Journal for Geometry and Graphics
Volume 15 (2011), No. 1, 1–18.

Algorithm for the Parameterization of Rational Curves Revisited

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Abstract. A rational parameterization of an algebraic curve yields a rational correspondence between this curve and the affine or projective line. One of the parameterization methods is based on finding all singular points and $d-3$ simple points of an implicitly given curve of degree d (see [17]). In this paper, we study some modifications of this well-known algorithm, which are then verified on several examples.

Key Words: Algebraic curve, singular points, genus, rational parameterization, quadratic transformation

MSC 2010: 51N35, 14H45, 68U05

1. Introduction

The choice of a suitable representation of geometric shapes (explicit, parametric, or implicit) is essential among others for the study and development of subsequent algorithms. Whereas for example Computer Graphics uses all aforementioned representations, Computer Aided Geometric Design (CAGD) focuses mainly on the piecewise rational parametric representation such as NURBS (Non-uniform B-spline) curves and surfaces, because they are used in many applications in Computer Aided Design and Manufacturing (CAD and CAM) — let us recall plotting and displaying curves and surfaces, computing transformations, finding offsets, determining curvatures, e.g., for shading, etc. (see [6]).

It is well-known, that for general curves and surfaces exact rational parametric representations do not exist and approximate techniques are therefore needed. Moreover, the existence of a rational parametrization of a geometric shape still does not guarantee the rationality of derived objects (like, e.g., classical or general offsets). In what follows, we will focus solely on planar algebraic curves and their exact rational parameterizations. The zero genus is a necessary and sufficient condition for the rationality. A rational parameterization of an algebraic

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ISSN 1433-8157/\$ 2.50

curve yields a rational correspondence between this curve and the affine or projective line. A parameterization algorithm decides whether a parameterization exists and in the positive case produces one particular result. This paper recalls one of the fundamental methods presented in [13] which computes a rational parameterization by intersecting a generic element of a certain 1-parameter pencil of associated curves with the original curve. The algorithm is based on finding all singular points and $d-3$ simple points of an implicitly given curve of degree d; the geometric fundamentals can be found in [17]. We discuss related theoretical results and suggest some particular modifications simplifying the computation. All the theoretical improvements have been implemented and the adapted algorithm was tested on several examples.

Of course, there exist other parameterization algorithms using for instance integral basis computation, canonical divisor or birational transformation of the curve into curve of degree two or one. Reader interested in these techniques can find more details e.g. in [8, 10, 13, 14, 15, 16, 19]. There are also algorithms devoted to turning an arbitrary parameterization into a proper (i.e., birational) one (cf., e.g., [11]). It is a subject for our further research to implement (some of) improvements presented in this paper also to these alternative algorithms.

The main contribution of the paper is a formulation of several improvements of one of the most often applied parameterization algorithm. All the gained results emphasize the indispensable role of classical geometric methods for modern applications. We strongly believe that further thorough analysis of classical geometric fundamentals may lead to next modifications and improvements and thus to offer more effective algorithms.

2. Preliminaries

We start with recalling some fundamental properties of rational curves and Cremona transformations. More details can be found, e.g., in [2, 4, 5, 13, 17, 18].

2.1. Algebraic curves and rational parameterizations

Let K be an algebraically closed field of characteristic zero. The affine or projective space of dimension n over the field K will be denoted by $\mathbb{A}^n$ or $\mathbb{P}^n$, respectively. An *affine plane algebraic curve* $\mathcal{C}_a$ in $\mathbb{A}^2$ is the set of zeros of a square-free polynomial $f(x, y) \in K[x, y]$, i.e.,

$$\mathcal{C}_a = \{(a_1, a_2) \in \mathbb{A}^2 \mid f(a_1, a_2) = 0\},$$

where f is the so-called *defining polynomial* of $\mathcal{C}_a$. The algebraic degree d of f is called the *degree* of $\mathcal{C}_a$. In case f is irreducible, we speak about the *irreducible curve* $\mathcal{C}_a$.

Analogously, a *projective plane algebraic curve* $\mathcal{C}$ in $\mathbb{P}^2$ is the set of zeros of a homogenous *defining polynomial* $F(x, y, z) \in K[x, y, z]$, i.e.,

$$\mathcal{C} = \{(a_1 : a_2 : a_3) \in \mathbb{P}^2 \mid F(a_1, a_2, a_3) = 0\}.$$

Clearly, for the defining polynomial $f(x, y)$ of some affine algebraic curve $\mathcal{C}_a$ we can construct the associated homogenous polynomial $F(x, y, z)$ describing the *associated* projective curve $\mathcal{C}$ just by multiplying each term of degree k by z^{d-k}, where d is the degree of $\mathcal{C}_a$.

An irreducible affine curve $\mathcal{C}_a$ is called *rational* if there exist rational functions $\phi(t), \chi(t) \in K(t)$ such that

(i) for almost all $t_0 \in K$, $(\phi(t_0), \chi(t_0))$ is a point on $\mathcal{C}_a$, and

(ii) for almost all $(x_0, y_0) \in \mathcal{C}_a$ there exists $t_0 \in K$ such that $(x_0, y_0) = (\phi(t_0), \chi(t_0))$.

Then, $(\phi, \chi)(t)$ is called a *rational parameterization* of $\mathcal{C}_a$. Similarly we speak about a *rational parameterization* $(\phi, \chi, \psi)(t)$ of a projective curve $\mathcal{C}$, where $\phi(t), \chi(t), \psi(t)$ are polynomials from $K[t]$.

The point P on $\mathcal{C}$ is called a *singular point* or a *singularity* of $\mathcal{C}$ iff

$$\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial z}(P) = 0.$$

If $P \in \mathcal{C}$ is not singular then it is called *simple* or *regular* on $\mathcal{C}$. Consider a number m_P such that for all $i + j + k < m_P$ the partial derivatives

$$\frac{\partial^{i+j+k} F}{\partial x^i \partial y^j \partial z^k}(P)$$

vanish at P but at least one of the partial derivatives of order m_P does not vanish at P. Then m_P is called the *multiplicity* of P on $\mathcal{C}$.

Furthermore, an m-fold singular point P on $\mathcal{C}$ is called *ordinary* iff all m tangent lines (see for instance [13] for a standard definition of tangent lines) of $\mathcal{C}$ at P are distinct. Otherwise, i.e., if at least two of them are coincident, the singularity is called *non-ordinary*.

One of the most important invariants associated with a singular point P is its *delta invariant* δ_P measuring the number of double points concentrated at this point — in other words, a singular point with delta invariant δ_P concentrates δ_P ordinary double points at P (for more details see [2]). Then, the *genus* of an algebraic curve $\mathcal{C}$ of degree d is defined as

$$\text{genus}(\mathcal{C}) = \frac{1}{2}(d-1)(d-2) - \sum_{i=1}^{n} \delta_{P_i},$$

where $P_1, \ldots, P_n$ are singularities of $\mathcal{C}$ possessing the delta invariants $\delta_{P_1}, \ldots, \delta_{P_n}$, respectively. The genus of the affine curve $\mathcal{C}_a$ is equal to the genus of its associated projective curve $\mathcal{C}$.

