

On Polygons Inscribed in Other Polygons

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Abstract. In this article we discuss closure properties of polygonal paths with vertices on the sides of a given polygon p and sides parallel to given directions v_i . In particular it is investigated the question of closedness and periodicity of such paths, which for triangles is shown to be equivalent to Ceva's theorem.

Key Words: Thomsen's figure, Ceva's theorem, inscribed polygons

MSC 2010: 51M04, 51N10

1. Introduction

The basic configuration in this article is a polygon $p = P_0 \dots P_{n-1}$ and a set of directions represented by unit vectors $\{v_0, \dots, v_{n-1}\}$. The lines extending the sides of the polygon are denoted by $\alpha_i = P_i P_{i+1}$. In addition there are considered points $\{X_0, \dots, X_k\}$, each lying on a side of p : $X_0 \in \alpha_0, X_1 \in \alpha_1, \dots$. The basic assumption is that the polygonal path $q_k = X_0 X_1 \dots X_k$ inscribed in p has its sides $\beta_i = X_i X_{i+1}$ parallel to the given v_i . Thus, q_k can be created by starting at a point $X_0 \in \alpha_0$ and projecting it parallel to v_0 to the point $X_1 \in \alpha_1$, then projecting this parallel to v_1 to the point $X_2 \in \alpha_2$, etc. We assume that v_i is not parallel to sides α_i or α_{i+1} .

The problems of concern here are whether there are such *closed* paths q_k inscribed in p , whether these are unique and which conditions on the v_i 's guarantee the existence of such paths. In the sequel we use the terms *inscribed polygon* and *path* interchangeably. *Closed* or *periodic* is called a path whose last point $X_k = X_0$. Closed paths having exactly one point X_i on each side α_i of the enclosing polygon p are called *simply periodic*. Closed paths having two points on each side of p are called *doubly periodic* and so on. The next well known figure of a triangle with inscribed hexagons, whose sides are parallel to the triangle's sides, give a glimpse of the problems at hand.¹

Figure 1 shows *Thomsen's figure* [1, p. 36], [6, p. 99], created by starting at a point X_0 on the side $\alpha_0 = P_0 P_1$ of the triangle $P_0 P_1 P_2$ and applying the previously described procedure

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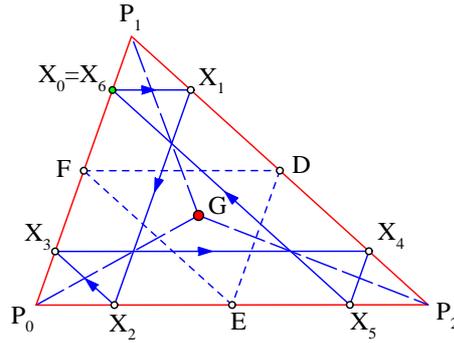


Figure 1: Thomsen's figure

with directions v_i parallel to the sides of the triangle. The point returns after six such parallel projections back to its initial position, thus creating a *doubly periodic* inscribed hexagon in the triangle. When the starting point coincides with the middle of the corresponding side ($X_0 = F$), then we obtain the medial triangle and this is the unique position of X_0 on P_0P_1 for which the hexagon degenerates to a triangle.

A generalization of this property, valid for all closed polygons with an odd number of sides, will be proved in Section 5. Section 2 deals with the properties of the generic case known from the time of PAPPUS (see [9, I, Sect. 498, p. 285], [3, p. 61]). For the sake of completeness we treat in brief this case, too. The discussion in that section overlaps with some parts of [8], though the notation there is somewhat less convenient than here. Section 3 deals with some elementary facts and related calculations which lay the basis of the results concerning the case of triangles. The subsequent sections investigate some special non-generic cases of inscribed polygons with particular attention to configurations producing doubly periodic inscribed polygons, which is a peculiarity occurring only in the case of odd-sided polygons. Section 4 gives a complete account in the case of triangles. Section 5 discusses in brief the case of *complementary* inscribed polygons in odd-sided ones. Section 6 discusses the class of *conjugate* inscribed polygons, which delivers many examples of doubly periodic inscribed polygons. This section, whose prevailing concept is the *affine reflection* ([6, p. 203], [10, II, p. 109]), has also some overlaps with the discussion in [8], though the focus here is directed to another aspect of the configuration under examination than the one discussed there. Finally, Sections 7 and 8 deal in brief with some special configurations delivering interesting examples of doubly periodic polygons.

2. The generic case

In this section, using the notation and the conventions made in the introduction, we look a little bit closer at the relation of X_n to X_0 , i.e., the place at which the polygonal path $q_n = X_0X_1 \dots X_n$ returns to the initial side containing the starting point $X_0 \in \alpha_0$.

Proposition 1. *In the generic case, i.e., assuming no special conditions on the directions $\{v_i\}$, the locus of points X^* of intersection between the first side β_0 and the last side β_{n-1} , while X_0 varies on line α_0 , is a line L intersecting α_0 at a point X^{**} . The polygonal path starting at $X_0 = X^{**}$ closes and defines the unique inscribed polygon $X_0 \dots X_{n-1}$ with sides respectively parallel to the given fixed directions v_i .*

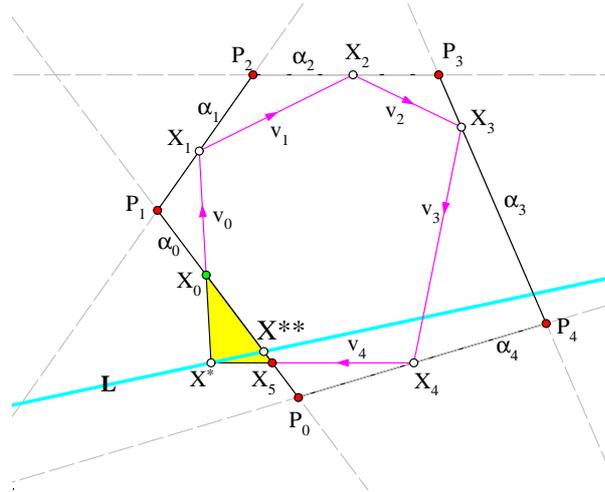


Figure 2: The locus of the point X^* of intersection between β_0 and β_{n-1}

Proof: The basic fact is that, using affine coordinates on the lines α_i , the respective coordinates of any pair (X_i, X_j) are related by a bijective linear function. This is obvious from the way each X_{i+1} is defined by its predecessor X_i and the corresponding v_i . From the definitions follows (as seen in Figs. 2 and 3) that all triangles of the form $X_i P_i X_{i+1}$ have constant angles, hence the relation of X_{i+1} to X_i is linear. When X^* is defined as the intersection of the last side $\beta_{n-1} = X_{n-1} X_n$ with the first side $\beta_0 = X_0 X_1$ of the inscribed path, for the same reason the dependency between X^* and X_0 is linear, and X^* traces a line. \square

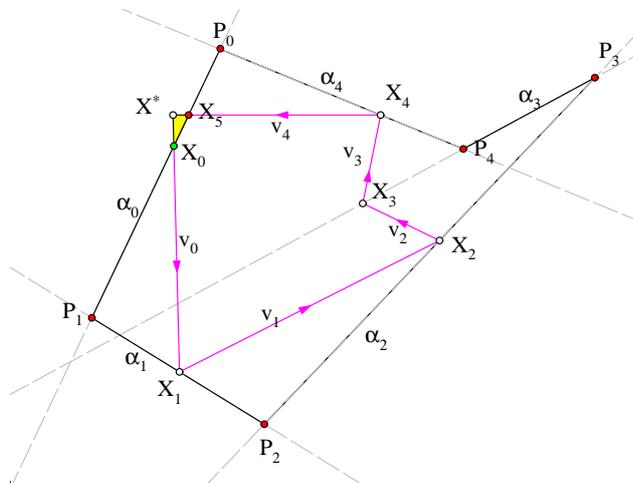


Figure 3: Linear dependency of X_i and X_j

In order to be more specific later, we need the precise formulas for the aforementioned linear dependencies. There is a very simple basic relation that rules the generic as well as the non-generic cases of inscribed polygons, to which belong the doubly periodic ones. From this relation follows, among other things, that there are no periodic polygons with period $k > 2$. In other words, the procedure under study produces either simply periodic or doubly periodic polygons, or paths that never close.

