Journal for Geometry and Graphics
Volume 16 (2012), No. 2, 123–128.

Gauss-Newton Lines and Eleven Point Conics

Roger C. Alperin

Department of Mathematics, San Jose State University
San Jose, CA 95192 USA
email: alperin@math.sjsu.edu

Abstract. We give a projective version of the Gauss-Newton line for a quadrilateral and its dual form for a quadrangle. These new lines are related to an eleven point conic.

Key Words: triangle geometry, Gauss-Newton line, eleven point conic

MSC 2010: 51M04, 51N15

1. Projective form of Gauss-Newton line

The *complete quadrilateral* consists of four given lines (quadrilateral) and their six intersection points. The *diagonals* are lines which join opposite points of the complete quadrilateral (on distinct quadrilateral lines). There are three such diagonals, and their midpoints lie on the *Gauss-Newton line*.

We give a projective form of the Gauss-Newton line and then relate this to the eleven point conic. In our projective form we consider an auxiliary line L. We intersect the diagonals with L and take the harmonic conjugate of each intersection point with respect to the two opposite points on the associated diagonal of the complete quadrilateral. These three harmonic conjugates lie on line. In Fig. 1 we illustrate this *projective form* of the Gauss-Newton line: the six vertices of the quadrilateral are labelled A, C, D, E, F, G; the diagonals meet L at points labelled by the lower case version of the vertices (in alphabetical order) on the diagonal line: ac, df, eg; the harmonic conjugates (labelled by reversing the order) ca, fd, ge lie on the projective Gauss-Newton line. If we let L be the line at infinity, then harmonic conjugates of intersections with L are the same as midpoints of the corresponding vertices on the side; thus we get the classical Gauss-Newton line.

Next we consider the dual situation. Starting from a quadrangle we obtain three 'Gauss-Newton' lines meeting at a point. We take four points determining a quadrangle $\square DEFG$ and the six lines of its complete quadrangle; its three diagonal points, $\triangle ABC$, obtained from the intersections of the three pairs of opposite sides of the quadrangle. We proceed as before: intersect the six sides of the quadrangle with the line L and take the harmonics with respect to the two associated quadrangle points on the line. These six points lie on the (eleven point) conic $K = K_{L,\square}$ determined by the quadrangle and line.