The main difficulty in computing genera of algebraic curves consists in determining delta invariants of all singular points. Nevertheless, the situation becomes considerably simpler for ordinary singularities — in this case, the delta invariant is given by $\delta_P = m_P(m_P - 1)/2$. Hence for curves with only ordinary singularities (*ordinary curves*) we can use the formula

$$\text{genus}(\mathcal{C}) = \frac{1}{2}\left((d-1)(d-2) - \sum_{i=1}^{n} m_{P_i}(m_{P_i} - 1)\right).$$

Finally, we recall the fundamental theorem for the theory of rational curves (cf. [18, p. 229].

Theorem 2.1. *An algebraic curve $\mathcal{C}$ is rationally parameterizable if and only if its genus is equal to zero.*

2.2. Cremona transformations and neighbourhood graphs

Birational transformations of the projective plane $\mathbb{P}^2$, forming the so-called *Cremona group*, are called *Cremona transformations*. By Noether's theorem (see [7]) each Cremona transformation of $\mathbb{P}^2$ can be expressed as a composition of quadratic transformations. A *standard quadratic transformation* Q is a Cremona transformation defined as

$$Q : \mathbb{P}^2 \to \mathbb{P}^2 : (x : y : z) \mapsto (yz : xz : xy). \tag{1}$$

The points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ are the *fundamental points* of the quadratic transformation Q. The lines $x = 0$, $y = 0$, $z = 0$ are the *irregular lines* of Q. It holds:

1. Every point of $\mathbb{P}^2$ except the fundamental points of Q is transformed into a unique point. The transformation Q is not defined for the fundamental points.
2. An arbitrary non-fundamental point (i.e., any point different from the fundamental points) lying on one of the irregular lines ($x = 0$, $y = 0$, $z = 0$) of the transformation Q is transformed to the point $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, respectively.
3. A point P not lying on any irregular line of Q is mapped to $P' = Q(P)$ which is not a point of any irregular line of Q^{-1}. It holds $P = Q^{-1}(P')$.

A *quadratic transform* of $F(x, y, z)$ is an irreducible factor $F'(x, y, z)$ of the polynomial $F(yz, xz, xy)$ which is not an irregular line. Curves defined by the polynomials F and F' are in the birational correspondence (except finitely many points). For the later use, we state the theorem whose proof can be found in [17, pp. 77–80].

Theorem 2.2. *Let $\mathcal{C}$ be a projective curve of degree d defined by the polynomial $F(x, y, z)$ containing the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ of multiplicities m_1, m_2 and m_3, respectively. Next, let $F'(x, y, z)$ be the quadratic transform of $F(x, y, z)$ and $\mathcal{C}'$ the curve defined by $F'(x, y, z)$. If none of the tangent lines at these three points is irregular then*

1. $F'(x, y, z) = \dfrac{F(yz, xz, xy)}{x^{m_1} y^{m_2} z^{m_3}}$ *is of degree $2d - m_1 - m_2 - m_3$.*
2. *There exists a one to one correspondence between the tangent lines to $\mathcal{C}$ at the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ and the non-fundamental intersections of $\mathcal{C}'$ with the irregular lines $x = 0$, $y = 0$, $z = 0$, which preserves multiplicities.*
3. *The points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ of the curve $\mathcal{C}'$ have multiplicities $d - m_2 - m_3$, $d - m_1 - m_3$, $d - m_1 - m_2$, respectively. The associated tangent lines are different from irregular lines and they are in the correspondence with non-fundamental intersections of the curve $\mathcal{C}$ with the lines $x = 0$, $y = 0$, $z = 0$.*
4. *The character of an m-fold point not lying on any of irregular lines is preserved.*

One of the most fruitful techniques in the theory of singularities of algebraic curves is involving so-called *neighbouring graphs*, introduced in [13]. Let P be a point on an irreducible curve $\mathcal{C}$. Using a suitable projective transformation T we can map P to the fundamental point $O = (0 : 0 : 1)$ and guarantee that none of the tangent lines of $\mathcal{C}'$ is an irregular line.

Then, the *first neighbourhood* P_α of P with respect to T is the set of all intersections of the quadratic transform $Q(\mathcal{C}')$ of the curve $\mathcal{C}'$ with the line $z = 0$, which are not fundamental points. The multiplicity and character of the neighbouring points P_α of P in its first neighbourhood are defined as the multiplicity and character of points P_α on the curve $Q(\mathcal{C}')$. The *second neighbourhood* of P is defined as the union of all the first neighbourhoods of P_α. In a similar way, we can consider the neighbourhoods of any arbitrary order. A *neighbourhood tree* of P with respect to a sequence of transformations $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ is such tree that has P as its root and the neighbourhood trees of P_α as its subtrees, where P_α are singular neighbouring points at the first neighbourhood of P. A *neighbourhood graph* of $\mathcal{C}$ with respect to $\mathcal{T}$ is the union of all the neighbourhood trees of singular points of $\mathcal{C}$ (cf. [13, 17]).

In particular, the neighbouring point of any simple point is again a simple point and the first neighbourhood of an ordinary m-fold point contains m simple points (see Table 1).

The delta invariant of a given singularity $P \in \mathcal{C}$ can be efficiently computed using the associated neighbouring points, in particular $\delta_P = \sum_{P \in \mathcal{M}} m_P(m_P - 1)/2$, where $\mathcal{M}$ is the

Table 1: Properties of the neighbouring points

Point P	*Neighbouring points* P_α
Simple point	Simple point
Ordinary m-fold point	m simple points
Non-ordinary m-fold point with tangent lines $(a_1x - b_1y)^{m_1} = 0, \dots, (a_sx - b_sy)^{m_s} = 0$	Points $\{P_i = (a_i : b_i : 0)\}_{i=1,\dots,s}$ of multiplicities m_i

neighbouring tree of P with respect to some sequence of transformations $\mathcal{T}$ (see [13, pp. 613–614]). Hence, we can compute the genus using the formula

$$\text{genus}(\mathcal{C}) = \frac{1}{2}\left((d-1)(d-2) - \sum_{P \in \mathcal{N}} m_P(m_P - 1)\right),$$

where $\mathcal{N}$ is the neighbourhood graph of $\mathcal{C}$ with respect to a sequence of transformations $\mathcal{T}$.

3. The classical parameterization algorithm

In this section, we recall the algorithm for computing rational parameterizations of algebraic curves which is based on the theory from [17] and then studied in more details and modified in [13]. The next sections are devoted to some partial improvements of this classical algorithm.

Definition 3.1. Let $\mathcal{C}$ be a curve of degree d. We consider a linear system of curves L^n of degree n such that:

(1) Every curve from the system L^n is of the degree $n \in \{d-2, d-1, d\}$.

(2) Every m-fold singular point of $\mathcal{C}$ is $(m-1)$-fold point L^n.

(3) Every s-fold neighbouring point of $\mathcal{C}$ is an $(s-1)$-fold neighbouring point of each curve from L^n.

(4) All curves from L^n do not have common component with $\mathcal{C}$.

Then, L^n is called a *system of adjoint curves* of degree n to the given curve $\mathcal{C}$.

The system of adjoint curves fulfilling the condition

(5) There exist $nd - (d-1)(d-2) - 1$ simple points of $\mathcal{C}$ being simple points of each curve from L^n

is called *1-parameter system of adjoint curves* of degree n to $\mathcal{C}$. Such system of curves will be denoted by $L^n(t)$ (since it depends on one parameter t).

The introduced system $L^n(t)$ fulfills the following property (cf. [13] for the proof), which is efficiently used in the parameterization process.