Proposition 2. *In the generic case, i.e., assuming no special conditions on the directions $\{v_i\}$, the only periodic polygon path $X_0 \dots X_m$ (with $X_m = X_0$) is the one given by the previous proposition (with $m = n - 1$) and there are no k -periodic paths with $k > 1$.*

Proof: In fact, in affine coordinates along line α_0 the relation of X_n to X_0 is given by a bijective linear function, called *shift-function* in the sequel,

$$t' = f(t) = at + b.$$

Thus, a k -fold application of the function amounts to

$$f^k(t) = a^k t + (a^{k-1} + \dots + a + 1)b.$$

If, for a specific t_0 we have a recurring polygonal path, then it must be $t_0 = f^k(t_0)$, which by substitution in the previous formula gives

$$t_0(1 - a^k) = (a^{k-1} + \dots + a + 1)b.$$

For $a = -1$ and $k = 2$ this is satisfied for every initial value t_0 . Thus, in such cases, there result doubly periodic inscribed polygons for every initial position X_0 on α_0 . We will see in the next section that this is the case at Thomsen's figure (see Fig. 1) and its generalization for odd-sided polygons (see Section 5). For $a \neq -1$ we can always simplify and obtain

$$t_0(1 - a) = b.$$

For $a \neq 1$ this gives just a unique simply periodic solution (the intersection of line L with side α_0), hence there are no k -periodic polygons with $k > 1$ in this case. For $a = 1$ we have the non-generic case in which $f(t) = t + b$ is a translation. For $b \neq 0$ there are no periodic inscribed polygons; for $b = 0$ the resulting inscribed polygon is periodic for every initial value $X_0 \in \alpha_0$. We return to some instances of this case in Section 6. \square

3. The linear relations

In the following we identify the vertices $\{P_i, X_j\}$ with corresponding position vectors of the plane and assume that the sides $\alpha_i = P_i P_{i+1}$ of the polygon of reference $p = P_0 \dots P_{n-1}$ are respectively parallel to the unit vectors u_0, \dots, u_{n-1} . As already tacitly assumed, indices exceeding $n - 1$ are reduced modulo n . Points $X_i \in \alpha_i$ are related to each other by the following relations

$$X_{i+1} = P_{i+1} + t u_{i+1} = X_i + t' v_i \implies t = \frac{\langle X_i - P_{i+1}, J(v_i) \rangle}{\langle u_{i+1}, J(v_i) \rangle}.$$

Here $\langle \dots, \dots \rangle$ denotes the usual inner product and $J(X)$ denotes the $\pi/2$ -rotation of vectors X , satisfying $\langle X, J(Y) \rangle = -\langle J(X), Y \rangle$, due to $J^2(X) = -X$. This leads to the representation of X_i through the formulas

$$\begin{aligned} X_{i+1} &= Q_{i+1} + f_i(X_i) u_{i+1}, \\ Q_{i+1} &= P_{i+1} - \frac{\langle P_{i+1}, J(v_i) \rangle}{\langle u_{i+1}, J(v_i) \rangle} u_{i+1}, \\ f_i(X_i) &= \frac{\langle X_i, J(v_i) \rangle}{\langle u_{i+1}, J(v_i) \rangle}. \end{aligned}$$

Remark 1. Obviously the existence of X_i 's and consequently the existence of the inscribed polygon is guaranteed only when $\langle u_{i+1}, J(v_i) \rangle \neq 0$. This is equivalent to the *non-parallelity* of side α_{i+1} to the vector v_i .

A simple substitution into the formulas reveals the inductive relations

$$\begin{aligned} f_i(X_i) &= a_i f_{i-1}(X_{i-1}) + b_i, \\ a_i &= \frac{\langle u_i, J(v_i) \rangle}{\langle u_{i+1}, J(v_i) \rangle}, \\ b_i &= \frac{\langle Q_i, J(v_i) \rangle}{\langle u_{i+1}, J(v_i) \rangle}. \end{aligned}$$

From the above follows inductively the relation

$$f_i(X_i) = (a_1 \dots a_i) f_0(X_0) + (b_i + a_i b_{i-1} + a_i a_{i-1} b_{i-2} + \dots + a_i \dots a_2 b_1).$$

Taking the origin of coordinates on α_0 at $P_0 (= P_n)$ and setting $X_0 = t u_0$, we find the coefficients of the linear function expressing $X_n = t' u_0$ in terms of t :

$$\begin{aligned} t' &= f(t) = at + b, \\ a &= a_0 \dots a_{n-1} = \frac{\langle u_0, J(v_0) \rangle}{\langle u_1, J(v_0) \rangle} \cdot \frac{\langle u_1, J(v_1) \rangle}{\langle u_2, J(v_1) \rangle} \dots \frac{\langle u_{n-1}, J(v_{n-1}) \rangle}{\langle u_0, J(v_{n-1}) \rangle}, \\ b &= (b_{n-1} + a_{n-1} b_{n-2} + a_{n-1} a_{n-2} b_{n-3} + \dots + a_{n-1} \dots a_2 b_1). \end{aligned}$$

From these it is seen that $f(t)$ is a non-constant function precisely when the projection directions v_i are non-parallel to the corresponding sides of the polygon, i.e., non-parallel to u_i .

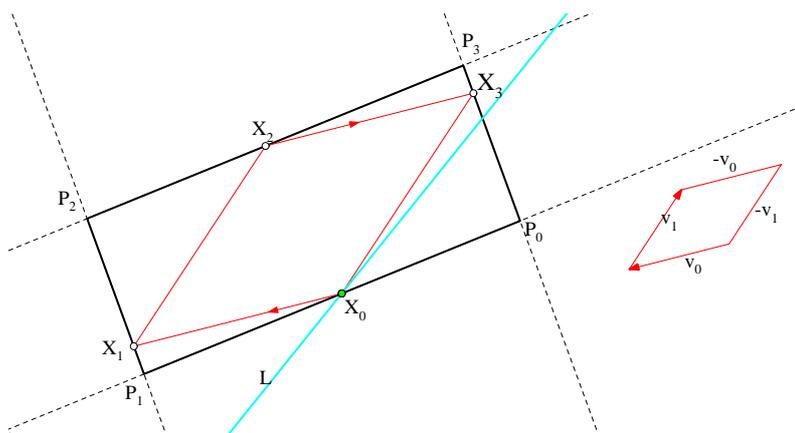


Figure 4: Parallelogram inscribed with given directions of sides

Although the two subsequent sections are applications of these formulas, we cannot withstand the temptation to discuss a simple example illustrating their usefulness:

Let two parallelograms be given (see Fig. 4). Our aim is to inscribe into the first $p = P_0P_1P_2P_3$ a third parallelogram $q_4 = X_0X_1X_2X_3$, whose sides are respectively parallel to the sides of the second parallelogram. By the results of the previous section there is precisely one solution in the generic case, depicted in Fig. 4.