ISSN 1433-8157/\$ 2.50

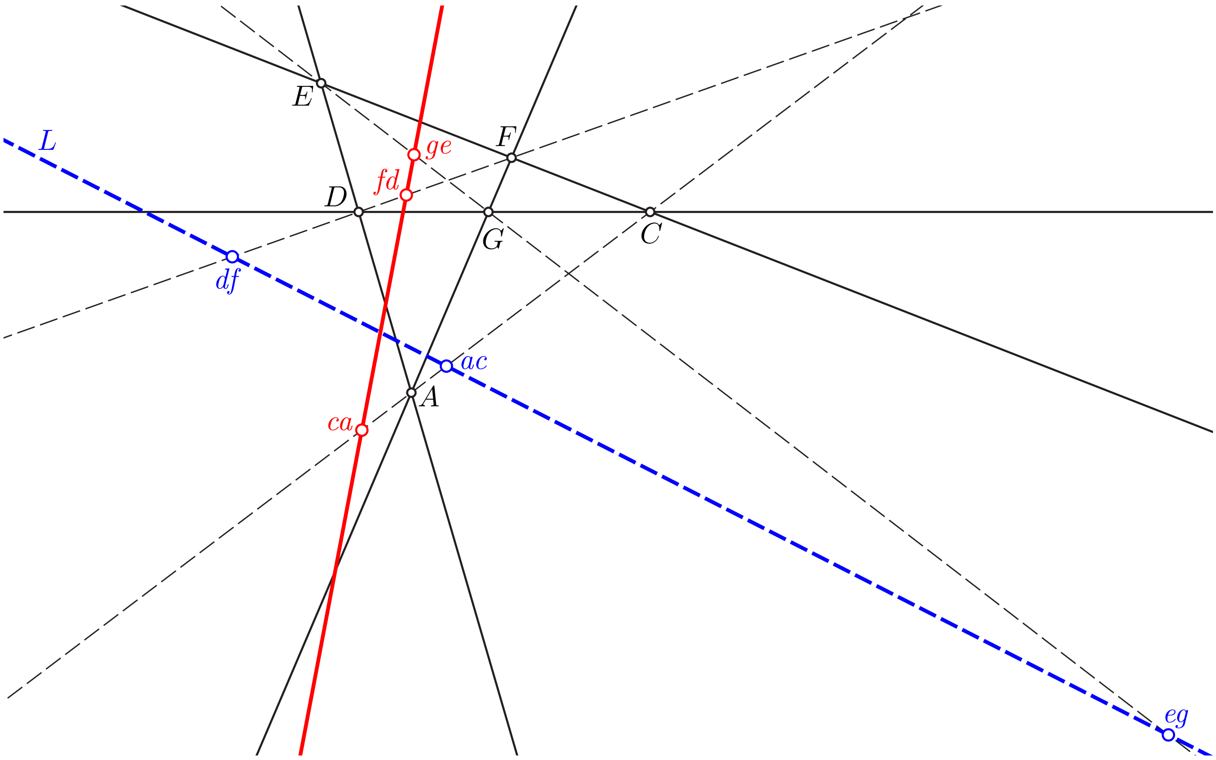

Figure 1: Gauss-Newton line of a complete quadrilateral

In Fig. 2 we display a complete quadrangle with vertices D, E, F, G, and their intersections on the line L, labelled by the corresponding lower case. The harmonic conjugates are shown on K.

The intersections of the three pairs of sides, at the diagonal vertices $\triangle ABC$, with the line L determine 3 pairs of points; the harmonic conjugates of these pairs (with respect to their side quadrangle vertices) determine the three Gauss-Newton lines. In Fig. 3 we show the three Gauss-Newton lines of the complete quadrangle; these lines meet at Z, the pole of L with respect to K.

We discuss the details of this in the next sections. We first review the construction of the eleven point conic.

2. Quadrangle and eleven point conic

We first review some terminology related to quadratic transformations and quadrangles.

Given a quadrangle $\square = \square DEFG$, these are the common points of a *pencil of conics* Π; the diagonal triangle $\triangle = \triangle ABC$ of this $\square$ are the additional vertices lying on the opposite sides of the complete quadrangle formed by the six sides of $\square$.

The polar line of a point P with respect to a conic K is the locus of the harmonic conjugates of P with the intersections of a line at P and the conic K. Two points are called *conjugate* if the polar of one passes through the other point. The conjugate relation is symmetric.

The polar of a point X in the pencil Π gives a pencil of lines meeting at the transform X^*. The transformation $\pi\colon\ X \to X^*$ is a *quadratic transformation* of the plane. It is singular on $\triangle$; it is undefined on the vertices of $\triangle$ and its sides are transformed to the opposite vertex.

