a principal contact element net:
Points on a circle (left), planes tangent to a cone of revolution (middle),
normal lines on a hyperboloid of revolution (right)

("discrete rotating motions"). Smooth motions with this property naturally arise as gliding motions along principally parametrized surfaces. Their discrete counterparts, rotating motions with discrete curvature line trajectories, are the main topic of this article.

In Section 2 we recall the notion of principal contact element nets — families of contact elements (point plus oriented tangent plane), indexed by $\mathbb{Z}^n$, such that neighbouring contact elements have a common tangent sphere. Principal contact element nets have been introduced in [2] as a comprehensive concept that captures different notions of discrete principal parametrizations (circular nets and conical nets, see [3, Section 3.1] and [8]). Indeed, the points of a principal contact element net form a circular net (the elementary quadrilaterals are circular; Fig. 1, left), while its planes form a conical net (four planes meeting in a vertex are tangent to a cone of revolution; Fig. 1, center).

A concise formulation of all calculations and formulas in this article is possible within the dual quaternion calculus of spatial kinematics (Section 2.2). In Section 2.3 we recall fundamental results on "rotation quadrilaterals" [10], the elementary building blocks of rotation nets.

The major contributions of this article are presented in Section 3. Just as a smooth surface gives rise to many gliding motions, parametrized by principal lines and the rotation angle about surface normals, a principal contact element net occurs in many ways as trajectory of a discrete rotating motion. The main result is a proof of the multidimensional consistency of discrete rotating motions with two independent principal contact element trajectories. The two trajectory surface are related by discrete version of the classic Bäcklund transform for pseudospherical surfaces.

2. Preliminaries

2.1. Curvature line discretizations

A parametrization of a smooth surface in $\mathbb{R}^3$ is called a curvature line parametrization or principal parametrization if infinitesimally neighbouring surface normals along both families of parameter lines intersect. Generically, this parametrization is unique (up to re-parametrization of the individual parameter lines). The property of concurrent neighbouring normals is preserved in its usual discretizations. The most prominent example of discrete curvature lines are circular nets — quadrilateral nets such that any elementary quadrilateral has a circumcircle, see for example [3, Section 3.1]. An alternative discretization are conical nets — quadrilateral nets such that the planes meeting in a vertex are tangent to a cone of revolution [8]. Neighboring circle axis and neighbouring cone axis in circular and conical nets intersect and can serve as discrete surface normals. In [2] principal contact element nets were introduced as a generalization of both, circular and conical nets.

Definition 1. An *oriented contact element* is a pair (p, ν) consisting of a point p and an oriented plane ν incident with p. The oriented line N orthogonal to ν and incident with p is called the *axis* or *normal* of the contact element, p is its *vertex* and ν its *tangent plane.*

Contact element nets are quadrilateral nets of oriented contact elements, that is, they are maps from $\mathbb{Z}^n$ to the space of oriented contact elements. The image of $\mathbf{i} \in \mathbb{Z}^n$ is denoted by $(p_\mathbf{i}, \nu_\mathbf{i})$. In Definition 2 below we adopt the notation of [3] where τ_i indicates a shift of indices in the i-th coordinate direction. For example $\tau_1 p_{120} = p_{220}$, $\tau_2 p_{120} = p_{130}$, etc.

Definition 2. A *contact element net* is a map $\mathbf{i} \mapsto (p_\mathbf{i}, \nu_\mathbf{i})$ from $\mathbb{Z}^n$ to the space of oriented contact elements. A *principal contact element net* (Fig. 1) is a contact element net such that any two neighbouring contact elements $(p_\mathbf{i}, \nu_\mathbf{i})$, $\tau_i(p_\mathbf{i}, \nu_\mathbf{i})$ have a common oriented tangent sphere.

In other words, neighbouring normals $N_\mathbf{i}$ and $\tau_i N_\mathbf{i}$ in a principal contact element net intersect in a point $z^i_\mathbf{i} = N_\mathbf{i} \cap \tau_i N_\mathbf{i}$ which is at the same oriented distance from $p_\mathbf{i}$ and $\tau_i p_\mathbf{i}$. The points $p_\mathbf{i}$ in a principal contact element net constitute the vertices of a circular net, the oriented planes $\nu_\mathbf{i}$ are the face planes of a conical net. The contact element axes $N_\mathbf{i}$ form a discrete line congruence [3, Section 2.2] such that any elementary quadrilateral lies on a hyperboloid of revolution (Fig. 1, right). Discrete line congruences of this type have interesting properties but have not yet been discussed in literature.