Things become more important in the non-generic case in which the coefficient a of the shift function has the value 1 (see Fig. 5). In this case line L is parallel to the side $\alpha_0 = P_0P_1$ and if $L \neq \alpha_0$ there is no inscribed parallelogram as required. The condition $a = 1$ reduces by the previous formulas and the present circumstances to

$$1 = \frac{\langle u_0, J(v_0) \rangle}{\langle u_1, J(v_0) \rangle} \cdot \frac{\langle u_1, J(v_1) \rangle}{\langle u_2, J(v_1) \rangle} \cdot \frac{\langle u_2, J(v_2) \rangle}{\langle u_3, J(v_2) \rangle} \cdot \frac{\langle u_3, J(v_3) \rangle}{\langle u_0, J(v_3) \rangle} = \left(\frac{\langle u_0, J(v_0) \rangle}{\langle u_1, J(v_0) \rangle} \right)^2 \cdot \left(\frac{\langle u_1, J(v_1) \rangle}{\langle u_0, J(v_1) \rangle} \right)^2,$$

and since $\langle v, J(u) \rangle$ is the sine of the oriented angle (u, v) , it is readily seen that this is equivalent to P_1 being the middle of X_0Y_0 . Thus, with one exception, only impossible cases occur when the directions (v_0, v_1) define a parallelogram (half of which is $X_0X_1Y_0$) with diagonals parallel to the sides of p ; in other words, when (v_0, v_1) are harmonic conjugate to (u_0, u_1) . The one exception is that for which (v_0, v_1) are parallel to the diagonals of p . In that case line L coincides with the side α_0 , and every point $X_0 \in \alpha_0$ delivers an inscribed parallelogram as required.

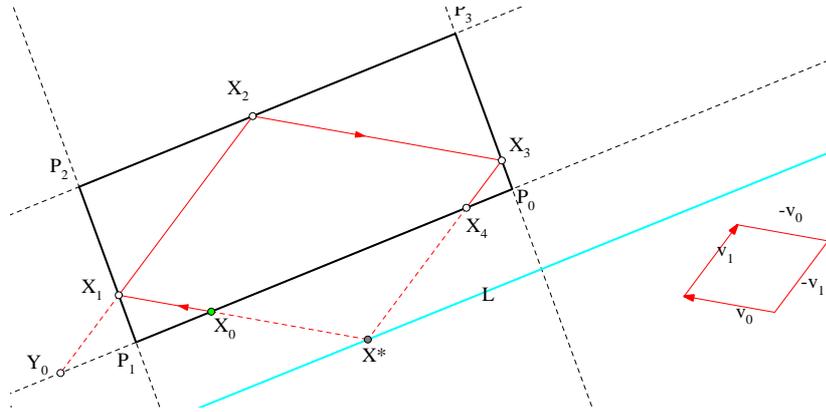


Figure 5: Impossibility to inscribe a parallelogram

4. The triangle

The first in our series of non-generic examples is that of a triangle. Our starting point is a generalization of Thomsen’s figure in another direction, which reveals a connexion of the configuration to the subject of pivoting ([2, Vol. I, p. 326]):

From a point D draw parallels to the sides of the triangle ABC and define the diagonals $\{EF, GH, IJ\}$ of the resulting parallelograms (see Fig. 6). We call them the *conjugate directions* of point D with respect to (w.r.t.) the triangle ABC .

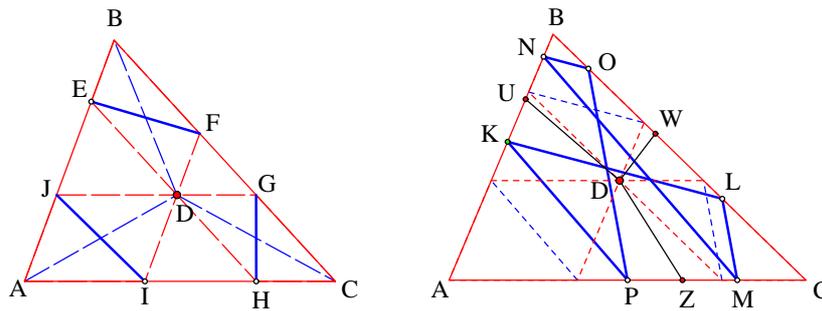


Figure 6: Doubly periodic inscribed polygons in triangles

Proposition 3. *Given is a triangle ABC and a point D not lying on its side-lines. Then, for every point $X_0 \in \alpha_0 = AB$ the inscribed polygon starting at X_0 and having its sides respectively parallel to the conjugate directions of D w.r.t. ABC is closed. Furthermore, for exactly one position U on the side $\alpha_0 = AB$ the corresponding polygon is a triangle. For all other $X_0 \neq U$ the corresponding inscribed polygon is a doubly periodic inscribed hexagon.*

Proof: In fact, take an arbitrary point K on side AB and draw successively the sides KL , LM , MN , NO , OP , PK parallel to the segments EF , GH , IJ , repeating in cyclic order. The resulting polygon returns to K and closes (see Fig. 6). To prove it, we adapt our notations to this situation taking u_0, u_1, u_2 as the unit vectors along the directed sides AB, BC, CA , respectively. The unit vectors v_0, v_1, v_2 are respectively taken along the directed segments EF, GH, IJ . Then the coefficient a of the shift function $f(t) = at + b$ is given by the expression

$$a = \frac{\langle u_0, J(v_0) \rangle}{\langle u_1, J(v_0) \rangle} \cdot \frac{\langle u_1, J(v_1) \rangle}{\langle u_2, J(v_1) \rangle} \cdot \frac{\langle u_2, J(v_2) \rangle}{\langle u_0, J(v_2) \rangle}.$$

It is then seen that the condition $a = -1$, leading to the proof, is equivalent to Ceva's theorem for the three cevians through D . In fact, introducing the three unit vectors w_0, w_1, w_2 respectively along the directed cevians BD, CD, AD , we can express the ratios in terms of the w_i 's. This is because each of these cevians, which are simultaneously diagonals of corresponding parallelograms, is harmonic conjugate to the other diagonal of the parallelogram transferred at the corresponding vertex. For example, the cevian and diagonal BD of the parallelogram $EBFD$ is harmonic conjugate to the other diagonal EF parallel transferred at B and with respect to the two sides BA, BC . Using these relations and the fact that, for unit vectors, the expression $\langle J(u), v \rangle$ equals the sine of the oriented angle (u, v) , we see that

$$\frac{\langle u_i, J(v_i) \rangle}{\langle u_{i+1}, J(v_i) \rangle} = \frac{\langle u_i, J(w_i) \rangle}{\langle u_{i+1}, J(w_i) \rangle}.$$