Theorem 3.2. *Let $\mathcal{C}$ be an irreducible rational curve of degree d and $L^n(t)$ be the 1-parameter system of adjoint curves to $\mathcal{C}$. Then the coefficients of $L^n(t)$ are polynomials in t. Almost every curve from this system intersects $\mathcal{C}$ in one additional point and for almost every simple point on $\mathcal{C}$, which is not fixed, there exists a curve from $L^n(t)$ passing through that point.*

The algorithm for parameterizing curves with arbitrary singularities is summarized in Algorithm 1 (cf. [13] and references therein for further details).

Algorithm 1 Parameterization of irreducible plane curves

1. INPUT: A curve $\mathcal{C}$ defined by a polynomial $F(x,y,z)$.
2. Find all singular points $P_1,\ldots,P_s \in \mathcal{C}$ having multiplicities $m_{P_1},\ldots,m_{P_n}$.
3. Find the neighbouring graph $\mathcal{N}$ of $\mathcal{C}$ with respect to a sequence of suitable transformations $\mathcal{T}$.
4. Compute the genus of $\mathcal{C}$. If genus($\mathcal{C}$) $= 0$ then $\mathcal{C}$ is rational. Otherwise a rational parameterization of $\mathcal{C}$ does not exist.
5. Choose $n \in \{d-2, d-1, d\}$ and construct the system of adjoint curves L^n to $\mathcal{C}$.
6. Find $nd-(d-1)(d-2)-1$ simple points and guarantee that the curves from L^n are passing through those simple points — obtain 1-parameter system $L^n(t)$.
7. Compute the coordinates of the intersection point $R=(x(t),y(t))$ of $\mathcal{C}_a$ and $L^n_a(t)$, where $\mathcal{C}_a$ and $L^n_a(t)$ are the affine versions of $\mathcal{C}$ and $L^n(t)$, respectively.
8. OUTPUT: Pair of functions $(x(t),y(t))$ represents the rational parameterization of $\mathcal{C}_a$.

Remark 3.3. In the following sections, we will present some improvements of the classical algorithm. Namely, we show the improvements of the steps 2 and 6 of Algorithm 1.

Example 3.4. Let us consider a curve $\mathcal{C}$ (see Fig. 1, left) defined by the polynomial

$$F(x,y,z) = 2x^4 - 2yx^3 + y^2x^2 - z^2x^2 + 2yz^2x - y^2z^2.$$

Solving the system of equations $\partial F/\partial x = 0$, $\partial F/\partial y = 0$, $\partial F/\partial z = 0$, we arrive at one ordinary double point $P_1 = (0:1:0)$ and one non-ordinary double point $P_2 = (0:0:1)$, where the double tangent line of $\mathcal{C}$ at P_2 has the equation $x-y=0$. The intersection of $Q(\mathcal{C})$ with the line $z=0$ yields one neighbouring double point $P_3 = (1:1:0)$ in the first neighbourhood of P_2. This point is ordinary and therefore another neighbouring point does not exist. Hence, the neighbouring graph contains only points P_1, P_2 and P_3.

We have to find one simple point on $\mathcal{C}$ to be able to construct a 1-parameter system $L^2(t)$ of adjoint curves to $\mathcal{C}$; we choose the point $P_3 = (-168:-744:175)$. Thus, we arrive at

$$H(x,y,z,t) = 31tx^2 - 25x^2 - 7tyx + 7zx - 7yz.$$

Finally, the coordinates of the "free" intersection point of $\mathcal{C}_a$ and $L^2_a(t)$ yield

$$x(t) = \frac{2\,(12t^2-25t+12)}{25t^2-48t+25}, \quad y(t) = \frac{2\,(372t^4-1375t^3+1826t^2-1025t+204)}{7\,(25t^4-48t^3+48t-25)}, \quad t \in \mathbb{R}.$$

A simplified version of Algorithm 1 (only for ordinary curves) can be found in [1] or [18]. In this case, it is not necessary to construct the neighbouring graph $\mathcal{N}$ of $\mathcal{C}$ — the system of adjoint curves is given only by the singularities and by suitable simple point(s).

However there exist special curves that can be parameterized by this simplified algorithm despite containing non-ordinary singular points — see Corollary 3.5. This idea is based on the following relation, whose proof is omitted here for the sake of brevity

$$\text{genus}(\mathcal{C}) \le \frac{1}{2}\left((d-1)(d-2) - \sum_{i=1}^{n} m_{P_i}(m_{P_i}-1)\right),$$

where $P_1,\ldots,P_n$ are singularities of $\mathcal{C}$ possessing the multiplicities $m_{P_1},\ldots,m_{P_n}$.

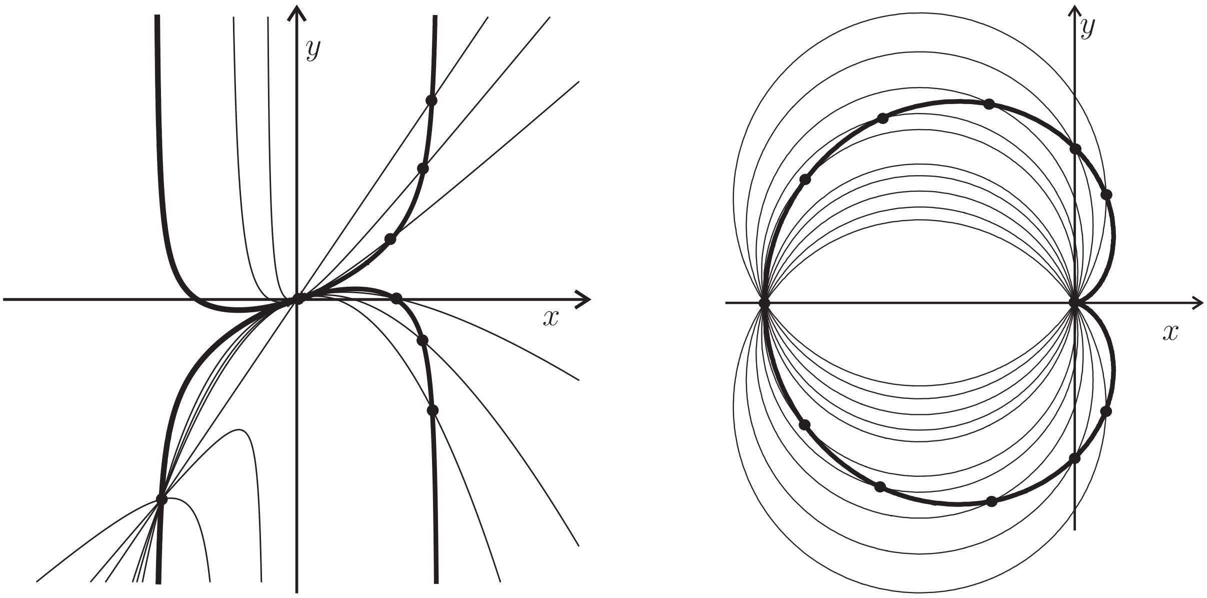

Figure 1: The curve from Example 3.4 (left) and the cardioid from Example 3.6 (right), both with the corresponding 1-parameter systems of adjoint curves.