In this article we consider a kinematic generation of principal contact element nets in $\mathbb{R}^3$. A concise analytic description can be obtained by means of the dual quaternion calculus of spatial kinematics which shall be introduced now.

2.2. Kinematics and dual quaternions

Quaternions and dual quaternions are important tools in theoretical and applied kinematics, see for example the description in [5], [6, Section 4.5] or [11, Chapter 9]. In our study they turn out to be a versatile tool as well. Equations in dual quaternion form are concise, of low degree, accessible to geometric interpretations and free of the need of case distinctions. We assume that the reader is familiar with quaternion algebra (see for example [5] or [6, Chapter 4]) and describe only the extension to dual quaternions and its application to spatial kinematics.

A dual quaternion is an object of the form $a = \widetilde{a} + \varepsilon\widehat{a}$ where primal part $\widetilde{a}$ and scalar part $\widehat{a}$ are ordinary quaternions and ε is the dual unit satisfying $\varepsilon^2 = 0$. The addition of dual quaternions is performed component-wise for primal and dual parts. The dual quaternion multiplication $\star$ extends the multiplication of ordinary quaternions. It is associative, distributive and the quaternion units 1, i, j, k commute with ε. These properties define the dual quaternion multiplication uniquely.

A dual quaternion $a = \widetilde{a} + \varepsilon\widehat{a}$ can be identified with a vector $a = (\widetilde{a}, \widehat{a}) = (a_0, \dots, a_7)$ in $\mathbb{R}^8$. *Vector part* and *dual part* of a are

$$\begin{aligned} \mathrm{V}\,a &= (0, a_1, a_2, a_3, 0, a_5, a_6, a_7) = \mathrm{V}\,\widetilde{a} + \varepsilon\,\mathrm{V}\,\widehat{a}, \\ \mathrm{S}\,a &= (a_0, 0, 0, 0, a_4, 0, 0, 0) = \mathrm{S}\,\widetilde{a} + \varepsilon\,\mathrm{S}\,\widehat{a}. \end{aligned} \tag{1}$$

A normalized dual quaternion $a = \widetilde{a} + \varepsilon\widehat{a}$ satisfies two conditions:

$$\|\widetilde{a}\|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \quad \text{and} \quad \langle a, a\rangle = 0 \tag{2}$$

where

$$\langle a, b\rangle := \sum_{i=0}^{3} (a_i b_{i+4} + a_{i+4} b_i). \tag{3}$$

The second equation in (2) is the well-known Study condition.

Plücker coordinate vectors $P = (p_0, \dots, p_5)$ for straight lines (see for example [7, Chapter 2]) can be embedded in the space of dual quaternions as

$$P \hookrightarrow (0, p_0, p_1, p_2, 0, p_3, p_4, p_5). \tag{4}$$

In this case the Study condition (2) reduces to the Plücker condition such that normalized Plücker coordinates (characterized by $p_0^2 + p_1^2 + p_2^2 = 1$) become normalized dual quaternions of vanishing scalar part. In this article we do not distinguish between a straight line and its Plücker coordinates, embedded in the space of dual quaternions.

Normalized dual quaternions with non-vanishing primal part constitute a two-fold covering of the group of proper Euclidean displacements. The action of a normalized dual quaternion a on a point with coordinate vector (x_1, x_2, x_3) and a line P (both in the moving space) is given by

$$1 + \varepsilon x' = a_\varepsilon \star (1 + \varepsilon x) \star a^{-1}, \quad P' = a_\varepsilon \star P \star a_\varepsilon^{-1}, \tag{5}$$

where $x = (0, x_1, x_2, x_3)$ is a vector valued ordinary quaternion, $a_\varepsilon := \widetilde{a} - \varepsilon\widehat{a}$, and the prime indicates coordinates in the fixed space. The inverse displacement is described by the inverse dual quaternion a^{-1} defined by the equation $a \star a^{-1} = 1$. It is uniquely defined for all dual quaternions with non-vanishing primal part.