Using these facts, the expression for the coefficient a becomes

$$a = \frac{\langle u_0, J(w_0) \rangle}{\langle u_1, J(w_0) \rangle} \cdot \frac{\langle u_1, J(w_1) \rangle}{\langle u_2, J(w_1) \rangle} \cdot \frac{\langle u_2, J(w_2) \rangle}{\langle u_0, J(w_2) \rangle}.$$

Then it is readily seen that the condition $a = -1$ is a vectorial form of Ceva's theorem. Notice that Thomsen's figure is the special case for which point D coincides with the centroid of the triangle. \square

Remark 2. The established form $f(t) = -t + b$ of the shift function implies some more properties than those mentioned in the proposition. It shows namely that the middles of the sides of the hexagons are located on the three cevians through D . It shows also that the middles of the segments defined by two vertices of the hexagons on the sides of the triangle are fixed. These are the points U, W, Z in Fig. 6. They are the vertices of a triangle which is similar to the *pre-cevian* ([11, p. 100]) triangle of ABC with respect to D (see Fig. 7). By its definition, this is a triangle $A'B'C'$ whose cevian with respect to D is the triangle of reference ABC .

Proposition 4. *Given is a triangle ABC and a point D not lying on its side-lines. The conjugate directions of D w.r.t. ABC are parallel to the sides of the pre-cevian triangle $A'B'C'$ of D w.r.t. ABC .*

Proof: In fact, draw the parallels from A and B respectively to IJ and EF and define their intersection C' and also the points $A' = AD \cap C'B$ and $B' = BD \cap AC'$ (Fig. 7). Line $A'B'$ contains point C . To see this, consider the pencil $A(B, C, B', D)$ of four lines through A which is harmonic, hence it intersects all lines not passing through A in four points forming a harmonic division. Thus, the points B, B_0, D, B' on the line BB' are harmonic. It follows

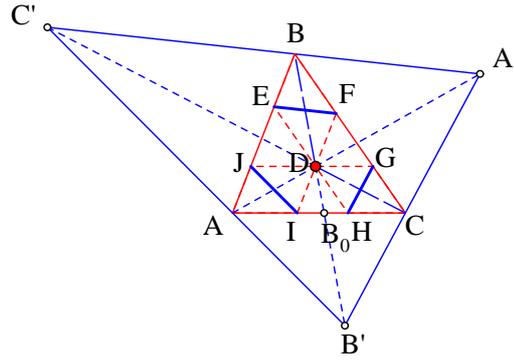


Figure 7: The pre-cevian $A'B'C'$ of ABC with respect to D

that the pencil $C(B, B_0, D, B')$ of lines at C is harmonic, too. Some pairs of lines of the two considered quadruples share a point on the line BC' , e.g., $C' = AB' \cap CD$ or B or $X = AC \cap C'B$. Hence, also the lines of the fourth pair meet on BC' , i.e., A' is the intersection of lines CB' and AD . The harmonicity at C shows that $A'B'$ is also parallel to GH . This completes the proof of the proposition. \square

Since Ceva's theorem stands in the middle of the subject and the theorem is a necessary and sufficient criterion for three lines being concurrent, it is not surprising that the previous construction exhausts all possibilities to inscribe periodic hexagons in triangles. In fact, the corresponding property is in some sense another form of Ceva's theorem. This is formulated in the next proposition referring to Fig. 8.

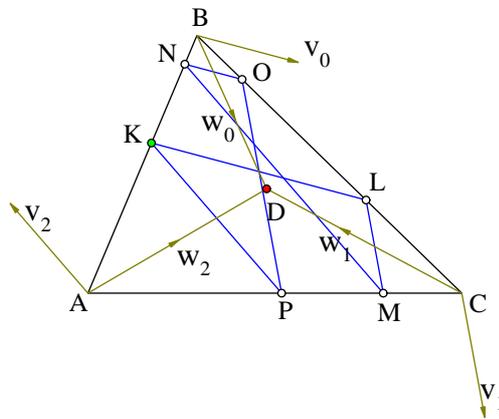


Figure 8: Periodicity and Ceva's theorem

Proposition 5. *Let $KLMNOP$ be a hexagon whose sides are respectively parallel to three directions (v_0, v_1, v_2) and have their end-points on adjacent sides of the triangle with vertices B, C, A . Let (w_0, w_1, w_2) be the corresponding harmonic conjugate directions at these vertices. Then the three lines $B + t w_0$, $C + t' w_1$ and $A + t'' w_2$ are concurrent at a point D .*

Proof: In fact, the assumption implies that the coefficient a of the corresponding shift function $f(t) = at + b$ is equal to -1 :

$$a = \frac{\langle u_0, J(v_0) \rangle}{\langle u_1, J(v_0) \rangle} \cdot \frac{\langle u_1, J(v_1) \rangle}{\langle u_2, J(v_1) \rangle} \cdot \frac{\langle u_2, J(v_2) \rangle}{\langle u_0, J(v_2) \rangle} = -1.$$

By the harmonicity relations this is equivalent to

$$\frac{\langle u_0, J(w_0) \rangle}{\langle u_1, J(w_0) \rangle} \cdot \frac{\langle u_1, J(w_1) \rangle}{\langle u_2, J(w_1) \rangle} \cdot \frac{\langle u_2, J(w_2) \rangle}{\langle u_0, J(w_2) \rangle} = -1,$$

which, as noticed earlier, is Ceva's condition for the concurrence of the three lines through the vertices. \square

Corollary 1. *A triangle $A_1B_1C_1$ inscribed in a second triangle ABC is cevian if and only if there is a hexagon inscribed in ABC , whose sides are respectively parallel to those of $A_1B_1C_1$.*

Remark 3. Given the triangle of reference ABC and the directions (v_1, v_2, v_3) , there are in general two triangles $A_1B_1C_1/A_2B_2C_2$ respectively inscribed/circumscribed in ABC both having sides parallel to the given $\{v_i\}$ (see Fig. 9). If now we rotate the system of $\{v_i\}$ so that their mutual angles remain fixed, then we arrive at the idea of *pivoting* a triangle $A'B'C'$ (could be called of *fixed similarity type*) inside/outside the fixed triangle ABC ([7, p. 109]). It turns out that when the triangle $A_1B_1C_1$ is *cevian*, i.e., the joins of the vertices AA_1, BB_1 and CC_1 intersect at a point D' , the same happens with the joins of the vertices AA_2, BB_2 and CC_2 of the external triangle $A_2B_2C_2$, which then intersect at another point D .

The last proposition shows that given the similarity type of the inscribed pivoting triangle, its position, for which the corresponding parallels to its sides build doubly periodic hexagons, is unique and coincides with the position for which the triangle becomes cevian.