Corollary 3.5. *Let $\mathcal{C}$ be an irreducible curve of degree d in $\mathbb{P}^2$ having arbitrary singular points $P_1, \ldots, P_n$ with the multiplicities $m_{P_1}, \ldots, m_{P_n}$. Then $\mathcal{C}$ is rational if*

$$(d-1)(d-2) = \sum_{i=1}^{n} m_{P_i}(m_{P_i} - 1).$$

Example 3.6. Let $\mathcal{C}$ be the cardioid (see Fig. 1, right) given by $F(x, y, z)$

$$\mathcal{C} : F(x, y, z) = (x^2 + y^2 + rxz)^2 - r^2 z^2 (x^2 + y^2).$$

This cardioid $\mathcal{C}$ contains one real double point $O = (0 : 0 : 1)$ and two complex conjugate double points $P_1 = (i : 1 : 0)$ and $P_2 = (-i : 1 : 0)$. There is one double tangent line $y = 0$ at O and $x = r/2$ at P_1 and P_2.

All of these three singularities are non-ordinary and have multiplicity 2. We use Corollary 3.5 for determining the rationality of $\mathcal{C}$.

$$(d-1)(d-2) = 3 \cdot 2 = 6 = 2 + 2 + 2 = \sum_{i=1}^{n} m_i(m_i - 1).$$

Hence, the curve $\mathcal{C}$ is parameterizable (the detailed description of the parameterization of cardioid can be found in [1]) and their parameteric expression is, e.g.,

$$x(t) = \frac{-2r^5t^4 + 2r^3t^2}{r^4t^4 + 2r^2t^2 + 1}, \quad y(t) = \frac{-4r^4t^3}{r^4t^4 + 2r^2t^2 + 1}, \quad t \in \mathbb{R}.$$

In [18], the authors parameterize the cardioid using the simplified algorithm for ordinary curves without checking their character. It must be emphasized that this approach does not work generally — for the cardiod, it gives the right solution only because of the fact that the first neighbourhood of each singular point contains only two simple points!

4. Birationally equivalent curve of lower degree

The complexity of the parameterization process depends significantly on the degree of the given curve. Thus, it seems to be more convenient to find a birationally equivalent curve of lower degree, which will be subsequently taken as the input of Algorithm 1. In what follows, we will present a method for the reduction of the degree using quadratic transformations.

The quadratic transformation was used in [17] solely for resolving non-ordinary singularities of an algebraic curve $\mathcal{C}$, specially for finding its neighbouring graph (cf. Section 2.2). During the process of resolving non-ordinary singularities, the degree of the curve became in most cases higher. Nonetheless, we show that the quadratic transformation can be used also for decreasing the algebraic degree.

Let us emphasize that the quadratic transform of a curve $\mathcal{C}$ has the degree $2d - m_1 - m_2 - m_3$, where d is the degree of $\mathcal{C}$ and m_1, m_2, m_3 are the multiplicities of the points $(1:0:0), (0:1:0), (0:0:1)$ on $\mathcal{C}$, respectively.

Proposition 4.1. *Let $\mathcal{C}$ be an algebraic curve containing singular points $P_1, \ldots, P_k$ of multiplicities $m_{P_1}, \ldots, m_{P_k}$, respectively. Then we can decrease the degree of $\mathcal{C}$ and preserve the field of its coefficients if there exist $i_1, i_2, i_3 \in \{1, \ldots, k\}$ such, that*

1. *$P_{i_1}, P_{i_2}, P_{i_3}$ have rational coordinates;*
2. *$P_{i_1}, P_{i_2}, P_{i_3}$ do not lie on a common line;*
3. *$m_{P_{i_1}} + m_{P_{i_2}} + m_{P_{i_3}} > d$.*

Proof: We transform the points $P_{i_1}, P_{i_2}, P_{i_3}$ of $\mathcal{C}$ to the points $(1:0:0), (0:1:0), (0:0:1)$ (this is always possible for linearly independent points) and obtain a curve $\mathcal{C}'$. Then, we apply the quadratic transformation on $\mathcal{C}'$ and get $\mathcal{C}''$. The curve $\mathcal{C}''$ has rational coordinates since both used transformations are rational and it possesses the degree $2d - (m_{P_{i_1}} + m_{P_{i_2}} + m_{P_{i_3}})$ which is lower than d as $m_{i_1} + m_{i_2} + m_{i_3} > d$. □

A rational parameterization of an algebraic curve yields a rational correspondence between this curve and the affine or projective line, which is the most prominent curve among birationally equivalent curves. The following proposition deals with a special class of curves having only one singular point (of course, these curves can be parameterized also using a pencil of lines with vertex at the singularity — see the classical parameterization algorithm).

Proposition 4.2. *Let $\mathcal{C}$ be an algebraic curve of degree d containing one $(d-1)$-fold point $P = (p_1 : p_2 : p_3)$ with rational coordinates. Then $\mathcal{C}$ can be transformed to a line by a certain sequence of projective and quadratic transformations.*

Proof: Every curve $\mathcal{C}$ of degree d with one $(d-1)$-fold rational point P contains infinitely many simple points with rational coordinates. We can obtain them by intersecting $\mathcal{C}$ with the line going through P. By this method, we obtain two simple points with rational coordinates, say P_1 and P_2. Next, we construct a projective transformation which sends the points P, P_1 and P_2 to the points $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$, respectively. We obtain $\mathcal{C}'$ by applying such projective transformation and finally we apply the quadratic transformation to $\mathcal{C}'$. The degree of $Q(\mathcal{C}')$ is $2d-(d-1)-1-1 = d-1$ and $Q(\mathcal{C}')$ contains the point $(0:0:1)$ possessing the multiplicity $d-1-1 = d-2$. Hence, we have got a curve of degree $d-1$ with one rational $(d-2)$-fold point $(0:0:1)$. Clearly, we can repeat this process until we arrive at a line. Let us note that 1-fold point is a regular point. □

Analogously, we can deal with curves containing only two rational singular points. We will show that curves of this kind whose degree can be decreased by the previous approach exist. For instance, let us consider a curve with one $(d-2)$-fold rational point — such curve can have in the best case another singular point with the multiplicity $\lfloor \frac{1}{2}\left(\sqrt{8d-15}+1\right)\rfloor$ (from the definition of genus and rationality). We have only to guarantee that it holds

$$d + \frac{1}{2}\left(\sqrt{8d-15}+1\right) - 2 > d. \tag{2}$$

It is obvious that (2) is fulfilled only for $d \geq 4$. This result was expectable since the rational curves of degree four are the curves of the lowest degree containing more than one singularity.

Example 4.3. Let us consider a curve $\mathcal{C}$ given by the polynomial

$$F(x,y,z) = -x^2y^2 + x^2z^2 + 2xy^2z - 2xz^3 + z^4.$$

The curve $\mathcal{C}$ is of degree 4 and contains three double points $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (1:0:1)$. Thus, we can find a birational equivalent of $\mathcal{C}$ with lower degree, see Proposition 4.1. We construct a transformation, which sends point P_3 to the point $(0:0:1)$ and obtain the curve $\mathcal{C}'$ with the defining polynomial $F'(x,y,z) = x^2z^2 + y^2z^2 - x^2y^2$. Now, we apply the quadratic transformation and get the curve $\mathcal{C}''$ defined by

$$F''(x,y,z) = x^2 + y^2 - z^2.$$

In addition, one could even find a line, which would be in birational correspondence with $\mathcal{C}''$ by finding three simple points with rational coordinates on $\mathcal{C}''$, then sending them to the fundamental points and finally applying the standard quadratic transformation.

5. Computing singular points

An efficient method for constructing a special family of singular points of a given curve was presented in [13]. This approach is based on describing all singular points of the given curve without giving their explicit expression. Clearly, this is very suitable especially for curves with many or complicated singularities. In this section, we recall a classical approach and then give some possible improvements or simplifications of the computational process.