Since a and $-a$ describe the same displacement, it is natural to identify proportional dual-quaternions, thus arriving at Study's kinematic mapping which associates proper Euclidean displacements with points of the Study quadric

$$\mathcal{S}\colon \langle x, x\rangle = 0. \tag{6}$$

The Study quadric is a hyperquadric in the seven-dimensional projective space P^7 over $\mathbb{R}^8$. Only points of an exceptional three-space E with equation $\widetilde{x} = (0, 0, 0, 0)$ do not occur as images of proper Euclidean displacements. We always identify Euclidean displacements with homogeneous coordinate vectors that describe points on the Study quadric (minus E).

2.3. Rotation quadrilaterals

In Section 3 we will consider special quadrilateral nets in the Study quadric. The geometry of an elementary quadrilateral

$$[a_{\mathbf{i}}, \tau_i a_{\mathbf{i}}, \tau_j a_{\mathbf{i}}, \tau_i\tau_j a_{\mathbf{i}}] = [a_0, a_1, a_2, a_3] \tag{7}$$

of a net of this type is discussed in this section. It follows the presentation of [10].

Definition 3. A quadrilateral $[a_0, a_1, a_2, a_3]$ on the Study quadric is called a *rotation quadrilateral* if its vertices and edges are contained in the Study quadric $\mathcal{S}$ in P^7.

The name "rotation quadrilateral" is justified by the observation that the edge through a_i and a_{i+1} is contained in $\mathcal{S}$ if and only if the relative displacement $r_{i,i+1} := a_{i+1} \star a_i^{-1}$ is a pure rotation (indices modulo four; see [12, Satz 19] or [5]). (We will frequently use results and formulas of [12] but, occasionally, adapt them to match our convention of quaternion multiplication which is slightly different from that used in [12].) The algebraic characterization of relative rotations is

$$\pi_5(r_{i,i+1}) = \pi_5(a_{i+1} \star a_i^{-1}) = 0 \tag{8}$$

where $\pi_5(x)$ is the projection onto the fifth coordinate (the dual scalar part) of a dual quaternion [12, Satz 13]. We denote the relative revolute axis of $r_{i,i+1}$ in the moving space by $R_{i,i+1}$.

The main result of [10] characterizes contact elements whose homologous images form an elementary quadrilateral in a principal contact element net.

Proposition 4 ([10])**.** *The only contact elements in the moving space whose homologous images with respect to a generic rotation quadrilateral form a non-degenerate elementary quadrilateral of a principal contact element net are those whose axes are transversal to the four relative revolute axes R_{01}, R_{12}, R_{23}, and R_{30}.*

Further elementary quadrilaterals of principal contact element nets have the normal $R_{i,i+1}$. Since their a_i and a_{i+1} images are identical, these quadrilaterals are degenerate. For a generic rotation quadrilateral there exist two (possibly complex or coinciding) transversals M and N of the four relative revolute axis. An important property of rotation quadrilaterals is stated in the following completion theorem:

Theorem 5. *Consider two skew lines M, N in the moving space and two displacements a_0, a_2 of a rotation quadrilateral. Generically, there exist two positions a_1, a_3 (possibly complex) such that $[a_0, a_1, a_2, a_3]$ is a rotation quadrilateral with relative revolute axes $R_{i,i+1}$ that intersect M and N.*

Proof: We consider a (yet undetermined) position x such that the relative displacements $x \star a_0^{-1}$, $x \star a_2^{-1}$ are rotations whose axes X_0, X_2 intersects M and N. It turns out that this problem amounts to solving a quadratic equation from which the Theorem's claim follows.

The Plücker line coordinate vector of the relative rotation axis X_i' in the fixed space is

$$X_i' = \mathrm{V}(x_i)_\varepsilon \tag{9}$$

(adapted from [12, Satz 13]). According to (5), the Plücker coordinate vector of the relative revolute axis X_i in the moving space is

$$X_i = (a_i)_\varepsilon^{-1} \star X_i' \star (a_i)_\varepsilon. \tag{10}$$

Hence, the sought position x has to satisfy the six linear equations

$$\langle (a_i)_\varepsilon \star \mathrm{V}(x \star a_i^{-1})_\varepsilon \star (a_i)_\varepsilon^{-1}, T \rangle = 0, \quad \pi_5(x \star a_i^{-1}) = 0 \tag{11}$$

with $i \in \{0, 2\}$ and $T \in \{M, N\}$, and the quadratic equation $\langle x, x \rangle = 0$. The solutions are the intersection points of a straight line with the Study quadric (6). $\square$