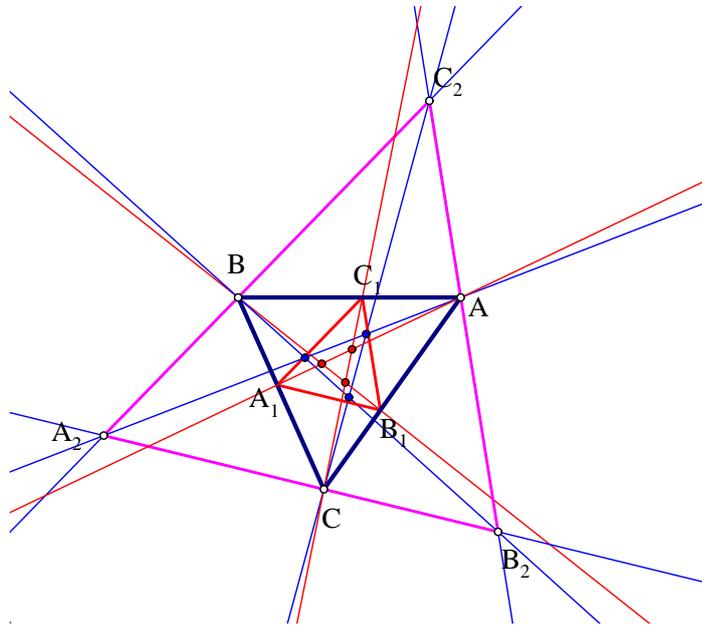


Figure 9: Pivoting inside/outside ABC

Remark 4. The established relations have also another interpretation in terms of *affine reflections* ([5, p. 203]). In fact, given the triangle ABC and a point D' , not lying on its side-lines, we have seen that the corresponding cevian triangle $A_1B_1C_1$ defines, through the middles of its sides, three other cevians intersecting at a point D (see Fig. 10).

Thus a system is created, which could be called *constellar system of affine reflections*, consisting of three affine reflections with respective axes AD, BD, CD , passing through the

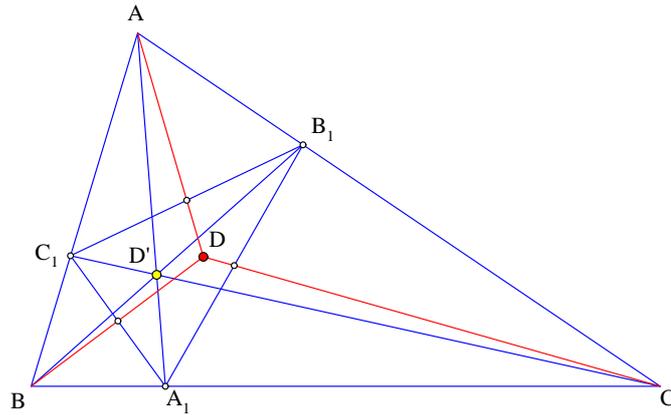


Figure 10: Constellar system of reflections

same point D and conjugate directions B_1C_1, C_1A_1, A_1B_1 , which define a triangular closed orbit. The latter means that there is a point (B_1) which, under the successive application of the three reflections, maps to itself. The established form of the function $f(t) = -t + b$ is equivalent to the fact, that such a system of three reflections has a composition which is also an affine reflection.

Remark 5. As noticed in Section 2, fixing the directions (v_1, v_2, v_3) , taking arbitrary points $X_0 \in \alpha_0 = BC$ and projecting them successively parallel to the given directions onto the sides, produces in each case a point $X^* = X_0X_1 \cap X_2X_3$ varying on a line L , which passes through the vertex (A) opposite to the side α_0 (see Fig. 12).

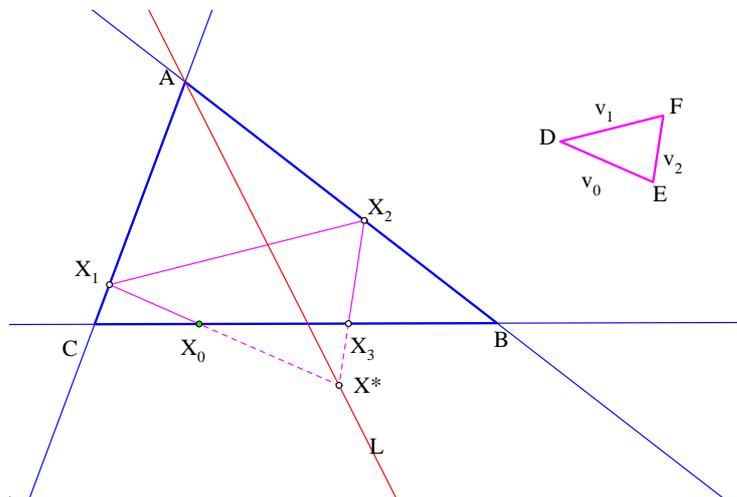


Figure 11: Line L passing through A

In one case there is no closed polygonal path at all. This happens when line L is parallel to α_0 . In this case the shift function $f(t) = t + b$ has the form of a translation. For every initial point $X_0 \in \alpha_0$ the final point X_3 is on α_0 at a fixed distance and direction from X_0 (see Fig. 12). Considering the directions of the v_i 's fixed relative to each other, but their system rotating, i.e., pivoting the triangle DEF with sides parallel to the v_i 's, this corresponds to the position of the pivoting triangle $A_1B_1C_1$ for which this becomes infinite. Equivalently, this is the case when the joining lines $\{AA_1, BB_1, CC_1\}$ (see Fig. 9) become parallel to the opposite

sides of the triangle of reference. In this case also the corresponding triangle $A_2B_2C_2$, which is circumscribed to ABC , degenerates to a point D'' coinciding with its pivot.

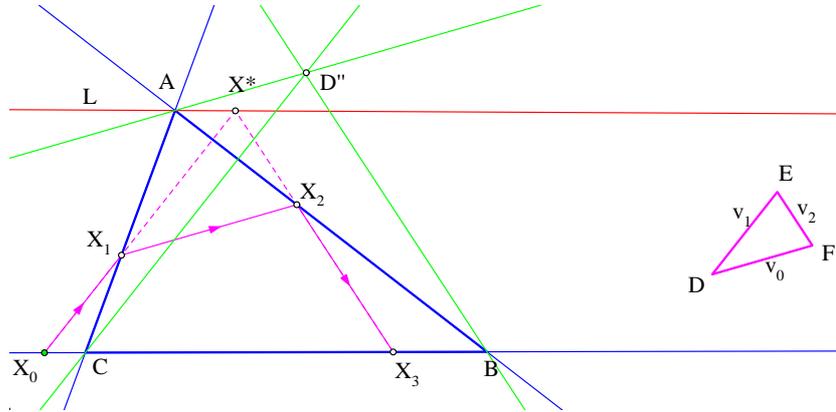


Figure 12: Line L parallel to $\alpha_0 = BC$

5. The complementary inscribed polygon

Referring to Proposition 3 and the notations introduced in its proof, Thomsen’s figure (see Fig. 1) results by making the particular choice $v_0 = -u_2$, $v_1 = -u_0$, $v_2 = -u_1$, which in a general notation can be written

$$v_i = -u_{(i+2) \bmod 3}, \quad i = 0, 1, 2.$$

In this case, it turns out that $a = -1$, and the shift function $f(t) = b - t$ satisfies $f^2(t) = t$. Consequently for every $t \neq \frac{b}{2}$ the polygonal path starting at $X_0 = tu_0$ is doubly periodic, whereas for $t = \frac{b}{2}$ we obtain the unique simply periodic path (corresponding to the medial triangle).