Definition 5.1. Let $\mathcal{C}$ be an irreducible curve defined by $F(x,y,z)$. We say that the set

$$\mathcal{F} = \{(p_1(\alpha) : p_2(\alpha) : p_3(\alpha))_{h(\alpha)=0}\},$$

where $p_1, p_2, p_3, h \in K[\alpha]$ with $\gcd(p_1,p_2,p_3) = 1$ and h is squarefree, is a *family of conjugate points* of $\mathcal{C}$ if

$$F(p_1(\alpha), p_2(\alpha), p_3(\alpha)) \equiv 0 \bmod h(\alpha).$$

To be able to construct such family of singular points we must have guaranteed that all singularities possess different x-coordinates. Hence, we must first examine this necessary condition.

5.1. Construction of a special affine transformation

In case the given curve $\mathcal{C}_a$ contains at least two singularities with the same x-coordinates we have to apply a special affine transformation

$$(x, y) \mapsto (x + my,\, y), \quad m \neq 0,$$

which gives a curve of the required quality. Clearly, we can choose an arbitrary m; we only have to check that the image curve fulfils the constrain. In [9], the authors describe a method based on subresultants (cf. Section 5.1.1). We present two further techniques applying the methods of automatic proving (cf. Section 5.1.2), or a numerical approach (cf. Section 5.1.3).

5.1.1. The Farouki-Sakkalis method

We consider a curve (an affine image of the original curve) defined by the polynomial $\overline{f}(x, y) = f(x + my, y)$ which satisfies the following two extra conditions:
(a) $\overline{f}$ is regular in y, i.e., $\overline{f}$ contains a monomial y^d, where d is the degree of $\overline{f}$;
(b) if the points (x, y_1) and (x, y_2) fulfil the condition $\overline{f} = \partial\overline{f}/\partial x = 0$ then $y_1 = y_2$.

We denote by

$$w(x, m) = \mathrm{Res}_y\left(f(x + my,\, y),\ \frac{\partial f}{\partial x}(x + my, y)\right),$$

where $\mathrm{Res}_\alpha(f, g)$ denotes the resultant of the polynomials f and g with respect to α. Next we denote by $s_k(m)$ the k-th subresultant of the polynomials w and $\partial w/\partial x$ with respect to x. Then the following theorem can be used for verifying the above mentioned conditions (a) and (b); see [9] for the proof and further details.

Theorem 5.2. *If m is such that $f_d(m, 1)s_k(m) \neq 0$, where $f_d(x, y)$ is the monomial of the highest degree d of the polynomial $f(x, y)$, then $f(x + my, y)$ satisfies the above mentioned conditions (a) and (b).*

5.1.2. Symbolic method based on automatic proving

Methods of automatic proving are suitable for such problems where the premises and conclusions are expressible through polynomials (which is our case). Subsequently, a suitable elimination technique, for example Gröbner bases, is applied. We recall that the system of polynomial equations $f_1 = 0, \ldots, f_s = 0$ does not have any solution over algebraically closed field K if and only if the reduced Gröbner basis of the ideal $\langle f_1, \ldots, f_s\rangle$ is equal to $\{1\}$ (cf. [3]).

Clearly, this can be immediately used for checking whether the polynomial $\overline{f}(x, y) = f(x + my, y)$ possesses the required feature or not. In other words, we want to find out if there exist singular points (x, y_1), (x, y_2), where $y_1 \neq y_2$. Such points do not exist if and only if the reduced Gröbner basis of the ideal

$$I = \left\langle \overline{f}(x, y_1),\ \overline{f}(x, y_2),\ \frac{\partial\overline{f}}{\partial x}(x, y_1),\ \frac{\partial\overline{f}}{\partial x}(x, y_2),\ \frac{\partial\overline{f}}{\partial y}(x, y_1),\ \frac{\partial\overline{f}}{\partial y}(x, y_2),\ 1 - w(y_2 - y_1)\right\rangle$$

is equal to $\{1\}$.

This method gives us a very compact condition on deciding the required quality of the singularities of the curve having the equation $f(x + my, y) = 0$. On the other hand, the main drawback is its complexity and hence speed (see Table 4).

5.1.3. A numerical method

Application of Gröbner bases computations is very elegant from the theoretical point of view but quite slow for particular practical implementations. Thus, we formulate another alternative simple method more suitable for including into algorithms.

First, we construct a polynomial in one variable whose roots give us the x-coordinates of all singular points of $\mathcal{C}_a$

$$h(x) = \frac{g}{\gcd(g, g')}, \quad \text{where} \quad g(x) = \gcd\left(Res_y\left(f, \frac{\partial f}{\partial x}\right), \, Res_y\left(f, \frac{\partial f}{\partial y}\right)\right).$$

Now, we apply a suitable numerical method and find all roots $\bar{x}_1, \ldots, \bar{x}_k$ of the polynomial $h(x)$. Next, we repeat the same approach for the y-coordinates of all singular points of $\mathcal{C}_a$ — we obtain $\bar{y}_1, \ldots, \bar{y}_s$. Finally, we choose $\varepsilon > 0$ and check whether there exist at least two different roots $\bar{y}_i, \bar{y}_j$ of the polynomial $h(y)$ satisfying

$$f(\bar{x}_\ell, \bar{y}_i) < \varepsilon \quad \text{and} \quad f(\bar{x}_\ell, \bar{y}_j) < \varepsilon.$$

5.2. Symbolic determination of singular points

A symbolic determination of singular points can be applied on any input curve having solely singularities with different x-coordinates. The main idea lies in a suitable decomposition of the set of all singularities into a union of special families of conjugate singular points. Such decomposition is called a *standard decomposition* of singularities (see [13] for more details).

5.2.1. Classical Sendra-Winkler method

As any projective plane is covered by three affine planes we may work w.l.o.g. only with affine curves. Then, all projective singular points of $\mathcal{C}$ can be found as the union of all singularities of the three associated affine curves. According to [13], we define the polynomials

$$C_i = \gcd\left(Res_y\left(f, \frac{\partial^i f}{\partial x^i}\right), \, Res_y\left(f, \frac{\partial^i f}{\partial x^{i-1} y}\right), \, \ldots, \, Res_y\left(f, \frac{\partial^i f}{\partial y^i}\right)\right). \tag{3}$$

Any polynomial of degree n contains at most an $(n-1)$-fold singular point and thus we can take $i = 1, \ldots, \deg(f) - 1$.

Each polynomial C_i should have the property that its roots give the x-coordinates of all singular points having multiplicity at least $i + 1$ (cf. [13]). However, the polynomial (3) does not fulfil this requirement. If P is a singular point of the multiplicity i on the curve defined by f then all the partial derivatives of f of all orders less then i have to vanish at P. Hence, it is necessary to modify the definition of C_i from [13], i.e., in what follows we consider

$$C_i = \gcd\left(Res_y\left(f, \frac{\partial f}{\partial x}\right), \, Res_y\left(f, \frac{\partial f}{\partial y}\right), \, \ldots, \, Res_y\left(f, \frac{\partial^i f}{\partial x^i}\right), \, Res_y\left(f, \frac{\partial^i f}{\partial y^i}\right)\right).$$

Next, we define polynomials B_i with the same roots as C_i but not containing multiple roots

$$B_i = \frac{C_i}{\gcd(C_i, C'_i)},$$

where C'_i denotes the derivative of C_i. Then, the polynomials $A_i = B_i / B_{i+1}$ give the x-coordinates of all $(i + 1)$-fold singular points of the curve defined by f.