We will actually need the result of Theorem 5 in the following form:

Corollary 6. *It is possible to complete a rotation quadrilateral whose relative revolute axes intersect two skew lines M, N in the moving space from three admissible input positions a_0, a_1, a_2. The input data is admissible if the two relative displacements $a_1 \star a_0^{-1}$ and $a_2 \star a_1^{-1}$ are rotations whose axes intersect M and N. In this case, the missing position a_3 is unique and real (if the input positions are real).*

3. Discrete rotating motions

Now we are ready to introduce the central concept of this article:

Definition 7. A *rotation net* (or a *discrete rotating motion*) is a quadrilateral net whose vertices and edges are contained in the Study quadric $\mathcal{S}$ in .

The elementary quadrilaterals of rotation nets are rotation quadrilaterals. Rotation nets of dimension two discretize two-parameter motions $x(t_1, t_2)$ with parameter lines $t_i = \text{const.}$ whose instantaneous screws have vanishing pitch, that is, they are actually rotations. The geometric interpretation in terms of the Study quadric is that not only the point $x(t_1, t_2)$ but also the tangents to the parameter lines are contained in the Study quadric. We call parametrized motions of this type *rotating* as well. The extension of this concept to more-dimensional motions is obvious.

Important examples of two-dimensional rotating motions are obtained from a principal parametrization $\mathbf{f}(t_1, t_2)$ of a surface $\Phi \subset \mathbb{R}^3$. The *Darboux frame* associated with this parametrization is the trihedron with base $\mathbf{f}$ and legs $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{n}$ where

$$\mathbf{u}_i = \frac{\partial \mathbf{f}/\partial t_i}{\|\partial \mathbf{f}/\partial t_i\|}, \quad \mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2. \tag{12}$$

The Darboux frame is orthonormal. As t_1 and t_2 vary in their respective domains, it moves along the surface Φ. The thus defined motion $x(t_1, t_2)$ is rotating and we may call it the Darboux motion of the principal parametrization. If we add the rotation angle w about the surface normal $\mathbf{n}(t_1, t_2)$ as a third parameter the motion $x(t_1, t_2, w)$ is still rotating (see [7, Section 7.1.5], in particular Remark 7.1.16). It is called a *gliding motion* along Φ. Similar results hold true for more-parameter motions. These observations motivate our study of the relation between discrete rotating motions and discrete principal curvature parametrizations.

3.1. Principal contact element nets and rotating motions

We are going to discuss possibilities for generating a principal contact element net $(p_{\mathbf{i}}, \nu_{\mathbf{i}})$ as trajectory of a discrete rotating motion. Any two neighbouring contact elements $(p_{\mathbf{i}}, \nu_{\mathbf{i}})$ and $\tau_i(p_{\mathbf{i}}, \nu_{\mathbf{i}})$ have a plane of symmetry $\beta^i_{\mathbf{i}}$ and, starting from one contact element, the complete net can be generated by successive reflections in the planes $\beta^i_{\mathbf{i}}$. The sequence of reflections between two contact elements is not unique but never leads to contradictions.

Sometimes (as for example in [4]) it is useful to assign an orthonormal frame $(X_{\mathbf{i}}, Y_{\mathbf{i}}, Z_{\mathbf{i}})$ to every element of a principal contact element net such that the origin coincides with $p_{\mathbf{i}}$, $Z_{\mathbf{i}}$ is the normal line and the $X_{\mathbf{i}}$- and $Y_{\mathbf{i}}$-axes correspond in the reflections at the planes $\beta^i_{\mathbf{i}}$. The thus obtained discrete line congruences $X_{\mathbf{i}}$ and $Y_{\mathbf{i}}$ can be regarded as discretizations of one family of principal curvature lines. It is obvious that by a simple change of orientation of certain $X_{\mathbf{i}}$- and $Y_{\mathbf{i}}$-axes the reflection at $\beta^i_{\mathbf{i}}$ can be replaced by a rotation about an axis perpendicular to $N_{\mathbf{i}}$ and $\tau_i N_{\mathbf{i}}$ and incident with $z^i_{\mathbf{i}} = N_{\mathbf{i}} \cap \tau_i N_{\mathbf{i}} \in \beta^i_{\mathbf{i}}$. The thus obtained rotation net can be considered as discretization of a Darboux motion. If we perturb every frame by a rotation through a certain angle about its $Z_{\mathbf{i}}$-axis, neighbouring frames still correspond in a rotation about an axis through $z^i_{\mathbf{i}}$ and in $\beta^i_{\mathbf{i}}$. This rotation net can be considered as a n-dimensional discrete gliding motion along a principal parametrization.