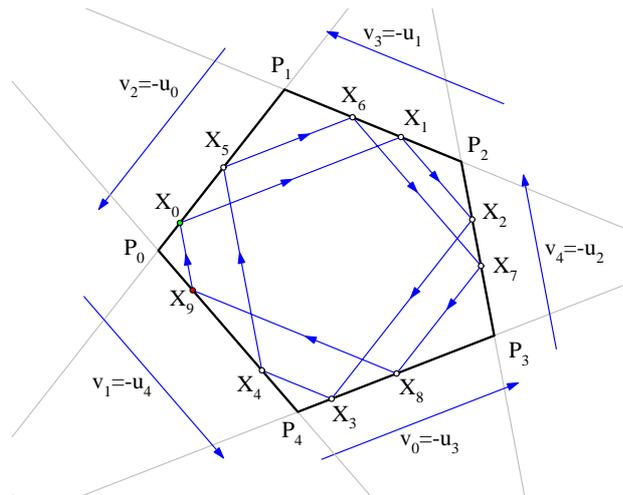


Figure 13: Doubly periodic inscribed polygons in odd-sided polygons

This example generalizes for polygons with an odd number of sides in a way illustrated by Fig. 13, for the case of a convex pentagon, and by Fig. 14 for the case of a non-convex and self-intersecting pentagon. In addition to the notations and conventions made in the introduction,

the next proposition uses the idea of *projection parallel to the sides* of the enclosing polygon. This is defined for polygons with an odd number of sides $n = 2k + 1$. The directions of the sides of the inscribed polygons are simply given by those of the *opposite* sides of the enclosing polygon:

$$v_i = -u_{(i+k+1) \bmod n}, \quad i = 0, 1, \dots, n - 1.$$

This means that every point X lying on side $\alpha_i = P_iP_{i+1}$ of p is projected parallel to v_i onto a point X' on the side α_{i+1} . By its definition, v_i is the direction of the side of the surrounding polygon which lies opposite to P_{i+1} . This is also the common vertex of the side-pair (α_i, α_{i+1}) .

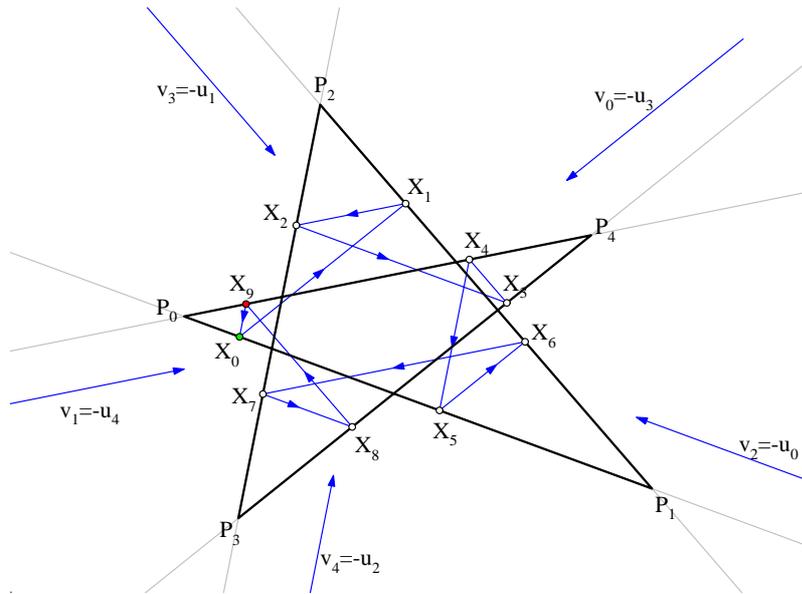


Figure 14: Doubly periodic inscribed polygons in a non-convex one

Proposition 6. *For polygons $p = P_0P_1P_2P_3 \dots P_{n-1}$ with an odd number n of sides, starting at any point $X_0 \in \alpha_0 = P_0P_1$ and projecting successively parallel to the sides generates a closed inscribed polygon. For a unique position $X_0 = X^*$ on side α_0 the corresponding inscribed polygon is simply periodic, whereas, for all $X_0 \neq X^*$ the corresponding inscribed polygon, starting at X_0 , is doubly periodic.*

Proof: The proof is analogous to that of the preceding remark concerning Thomsen’s figure. By assumption, the polygon has $n = 2k + 1$ sides, and the projecting directions satisfy $v_i = -u_{(i+k+1) \bmod n}$. Introducing these into the expression for the coefficient a of the function $f(t) = at + b$, discussed in Section 3, we find easily that this reduces to $a = -1$. Thus, the form of the linear function is now $f(t) = b - t$, and for every point $t \neq \frac{b}{2}$ we have $f(t) \neq t$ but $f^2(t) = t$, whereas for $t_0 = \frac{b}{2}$ we have $f(t_0) = t_0$, giving the simply periodic solution. \square

Figure 15 displays a case of the unique simply periodic inscribed pentagon defined by the preceding proposition. The figure shows also the fact, directly deducible from the formulas, that the vertices of the doubly periodic polygons on each side α_i lie symmetrically with respect to the vertex of this simply periodic polygon lying on the same side.

Remark 6. We call the simply periodic polygon, defined by the preceding proposition the *complementary* polygon of the enclosing polygon p . Among the infinite many polygons inscribed in a fixed odd-sided polygon the *complementary* one is a singularium in the sense

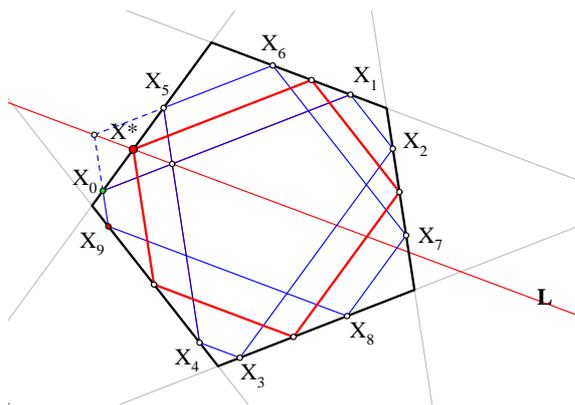


Figure 15: The *dual* inscribed polygon of an odd-sided polygon

that it has its sides parallel to those of the surrounding polygon p . The naming convention conforms to the traditional one ([4, p. 81]), which singles out the medial triangle among the multitude of triangles inscribed in a given triangle.

6. Conjugate polygons

Given a polygon $p = P_0P_1 \dots P_{n-1}$, there are several ways to define other related polygons. An interesting and wide class of such polygons represent the *conjugate polygons* of p w.r.t. points Q . These are defined by taking the harmonic conjugate lines β_i of Q with respect to the side-pairs (α_i, α_{i+1}) . Figure 16 displays such a conjugate polygon $s = S_0 \dots S_4$ for the case of a pentagon. The system of projecting directions (v_0, \dots, v_{n-1}) , which are unit vectors along the sides $\beta_i = S_iS_{i+1}$, defines the directions of projections generating the inscribed polygons.