Finally, we express the y-coordinates of all singularities depending on their x-coordinates. Using a suitable elimination technique, e.g., computing the reduced Gröbner basis of the ideal

$$\left\langle f(x,y),\ \frac{\partial f(x,y)}{\partial x},\ \frac{\partial f(x,y)}{\partial y},\ A_i(x)\right\rangle,$$

we arrive at the family of the conjugate singular points in the form

$$\mathcal{F} = \left\{ \left(\alpha : \frac{a_i(\alpha)}{b_i(\alpha)} : 1 \right)_{A_i(\alpha)=0} \right\}, \quad \text{where } a_i(\alpha), b_i(\alpha), A_i(\alpha) \in K[\alpha].$$

5.2.2. The improvement of the classical Sendra-Winkler method

In this part we design a modification based on determining all singular points at once without former computing their multiplicities. We define the polynomial

$$h(x) = \frac{g}{\gcd(g, g')}, \quad \text{where} \quad g(x) = \gcd\left(Res_y\left(f, \frac{\partial f}{\partial x}\right),\ Res_y\left(f, \frac{\partial f}{\partial y}\right)\right).$$

The roots of h are the x-coordinates of all singular points of $\mathcal{C}_a$. We factorize h over $\mathbb{Q}$

$$h = h_1 \cdots h_k$$

and find the y-coordinates of singular points depending on h_i. Hence, we obtain several different families of singular points such that each family contains singularities of the same multiplicity and character (for more details cf. [13]). All singularities of $\mathcal{C}$ are then given by

$$\mathcal{F} = \bigcup_j \left\{ (a_j(\alpha) : b_j(\alpha) : c_j(\alpha))_{h_j(\alpha)=0} \right\}, \quad \text{where } a_j(\alpha), b_j(\alpha), c_j(\alpha), h_j(\alpha) \in K[\alpha].$$

The multiplicities are determined by the following theorem (cf. [13]).

Theorem 5.3. *Let* $\mathcal{F} = \left\{ (m_1(\alpha) : m_2(\alpha) : m_3(\alpha))_{p(\alpha)=0} \right\}$ *be a family of conjugate points, where* $m_1, m_2, m_3, p \in K[x]$ *and* $F(x,y,z) \in K[x,y,z]$. *Then* $F(x,y,z)$ *vanishes at all points of the set* $\mathcal{F}$ *if and only if* $p(\alpha)$ *divides* $F(m_1(\alpha), m_2(\alpha), m_3(\alpha))$.

Example 5.4. We determine the families of affine singularities of the astroid $\mathcal{C}$ given by

$$f(x,y) = \left(x^2 + y^2 - 25\right)^3 + 675x^2y^2.$$

Using any method from Section 5.1, we certify that the curve given by $f(x+2y, y)$ possesses only singularities with the different x-coordinates. Thus, we will continue with the polynomial

$$\overline{f}(x,y) = \left((x+2y)^2 + y^2 - 25\right)^3 + 675y^2(x+2y)^2.$$

Then, the polynomial h has the form

$$h(x) = (x-10)(x-5)(x+5)(x+10)\left(x^2+25\right)\left(x^2+225\right).$$

Hence, we can find the corresponding y-coordinates. Finally, we apply the inverse affine transformation and find the families of the conjugate singular points of $\mathcal{C}$ in the form

$$\mathcal{F} = (0 : -5 : 1) \cup (5 : 0 : 1) \cup (-5 : 0 : 1) \cup (0 : 5 : 1) \cup$$
$$\cup \left\{ (-\alpha : -\alpha : 1)_{\alpha^2+25=0} \right\} \cup \left\{ \left(\frac{\alpha}{3}, -\frac{\alpha}{3}, 1 \right)_{\alpha^2+225=0} \right\}.$$

Using Theorem 5.3, we arrive at the multiplicity 2 of all singular points.

6. Determining simple points

One of the main computational drawbacks of Algorithm 1 is the necessity of determining a high number of simple points on a given curve $\mathcal{C}$. Generally, we have to determine $nd-(d-1)(d-2)-1$ simple points to parameterize a given curve of degree d using a 1-parameter system of adjoint curves of degree $n \in \{d-2, d-1, d\}$ (cf. Table 2).

The authors of [12] overcame significantly this limitation as they proved that it is enough to determine only one point for a 1-parameter system of adjoint curves of degree d, two points for a 1-parameter system of adjoint curves of degree $d-1$ and three points for a 1-parameter system of adjoint curves of degree $d-2$. This result is summarized again in Table 2.

Table 2: Number of necessary simple points depending on the degree of a chosen 1-parameter system of adjoint curves — original approach and improvement from [12]

System of adjoint curves	*Number of simple points for classical method*	*Number of simple points for Sendra-Winkler method*
$L^{d-2}(t)$	$d-3$	3
$L^{d-1}(t)$	$2d-3$	2
$L^{d}(t)$	$3d-3$	1

We show that only one simple point for determining a 1-parameter system of adjoint curves of any degree $n \in \{d-2, d-1, d\}$ is sufficient. The presented approach is based again on the construction of a special family of points — we show how a family of $nd-(d-1)(d-2)-1$ simple points can be determined by only one particular simple point on $\mathcal{C}$.

Let F define an algebraic curve $\mathcal{C}$ of degree d, $P=(p_1 : p_2 : p_3)$ be a simple point on $\mathcal{C}$ and L^n be a system of adjoint curves to $\mathcal{C}$ of degree $n \in \{d-2, d-1, d\}$. Next, we guarantee that the curves from L^n are passing through P. This system will be denoted by L^n_*. Then, we choose one particular curve $\mathcal{M}$ from L^n_* (given by $H(x,y,z)$) and complete the computation by determining the system of simple intersections of $\mathcal{M}_a$ and $\mathcal{C}_a$. Such system should contain $nd-(d-1)(d-2)$ simple points for $p_3 \neq 0$ and $nd-(d-1)(d-2)-1$ for $p_3=0$ (by Bezout's Theorem and the rationality of $\mathcal{C}$). First, we construct a polynomial $A(x)$ whose roots are exactly the common points of $\mathcal{C}_a$ and $\mathcal{M}_a$, i.e.,

$$A(x)=\frac{B}{\gcd(B,B')}, \quad \text{where} \quad B(x)=Res_y(f(x,y),h(x,y)),$$

with $f(x,y)=F(x,y,1)$ and $h(x,y)=H(x,y,1)$. Now, we construct a polynomial whose roots give the x-coordinates of all singular points of $\mathcal{C}_a$

$$C(x)=\frac{D}{\gcd(D,D')}, \quad \text{where} \quad D(x)=\gcd\left(Res_y\left(f,\frac{\partial f}{\partial x}\right), Res_y\left(f,\frac{\partial f}{\partial y}\right)\right).$$

Hence, we can construct a polynomial whose roots are exactly the x-coordinates of simple intersection points of $\mathcal{C}_a$ and $\mathcal{M}_a$

$$E(x)=A(x)/C(x).$$

Finally, we introduce the polynomial g to guarantee that the intersection of $\mathcal{C}_a$ and $\mathcal{M}_a$ does not contain the point P

$$g(x) = \begin{cases} \dfrac{E(x)}{x - p_1/p_3} & \text{for } p_3 \neq 0, \\ E(x) & \text{for } p_3 = 0. \end{cases}$$

Let us note that some of the new intersection points of $\mathcal{C}$ and $\mathcal{M}$ can lie on the line at infinity or two simple points can have the same x-coordinate. In this case, the polynomial $g(x)$ is not of degree $nd - (d-1)(d-2) - 1$ and thus we have to choose a different curve $\mathcal{M}$ from the system L^n_* and repeat the process.