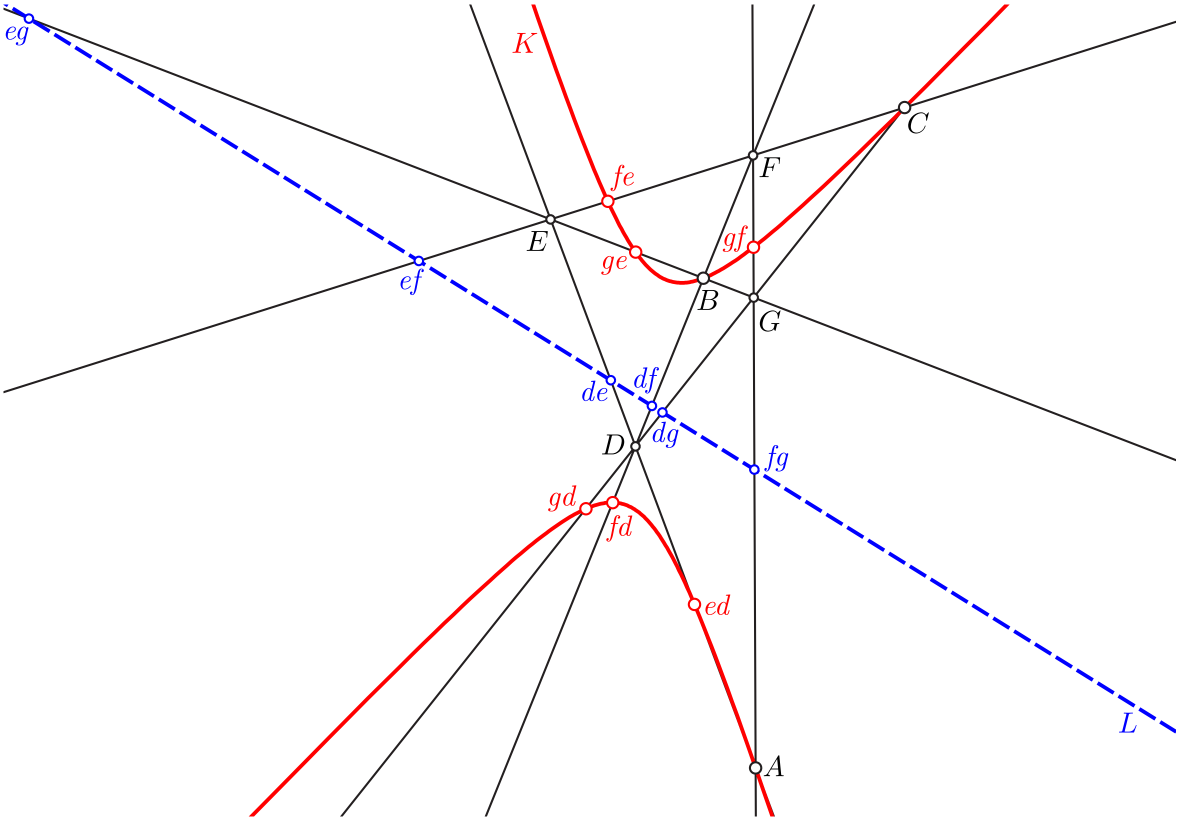

Figure 2: Eleven point conic

The quadratic transform can also be defined by taking two conics $K_1, K_2 \in \Pi$; let $N = Polar(K_1, X)$, then $X \to X^* = Pole(K_2, N)$ is the associated *Cremona transformation*. This definition is independent of the choice of two non-singular conics generating the pencil.

The Desargues involution (using the conic pencil Π) on the line L swaps the intersections of each conic in the pencil. The fixed points e, f of the Desargues involution correspond to the two conics $K_1 = K_e$, $K_2 = K_f$ of Π which are respectively tangent there. Using the second definition of the transform we find $N = L$ and so $e^* = f$. In general the polar of a point $P \in L$ in K_L meets L again at Q, the Desargues involute of P.

Each side of the complete quadrangle determined by $\square$ has two quadrangle points. These quadrangle points are fixed points of π (using the second definition of the transform). Thus the transform of a side through two fixed points is a reducible conic consisting of this side together with the $\triangle$ side opposite the triangle vertex of that side.

2.1. Eleven point conic [2, 3]

For any line L, not passing through A, B, C, the transform $\pi(L)$ is a conic K_L. Any such conic passes through the three singular points A, B, C since the line meets each of the sides of the triangle. Since π is an involution the set $L \cap \pi(L)$ is an invariant set of two points on K_L.

The six lines of the complete quadrangle determined by the quadrangle $\square$ meet the line L at six points which are then transformed to points of K_L. From the definition of the quadratic transform in terms of polars, we see that these six points are the harmonic conjugates of the intersections of the complete quadrangle with L on each of the respective sides.