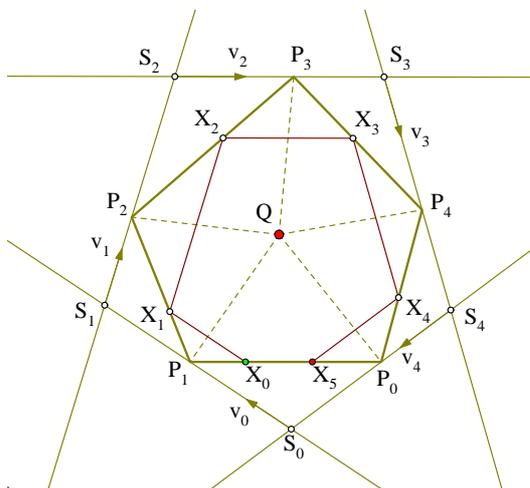


Figure 16: Conjugate $S_0 \dots S_5$ of $P_0 \dots P_5$ with respect to Q

The *shift-function* $f(t) = at + b$ giving the coordinate of X_n in terms of the coordinate of X_0 along side $\alpha_0 = P_0P_1$ (see Section 3) is ruled by the following proposition.

Proposition 7. *Given a polygon $p = P_0 \dots P_{n-1}$ and a point Q , the inscribed polygonal paths $X_0 \dots X_n$, with sides parallel to those of the conjugate polygon $s = S_0 \dots S_{n-1}$ of p w.r.t. Q , define a shift function of the form $f(t) = \pm t + b$, where the sign is -1 for odd-sided polygons ($n = 2k + 1$) and $+1$ for even sided polygons ($n = 2k$).*

Proof: The proof is given by interpreting the correspondences $X_i \mapsto X_{i+1}$ in terms of affine reflections. In fact, let the symbol $r_i = (QP_{i+1}, v_i)$ denote the affine reflection with axis along line QP_{i+1} and conjugate direction parallel to v_i (notation slightly different from the one in [10, II, p. 109]). By definition such a reflection maps a point X to X' , such that XX' is parallel to v_i and the middle of XX' is on QP_{i+1} . In terms of these maps the relation of X_n to X_0 is given by the composition

$$X_n = r_{n-1}(\dots(r_0(X_0))\dots).$$

The proposition follows immediately from next lemma. □

Lemma 1. *For odd-sided polygons the composition $r = r_{n-1} \circ \dots \circ r_0$ is a reflection with conjugate axis parallel to α_0 . For even-sided polygons r is a shear with axis parallel to α_0 .*

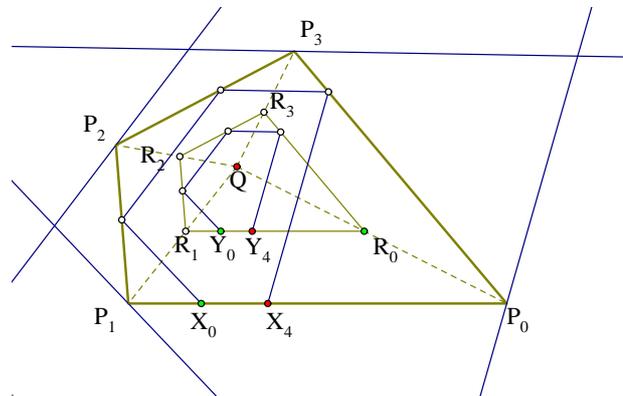


Figure 17: The affine map $r = r_{n-1} \circ \dots \circ r_0$

Proof: The proof of the lemma follows from the fact that line $\alpha_0 = P_0P_1$ remains invariant under r . The same happens for the corresponding sides of the *parallel* polygons $R_0 \dots R_{n-1}$ with vertices defined from those of $P_0 \dots P_{n-1}$ by defining for a fixed $s \neq 0$ (see Fig. 17):

$$R_i = (1 - s)Q + sP_i.$$

Thus, taking an affine coordinate system whose x -axis coincides with α_0 the affine map r has the representation

$$\begin{aligned} x' &= Ax + By + C, \\ y' &= y. \end{aligned}$$

Since r is a composition of affine reflections, its determinant is ± 1 hence $A = \pm 1$. In the case $A = -1$ the transformation of coordinates $\{u = x - (B/2)y - (C/2), v = y\}$ reduces the above matrix representation to $\{u' = -u, v = v\}$, which is the representation of a reflection with conjugate axis α_0 . In the case $A = +1$ it is easily seen that $B \neq 0$, since otherwise r would be a pure translation by a constant vector parallel to the x -axis. But it is easily seen that the image of $Y_0 = (1 - s)Q + sX_0$ is $Y_4 = (1 - s)Q + sX_4$, thus $Y_4 - Y_0 = s(X_4 - X_0)$ showing that r cannot be a pure translation. Thus $B \neq 0$ and in the case $A = 1$ the coordinate-change $\{u = x, v = y + (C/B)\}$ reduces the matrix representation to $\{u' = u + Bv, v' = v\}$, which is the representation of a shear with axis parallel to α_0 . □

Corollary 2. *Let $p = P_0 \dots P_{n-1}$ be a polygon and Q a point not lying on the side-lines of p . Let also $X_0 \dots X_n$ be a polygonal path inscribed in p and with sides parallel to those of the conjugate polygon $s = S_0 \dots S_{n-1}$ of p with respect to Q . Then, if n is odd all these paths are doubly periodic except one which is simple periodic. If n is even, then these paths are either all open or all of them are closed.*

Proof: Since Q is a fixed point of the affine map r , it follows that in both cases the axis of fixed points of r passes through Q . In the case of odd-sided polygons this implies that the intersection of the axis of r with α_0 delivers the unique simply closed polygonal path of the system, whereas for all other positions of $X_0 \in \alpha_0$ we obtain doubly periodic inscribed polygons. As we saw in Section 4, in the case of triangles the above procedure exhausts all possibilities to produce doubly periodic inscribed polygons, examples for $n \geq 5$, as those given in the next sections, show that there are more cases of periodic inscribed polygons in the case of greater (odd) numbers of sides.

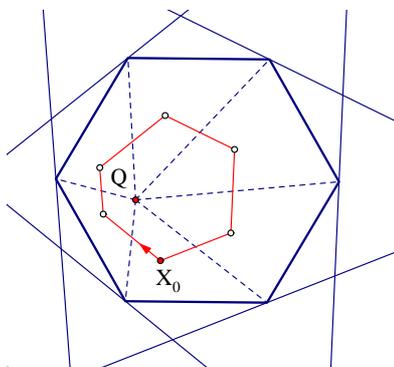


Figure 18: Conjugate-directions paths in regular hexagons

In the case of even sided polygons the previous analysis shows that the procedure of parallel projections along the sides of a conjugate polygon produces inscribed polygons, which are either all non-closed or all are closed. The latter happens in the exceptional case in which the shear r becomes the identity, i.e., in its normal-form representation $\{x' = x + By, y' = y\}$ the coefficient $B = 0$. \square

Remark 7. In [8] we showed that in the case of quadrilaterals the condition $B = 0$ happens exactly when the point Q lies on the Newton line of the quadrilateral. In more general cases the condition $B = 0$ is satisfied by the points of a certain algebraic curve whose equation and geometric properties depend on the enclosing polygon p . In some cases of non-generic polygons the corresponding curve can degenerate. For example, even sided polygons which are also point-symmetric satisfy the corresponding equality $B = 0$ for every point of the plane. This is due to Carnot's theorem, by which, for every point X , the successive symmetric to the vertices of the symmetric polygon come back to the original point ([1, p. 37]). Taking X on the conjugate axis of r_0 through the vertex P_1 and applying this theorem we obtain a closed path and this suffices to show that $B = 0$. Figure 18 illustrates this behavior in the case of a regular hexagon. The polygonal path closes for every possible position of the point X_0 .