These considerations show that *every principal contact element net occurs in multiple ways as trajectory of a rotation net* — just as any principal parametrization gives rise to infinitely many gliding motions. The rotation angle about the $Z_{\mathbf{i}}$-axis gives one degree of freedom per vertex.

3.2. Pairs of principal contact element nets

By Proposition 4, rotation nets with principal contact element nets as trajectories are characterized by the fact that all relative rotation axis in the moving space intersect a fixed line N whose images constitute the set of contact element normals. Clearly, every point $p \in N$ and the plane ν through p and orthogonal to N define a contact element (p, ν) whose trajectory is a principal contact element as well. In other words, principal contact element nets as trajectories of discrete rotating motions come in one-parameter families of parallel nets.

Calling two principal contact element nets *independent* if there exists a pair of contact elements with the same index $\mathbf{i} \in \mathbb{Z}^n$ but different normals, we aim at a kinematic generation of independent principal contact element nets as trajectories of rotation nets. In view of Proposition 4 we can hope at most for two independent trajectories. The characteristic property is that all relative rotation axes in the moving space intersect two fixed lines M and N. We will show that such rotation nets exist for arbitrary dimension of the underlying motion. This result is rather surprising since a naive counting of free parameters suggest only existence of 2-dimensional motions with this property.

Theorem 8. *A discrete n-dimensional rotating motion with two independent principal contact element nets as trajectories is uniquely defined by two skew axes M, N of the contact elements in the moving space and the values along the coordinate axes in $\mathbb{Z}^n$.*

For $n = 2$ this result follows directly from Corollary 6. Starting with a_{00}, a_{10}, and a_{01}, the position a_{11} is uniquely defined. From a_{10}, a_{20}, and a_{11} we can construct a_{21}, and so on. Two independent principal contact element net trajectories of a discrete rotating motion of

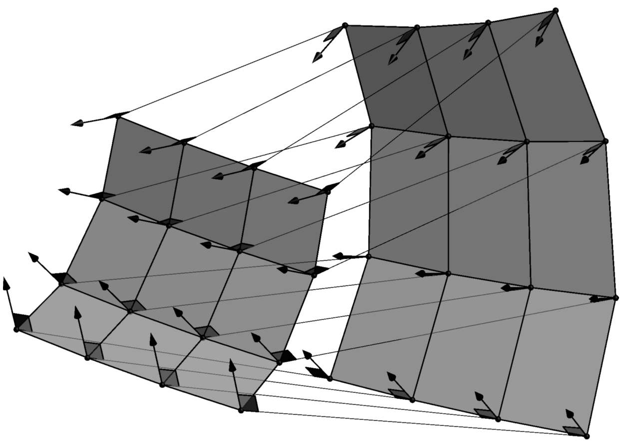

Figure 2: Two independent principal contact element net trajectories of a rotating motion

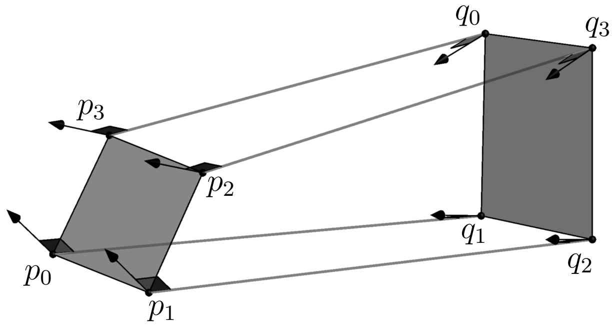

Figure 3: The figures formed by corresponding contact elements are congruent

dimension $n = 2$ are depicted in Fig. 2. The figures formed by corresponding points and face normals are congruent. Neighbouring figures correspond in a pure rotation (Fig. 3).