We have the polynomial $g(x)$ with roots giving the x-coordinates of affine simple intersection points of $\mathcal{C}$ with $\mathcal{M}$ and our goal is to find the corresponding y-coordinates. Hence, we compute the reduced Gröbner basis of the ideal

$$\langle f(x,y),\ h(x,y),\ g(x) \rangle.$$

Since there exists exactly one common intersection point for every root of the polynomial $g(x)$, the computed basis contains a polynomial being linear in y, say $a(x) - yb(x)$. From this polynomial we determine a family of $nd - (d-1)(d-2) - 1$ simple points on $\mathcal{C}_a$

$$\mathcal{F} = \left\{ \left(\alpha : \frac{a(\alpha)}{b(\alpha)} : 1 \right)_{g(\alpha)=0} \right\}, \quad \text{where } a(\alpha), b(\alpha), g(\alpha) \in K[\alpha].$$

Finally, applying Theorem 5.3 we ensure that every point from this family is lying on each curve from L^n — and obtain $L^n(t)$. Thus we arrive at the following theorem

Theorem 6.1. *Let $\mathcal{C}$ be a rational curve. Then only one simple point for determining a 1-parameter system of adjoint curves of any degree $n \in \{d-2, d-1, d\}$ to $\mathcal{C}$ is sufficient.*

At the end of this section, we demonstrate our method on a particular example for $n = d - 2$ (for which it is necessary to find 3 simple points in the original approach).

Example 6.2. Let $\mathcal{C}_a$ be an algebraic curve defined by

$$f(x,y) = x^4 + 2x^2y^2 + 10x^3y^2 - 25x^4y^2 + y^4 + 10xy^4.$$

We start with constructing the system of adjoint curves L^4_a

$$L^4: \quad x^2ya_1 + y^3a_1 + x^3a_2 + xy^2a_2 + xy^3a_3 + x^2y^2a_4 + x^3ya_5.$$

We find a simple point on $\mathcal{C}_a$, e.g., $P = \left(\frac{5}{6}, \frac{5}{8}\right)$, and guarantee that the curves from L^4 are going through P. It gives us the following condition $a_1 = -4a_2/3 - 3a_3/10 - 2a_4/5 - 8a_5/15$. Next, we choose one particular curve $\mathcal{M}_a$ from such system curves, e.g., for $a_2 = a_3 = 30$, $a_4 = a_5 = 0$

$$h(x,y) = 30x^3 - 49x^2y + 30xy^2 - 49y^3 + 30xy^3.$$

Using the above given steps, we compute the simple intersection points of $\mathcal{C}_a$ and $\mathcal{M}_a$ having the x-coordinates described as the roots of the polynomial

$$A(x) = -810000x^5 + 3618000x^4 - 4450464x^3 + 925500x^2 + 616225x.$$

The polynomial $C(x) = x$ gives us the x-coordinates of all singular points of $\mathcal{C}_a$. Hence, the x-coordinates of all simple intersections of $\mathcal{C}_a$ and $\mathcal{M}_a$ without the point P are given by

$$g(x) = \frac{A(x)}{C(x)\left(x - \frac{5}{6}\right)} = 135000x^3 - 490500x^2 + 332994x + 123245.$$

The reduced Gröbner basis of the ideal $\langle f(x,y), h(x,y), g(x)\rangle$ contains the polynomial $42682500x^2 - 67403160x + 14365728y - 18363505$. So, we obtain

$$\left\{\left(\alpha,\ -\frac{5\left(8536500\alpha^2-13480632\alpha-3672701\right)}{14365728},\ 1\right)_{135000\alpha^3-490500\alpha^2+332994\alpha+123245=0}\right\}.$$

7. Final examples and conclusion

In this section, we will test[2] our modifications of the classical parameterization algorithm on several random curves of different degrees and complexities (the lower indices in the following list of polynomials denote the algebraic degree of the studied curves). First, we will study the possibility to reduce the degree using the standard quadratic transformation. Next, we will study the computation times of all presented methods for determining singular points. We present the computation times of the three given methods for finding a suitable affine transformation and of the two methods for the symbolic determination of singularities (the computations of a suitable affine transformation via Farouki-Sakkalis and the Automatic proving method was interrupted after 1000 s for the polynomials p_{14}, p_{15} and p_{20}). All gained results are summarized in Tables 3, 4, and 5.

$$p_4 = x^4 + 10x^3 + 2y^2x^2 + 10y^2x + y^4 - 25y^2.$$

$$p_6 = x^6 + 3y^2x^4 - 75x^4 + 3y^4x^2 + 525y^2x^2 + 1875x^2 + y^6 - 75y^4 + 1875y^2 - 15625.$$

$$p_7 = 160x^4y - 115x^4 + 13824x^3y^4 - 17472x^3y^3 + 5424x^3y^2 - 22080x^2y^4 + 22496x^2y^3 - 5354x^2y^2 + 9552xy^4 - 5184xy^3 - 1251y^4.$$

$$p_8 = x^6y^2 + 4x^6y - 8x^6 + 8x^5y^2 - 60x^5 + 14x^4y^2 - 96x^4y - 155x^4 - 32x^3y^2 - 294x^3y - 170x^3 - 138x^2y^2 - 336x^2y - 68x^2 - 166xy^2 - 136xy - 68y^2.$$

$$p_9 = 12x^5y^4 + 24x^5y^2 + 12x^5 + 12x^4y^5 + 39x^4y^3 + 27x^4y + 52x^3y^4 + 48x^3y^2 + 72x^2y^5 + 108x^2y^3 + 48xy^4 + 108y^5.$$

$$p_{10} = x^5y^5 + 21x^5y^4 - 19x^5y^3 + x^5y^2 + x^5y + x^5 + 10x^4y^6 + 174x^4y^5 - 178x^4y^4 + 11x^4y^3 + 10x^4y^2 + 10x^4y + 40x^3y^7 + 552x^3y^6 - 607x^3y^5 - 5x^3y^4 + 83x^3y^3 + 25x^3y^2 + 80x^2y^8 + 816x^2y^7 - 890x^2y^6 - 233x^2y^5 + 338x^2y^4 - 9x^2y^3 + 80xy^9 + 528xy^8 - 452xy^7 - 600xy^6 + 597xy^5 - 95xy^4 + 32y^{10} + 96y^9 + 40y^8 - 452y^7 + 378y^6 - 81y^5.$$

$$p_{11} = 115x^7y^4 - 160x^7y^3 - 70x^7y^2 + 160x^7y - 45x^7 + 300x^6y^4 - 780x^6y^3 - 5084x^6y^2 + 12428x^6y - 6864x^6 - 20x^5y^4 - 11608x^5y^3 + 26636x^5y^2 + 2464x^5y - 19392x^5 - 5904x^4y^4 + 15168x^4y^3 + 43152x^4y^2 - 60096x^4y - 13824x^4 + 1120x^3y^4 + 51296x^3y^3 - 63936x^3y^2 - 55296x^3y + 17472x^2y^4 - 25152x^2y^3 - 82944x^2y^2 - 1920xy^4 - 55296xy^3 - 13824y^4.$$

$$p_{12} = 64x^8y^4 - 512x^8y^3 + 1776x^8y^2 - 3040x^8y + 2032x^8 - 256x^7y^4 + 1504x^7y^3 - 1760x^7y^2 - 5088x^7y + 9184x^7 + 112x^6y^4 + 2528x^6y^3 - 15968x^6y^2 + 15064x^6y + 15608x^6 + 1504x^5y^4 - 10592x^5y^3 - 1912x^5y^2 + 46576x^5y + 11784x^5 - 1680x^4y^4 - 12792x^4y^3 + 48176x^4y^2 + 42776x^4y + 3327x^4 - 5000x^3y^4 + 19056x^3y^3 + 57624x^3y^2 + 13308x^3y + 1848x^2y^4 + 34056x^2y^3 + 19962x^2y^2 + 7424xy^4 + 13308xy^3 + 3327y^4.$$