7. Diagonal generated polygons

Given a polygon $p = P_0P_1 \dots P_{n-1}$, its diagonals $d_0 = P_0P_2, d_1 = P_1P_3, \dots$ created by skipping one vertex, define a system of corresponding projection-vectors (v_0, \dots, v_{n_1}) . The corresponding polygonal paths (see Fig. 17) demonstrate a typical behavior. For odd-sided polygons they define a shift function of the form $f(t) = -t+b$, whereas for even sided polygons they define a shift function of the form $f(t) = t$. Thus, in the first case for all positions of $X_0 \in \alpha_0$ but one the resulting inscribed polygons are doubly periodic, whereas in the second case for all positions of X_0 the resulting inscribed polygon is closed.

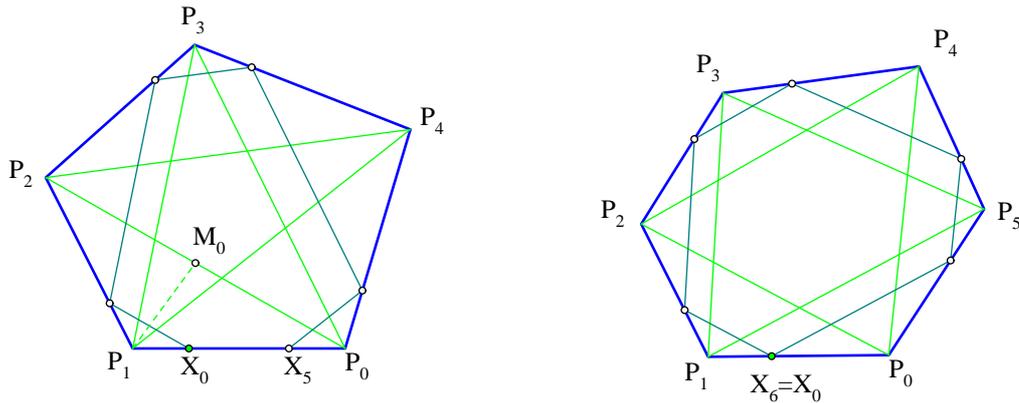


Figure 19: Projecting parallel to diagonals

In the odd-sided case the coefficient of the shift function is easily seen to be $a = -1$ by realizing that this function interchanges the positions of P_0 and P_1 , thus there are t_0, t_1 satisfying $t_1 = at_0 + b$ and $t_0 = at_1 + b$. Analogous is the proof for the shift function in the even-sided case. In that case namely points P_0, P_1 map to themselves, this implying again the stated property.

Remark 8. An argument similar to the one used in the proof of Lemma 1 shows that in the odd-sided case the composition $r = r_{n-1} \circ \dots \circ r_0$ of affine reflections $r_i = (P_{i+1}M_i, d_i)$ is an affine reflection, whereas in the case of even-sided polygons it is the identity. M_i denotes here the middle of the diagonal d_i .

8. Stellar diagonal generated polygons

A slightly more complicated non-generic case of inscribed polygons is the one in which the sides $\alpha_0, \dots, \alpha_{n-1}$ of the polygon $p = P_0P_1 \dots P_{n-1}$ are extended to build intersections and create the stellar-formed polygon with vertices $R_0 = \alpha_{n-1} \cap \alpha_1, R_1 = \alpha_0 \cap \alpha_2, \dots$ (see Fig. 20).

Let the system of projecting directions be that of unit vectors (v_0, \dots, v_{n-1}) respectively parallel to the sides $\beta_0 = R_0R_1, \beta_1 = R_1R_2, \dots$ of the stellar polygon. Define also the affine reflections $r_i = (P_{i+1}M_{i+1}, \beta_i)$, where M_{i+1} denotes the middle of the side β_i . The following proposition rules the closing properties of inscribed polygons in p generated by projecting points $X_0 \in \alpha_0$ parallel to the v_i 's.

Proposition 8. *For odd-sided polygons the composition $r = r_{n-1} \circ \dots \circ r_0$ is a reflection with conjugate axis parallel to α_0 . For even-sided polygons r is a shear whose axis is parallel to α_0 and passes through R_0 .*

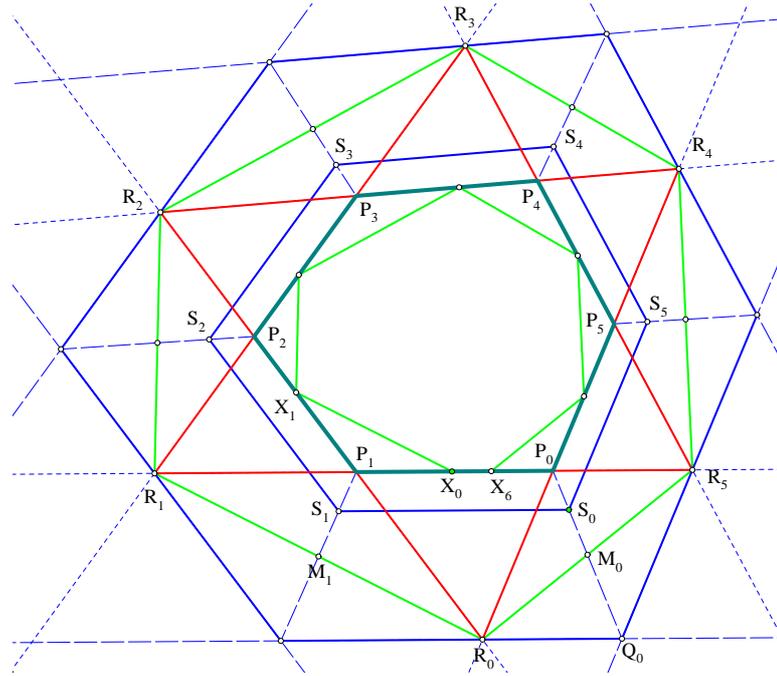


Figure 20: Projecting parallel to the stellar diagonal polygon

Proof: The proof is a variation of the one given for Lemma 1. In fact, consider the polygon $q = Q_0 \dots Q_{n-1}$ created by drawing parallels to the sides of p from the vertices R_i of the stellar polygon. Taking an arbitrary point S_0 on P_0Q_0 and drawing successively parallels to the sides of p , one can create polygons $s = S_0 \dots S_{n-1}$, which, depending on the position of S_0 , vary continuously from p to q . It is easily seen that the first side γ_0 of these polygons is preserved by r . Thus, selecting the x -axis along α_0 , we notice that r can be represented in coordinates through equations of the form

$$\begin{aligned} x' &= \pm x + By + C, \\ y' &= y, \end{aligned}$$

the sign being positive for even-sided polygons and negative for odd-sided ones. By a reasoning as the one for the proof of Lemma-1 we see that for odd-sided polygons r is a reflection whereas for even-sided polygons it is a shear. The statement on the axis in the even-sided case results by observing that R_0 maps onto itself under r . \square

The consequences from this proposition regarding the inscribed polygons are almost the same as those of Section 6 for the conjugate polygons.

Proposition 9. *For odd-sided polygons the inscribed polygons created by successively projecting along the diagonals of the stellar polygon are all but one doubly periodic. In the case of even-sided polygons the above procedure never closes.*

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