In case of $n \geq 3$ it is not immediately clear that this inductive construction works. We describe the situation only for $n = 3$; the problems in higher dimensions are similar. Consider the elementary cube a_{ijk} with i, j, $k \in \{0, 1\}$ (Fig. 4). The input data consists of a_{000}, a_{100}, a_{010}, and a_{001}. According to Corollary 6, it defines a_{110}, a_{101}, and a_{011}. Now there are three possibilities to construct the missing vertex a_{111}, from the three positions a_{100}, a_{110}, a_{101}, from the three positions a_{010}, a_{110}, a_{011}, or from the three positions a_{001}, a_{101}, a_{011}. We have to show that all thus constructed positions are actually identical. In the terminology of [3] this is called *3D-consistency* of rotation nets with two independent principal curvature trajectories. In Theorem 9, below, we will show that these nets are actually *nD-consistent.* This is a fundamental property in the discretization of differential geometric concepts and immediately implies Theorem 8.

Theorem 9. *Generic discrete rotation nets with two independent curvature line trajectories are nD-consistent.*

Proof: We have to show that a generic n-dimensional cube can be constructed from one vertex $a_{(0,\ldots,0)}$ and n-adjacent vertices $\tau_i a_{(0,\ldots,0)}$, $i \in \{1, \ldots, n\}$. Using the notation of the last paragraph before the Theorem we proof 3D-consistency at first. We already know that there exist points a_1, a_2 and a_3 such that

$$[a_{100}, a_{110}, a_{101}, a_1], \quad [a_{010}, a_{110}, a_{011}, a_2], \quad [a_{001}, a_{101}, a_{011}, a_3] \tag{13}$$

are rotation quadrilaterals with all required properties. We have to show that these points coincide whereupon we may set $a_{111} := a_1 = a_2 = a_3$.

If we require only one principal contact element net trajectory (for example with normal M and plane $\mu \perp M$), only six linear equations remain. They define a straight line $K \subset P^7$ which, in the sense of algebraic geometry, has at least two intersection points k_1, k_2 with the Study quadric. Since principal contact element nets are known to be nD-consistent [2], the k_1- and k_2-images of the contact element (p, μ) coincide. We denote this contact element by (p_K, μ_K) and conclude that the relative displacement between k_1 and k_2 is a pure rotation. This implies that the line K is actually contained in the Study quadric $\mathcal{S}$.

Denote by $L \subset \mathcal{S}$ the straight line of positions obtained in the same way but with M replaced by N and by (p_L, ν_L) the corresponding contact element. The proof of 3D-consistency will be finished if we can show that K and L have a point in common. Assume conversely that K and L are skew. Then they span a three-space whose intersection with $\mathcal{S}$ is a ruled quadric. We conclude that every position $k \in K$ can be rotated to a unique position $l \in L$. In other words, the contact element (p_K, ν_K) can be rotated in infinitely many ways into the contact element (p_L, ν_L). By elementary geometric reasoning this is only possible if both contact elements have a common tangent sphere. This contradicts the skewness of M and N. Hence, k and l intersect in a unique position a_{111} which implies the Theorem's statement for $n = 3$.

Now we consider 4D-consistency (see Fig. 5 which displays the projection of a 4D-cube). According to Corollary 6, the input data a_{0000}, a_{1000}, a_{0100}, a_{0010}, a_{0001} defines the positions a_{1100}, a_{1010}, a_{1001}, a_{0110}, a_{0101}, a_{0011}. By the previous discussion, we can construct the position a_{1110}, a_{1101}, a_{1011}, a_{0111} without a contradiction. Now we have four ways to construct the missing position a_{1111} by completing a 3D-cube. In Fig. 5 two of these cubes are highlighted.