[2]The methods were tested on the computer with processor AMD Phenom II X4 955, 3.2 GHz.

$$p_{14} = 12x^9y^5 + 27x^9y^4 + 72x^9y^3 + 147x^9y^2 + 112x^9y + 192x^9 + 87x^8y^5 + 252x^8y^4 + 705x^8y^3 + 820x^8y^2 + 1224x^8y + 192x^8 + 300x^7y^5 + 1131x^7y^4 + 2172x^7y^3 + 3318x^7y^2 + 1236x^7y + 144x^7 + 693x^6y^5 + 2440x^6y^4 + 4818x^6y^3 + 3180x^6y^2 + 892x^6y + 48x^6 + 1036x^5y^5 + 3645x^5y^4 + 4164x^5y^3 + 2167x^5y^2 + 300x^5y + 12x^5 + 1125x^4y^5 + 2772x^4y^4 + 2613x^4y^3 + 720x^4y^2 + 72x^4y + 744x^3y^5 + 1573x^3y^4 + 840x^3y^3 + 168x^3y^2 + 379x^2y^5 + 480x^2y^4 + 192x^2y^3 + 108xy^5 + 108xy^4 + 24y^5.$$

$$p_{15} = x^{10}y^5 + x^{10}y^4 - 14x^{10}y^3 + 26x^{10}y^2 - 19x^{10}y + 5x^{10} + 6x^9y^5 - 23x^9y^4 + 38x^9y^3 - 36x^9y^2 + 20x^9y - 5x^9 + x^8y^5 + 8x^8y^4 - 29x^8y^3 - 24x^8y^2 + 98x^8y - 54x^8 + 6x^7y^5 - 16x^7y^4 - 147x^7y^3 + 299x^7y^2 - 65x^7y - 77x^7 + x^6y^5 - 162x^6y^4 + 264x^6y^3 + 186x^6y^2 - 268x^6y - 31x^6 - 53x^5y^5 + 49x^5y^4 + 409x^5y^3 - 321x^5y^2 - 134x^5y + x^5 - 19x^4y^5 + 270x^4y^4 - 125x^4y^3 - 226x^4y^2 + 5x^4y + 58x^3y^5 + 26x^3y^4 - 184x^3y^3 + 10x^3y^2 + 21x^2y^5 - 71x^2y^4 + 10x^2y^3 - 10xy^5 + 5xy^4 + y^5.$$

$$p_{20} = x^{10}y^{10} + 11x^{10}y^9 + 36x^{10}y^8 + 66x^{10}y^7 + 81x^{10}y^6 - 149x^{10}y^5 - 550x^{10}y^4 - 48x^{10}y^3 + 648x^{10}y^2 + 176x^{10}y + 32x^{10} + 16x^9y^{10} + 117x^9y^9 + 318x^9y^8 + 504x^9y^7 - 786x^9y^6 - 4289x^9y^5 - 1020x^9y^4 + 6484x^9y^3 + 2272x^9y^2 + 400x^9y + 91x^8y^{10} + 508x^8y^9 + 1171x^8y^8 - 1538x^8y^7 - 13730x^8y^6 - 6112x^8y^5 + 28070x^8y^4 + 12776x^8y^3 + 2240x^8y^2 + 266x^7y^{10} + 1204x^7y^9 - 1313x^7y^8 - 23093x^7y^7 - 17447x^7y^6 + 68643x^7y^5 + 41192x^7y^4 + 7400x^7y^3 + 461x^6y^{10} - 396x^6y^9 - 21524x^6y^8 - 27472x^6y^7 + 103672x^6y^6 + 84091x^6y^5 + 15970x^6y^4 + 17x^5y^{10} - 10543x^5y^9 - 24561x^5y^8 + 98987x^5y^7 + 112892x^5y^6 + 23525x^5y^5 - 2121x^4y^{10} - 11718x^4y^9 + 58329x^4y^8 + 99786x^4y^7 + 23955x^4y^6 - 2322x^3y^{10} + 19386x^3y^9 + 56052x^3y^8 + 16650x^3y^7 + 2781x^2y^{10} + 18171x^2y^9 + 7560x^2y^8 + 2592xy^{10} + 2025xy^9 + 243y^{10}.$$

To sum up, we presented the classical algorithm for the rational parameterization of algebraic curves with zero genus, introduced in [13], and designed some of its possible modifications and improvements. Namely, the method for reducing the degree of algebraic curves using the standard quadratic transformation was presented and novel approaches to the symbolic determination of singular and simple points of a given curve were studied and discussed. All modifications were implemented in CAS Mathematica® and then tested on several examples which proved the functionality and effectiveness of the introduced methods.

Table 3: The degrees of birationally equivalent curves to given curves after one iteration step and after maximum iteration steps decreasing the degree using the quadratic transformation

Polynomials	*Initial degree*	*Degree after 1 step*	*Final degree after (•) steps*
p_4	4	4	4 (0)
p_6	6	6	6 (0)
p_7	7	4	1 (3)
p_8	8	6	1 (4)
p_9	9	5	5 (1)
p_{10}	10	5	1 (5)
p_{11}	11	7	1 (5)
p_{12}	12	8	4 (3)
p_{14}	14	9	9 (1)
p_{15}	15	10	10 (1)
p_{20}	20	10	1 (6)

Table 4: The computation times (in seconds) of three presented methods for determining a suitable affine transformation

Polynomials	*Farouki-Sakkalis*	*Automatic proving*	*Numerical method*
p_4	0.016	0.001	0.015
p_6	1.28	0.046	0.093
p_7	1.124	0.624	0.016
p_8	0.015	0.078	0.001
p_9	0.967	0.093	0.016
p_{10}	122.57	24.663	0.109
p_{11}	57.689	116.86	0.046
p_{12}	240.819	636.937	3.728
p_{14}	—	—	0.14
p_{15}	—	—	0.156
p_{20}	—	—	4.212

Table 5: The computation times (in seconds) of two presented methods for determining singular points on the given curves

Polynomials	*Sendra-Winkler*	*Improvement of Sendra-Winkler*
p_4	0.001	0.016
p_6	0.031	0.031
p_7	0.063	0.046
p_8	0.188	0.093
p_9	0.078	0.063
p_{10}	0.67	0.266
p_{11}	1.123	0.296
p_{12}	2.728	0.452
p_{14}	5.07	0.811
p_{15}	7.988	1.185
p_{20}	32.745	4.103

Acknowledgments

The first author was supported by the project CZ.1.05/1.1.00/02.0090, the second author was supported by the Research Plan MSM 4977751301. We thank all referees for their valuable comments, which helped us to improve the paper.

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Received July 9, 2010; final form March 25, 2011