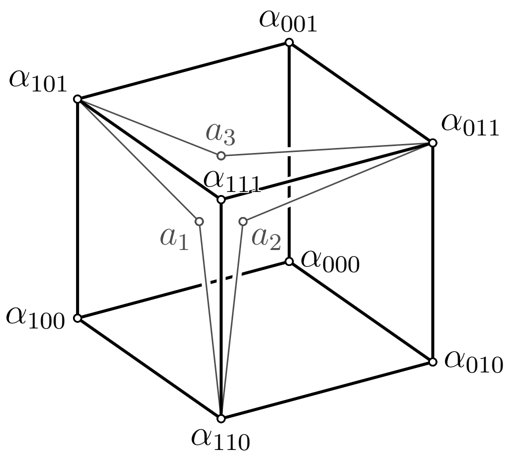

Figure 4: 3D-consistency

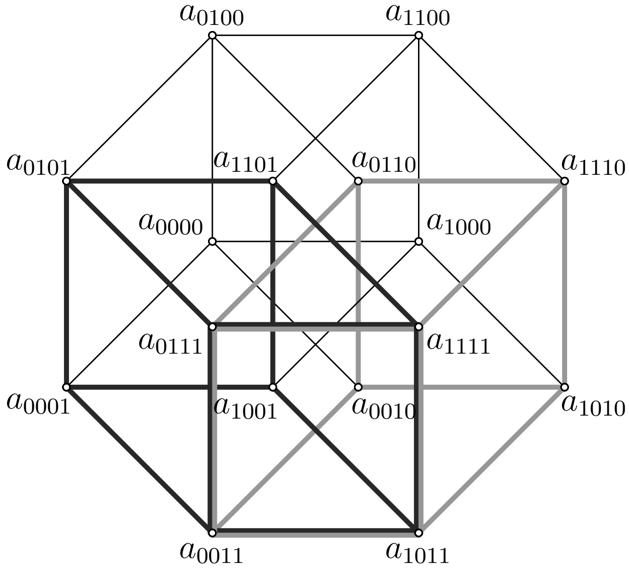

Figure 5: 4D-consistency

They share the common quadrilateral $[a_{0111}, a_{0011}, a_{1011}, a_{1111}]$. Thus, by Corollary 6, completing the 3D-cubes leads to the same position for a_{1111}. This situation does not change if we consider other pairs of 3D-cubes.

The same argument yield nD-consistency: Inductively it can be shown that the input data defines all positions uniquely with exception of one position $a_{\mathbf{1}}$. This position can be constructed by completing n cubes of dimension $n-1$. But any two of these cubes share a face of dimension $n-2 \geq 2$ which already uniquely determines $a_{\mathbf{1}}$. □

The construction of n-dimensional rotation nets as in Theorem 8 from the given input data is inductive. At every step it requires solving the equation system (11) augmented with the Study condition (2). The solution is generically unique and can be computed linearly. When prescribing the positions on the coordinate axes only a subset of (11) needs to be fulfilled.

4. Conclusion and future research

In an attempt to relate recent progress in the field of discrete differential geometry to kinematics this article introduces discrete rotating motions and studies possibilities to obtain curvature line discretizations as trajectories.

We are already in a position to present some of the implications of our study. Assume that $(x_{\mathbf{i}}, \nu_{\mathbf{i}})$ and $(y_{\mathbf{i}}, \mu_{\mathbf{i}})$ are independent principal contact elements obtained as trajectory surfaces of a discrete rotating motion. The figures formed by corresponding contact elements are congruent (Fig. 3) and clearly we can find two families of parallel trajectory surfaces with the same properties. Distinguished examples are obtained by choosing the contact points x and y on the common perpendicular of the contact element normals M and N in the moving space. In this case $x_{\mathbf{i}} \in \mu_{\mathbf{i}}$ and $y_{\mathbf{i}} \in \nu_{\mathbf{i}}$ and the trajectory surfaces satisfy all geometric properties of the classic Bäcklund transform for pseudospherical surfaces (surfaces of constant Gaussian curvature):

- Corresponding points are at constant distance (independent of $\mathbf{i} \in \mathbb{Z}^n$),

- corresponding tangent planes intersect in a constant angle (independent of $\mathbf{i} \in \mathbb{Z}^n$), and
- the connecting line of corresponding points lies in both tangent planes.

This observation leads to the conjecture that the trajectory surfaces are discrete surfaces of constant Gaussian curvature in the sense of [1]. Indeed, we have a proof for this which shall be published elsewhere. By a result of [1], this implies that the parallel trajectory surfaces are linear Weingarten surfaces. An analytic description of a Bäcklund transform but for a different discrete curvature line parametrization ("discrete O-surfaces") can be found in [9]. Our main results, the existence of discrete rotating motions with two independent principal contact element net trajectories and their multidimensional consistency, are meant to provide the basis for the description of Bäcklund transforms of pseudospherical principal contact element nets — in a forthcoming publication.

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Received August 8, 2010; final form August 26, 2010