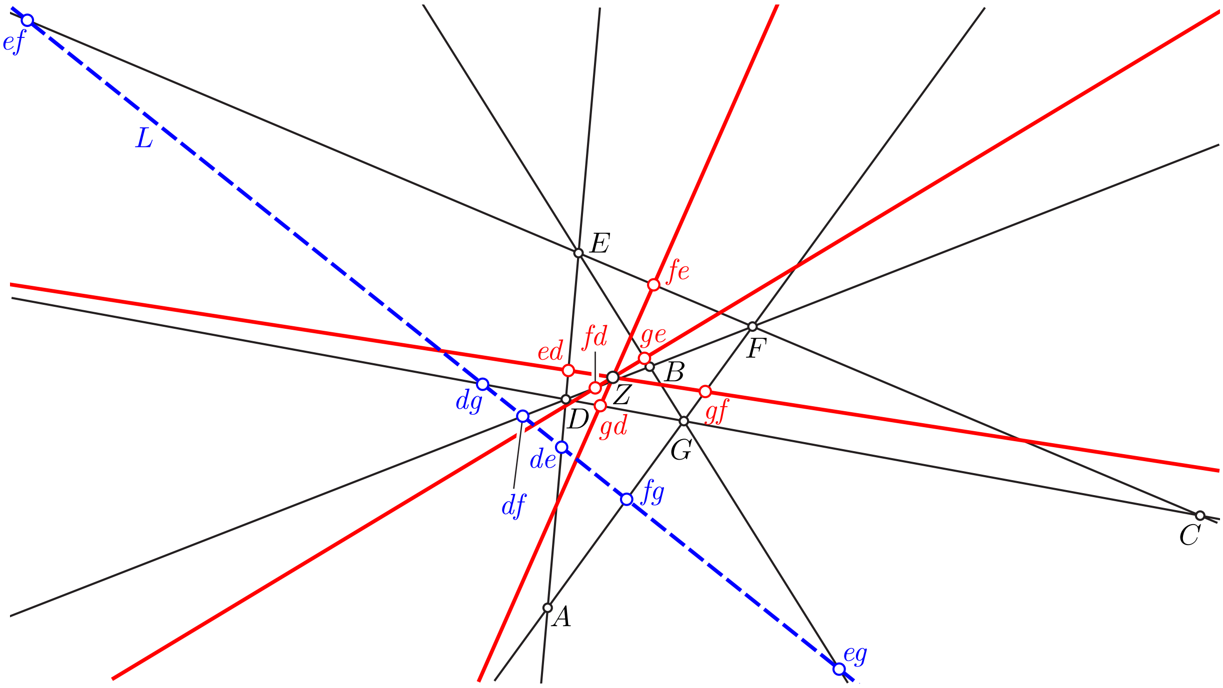

Figure 3: Gauss-Newton lines of a complete quadrangle

Together with e, and f, the fixed points of the Desargues involution, we have 11 points on the conic K. In addition we can define a 12th point as the pole of L with respect to the rectangular hyperbola through the quadrangle points. A special circumstance arises if the pencil Π is a pencil of rectangular hyperbolas.

We next explore some examples.

2.1.1. Steiner conic

The two triangles $\triangle ABC$ and $\triangle DEF$ are in perspective from the point G. Thus they are also in perspective from the (Desargues) tripolar line M.

The transform $K_M = \pi(M)$ is called a *Steiner conic*. It is tangent to the sides of $\triangle DEF$ at the points A, B, C, since three pairs of the eleven points are just three points.

The Gauss-Newton lines are the cevians of the triangle. The polar of the point G in the conic K_M is the line M.

2.2. Poncelet pencil

Given the triangle $\triangle ABC$ and the quadrangle points consisting of the incenter and excenters and L is any line through O then the 11 point conic pencil is a rectangular hyperbola. This gives a pencil of rectangular hyperbolas through $\triangle ABC$ and the orthocenter H [1].

2.2.1. L_∞

The transform of $\pi(L_\infty)$ is the locus of centers of the conics of Π; it is the conic denoted K_∞. If $M = L_\infty$ the Steiner conic is tangent to the sides of $\triangle DEF$ at the midpoints of its sides.

If Π is the pencil of rectangular hyperbolas through $HABC$ (H the orthocenter) then K_∞ is the nine point circle of $\triangle ABC$.

3. Coordinates

By a projective change of coordinates we may suppose the triangle $\triangle ABC$ has its vertices with projective coordinates $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and a quadrangle point $G=(1,1,1)$; then the remaining quadrangle points are $D=(-1,1,1)$, $E=(1,-1,1)$, $F=(1,1,-1)$. An irreducible conic K passing circumscribing $\triangle ABC$ has the simple equation

$$K:\ uyz+vxz+wxy=0$$

for certain constants u, v, w. In this form the Cremona transformation has the standard form

$$\pi:\ (x,y,z)\to(yz,\,xz,\,xy)$$

so that the transform of K is the line L having the equation $ux+vy+wz=0$.

Let us assume now that L is the given line whose eleven point conic is K. Take the two sides of $\square$ meeting at B, EG and DF. These lines are $x=z$ and $x=-z$, so they meet L respectively at the points $(-v,\,u+w,\,-v)$, $(-v,\,u-w,\,v)$. We now determine the harmonics of these points on their lines EG, DF with respect to the quadrangle points.

Using (the y-coordinate of) the points $-1,1$ on the line then the harmonic of the point t is $\frac{1}{t}$. Thus the harmonic of $(-v,\,u+w,\,-v)$ is $(u+w,\,-v,\,u+w)$. Similarly on the line DF the harmonic of $(-v,\,u-w,\,v)$ is $(w-u,\,v,\,u-w)$. Let M be the line passing through $(u+w,\,-v,\,u+w)$ and $(w-u,\,v,\,u-w)$.

The point of AC and L is $(w,0,-u)$ and its harmonic with respect to A,C is the same as using the line $y=0$; so we compute the harmonic conjugate of t with respect to $\infty,0$ to obtain $-t$. Thus the harmonic of $(w,0,-u)$ with respect to A,C is $(w,0,u)$.

Proposition 1. *The points $S=(u+w,\,-v,\,u+w)$ and $T=(w-u,\,v,\,u-w)$ lie on the conic K. The line M passing through S and T has the equation $uvx+(u^2-w^2)y-vwz=0$. The point $(w,0,u)$ and also $(u(v^2+w^2-u^2),\ v(u^2+w^2-v^2),\ w(u^2+v^2-w^2))$, the pole of L with respect to the conic K, lie on the line M.*

Proof: By definition of the eleven point conic, it passes through the harmonic conjugates of the intersection points of the six sides of complete quadrangle determined by the quadrangle $\square DEFG$. The points S,T of the statement are two of these six.

Using the points S,T it is easy to see that the line M has the equation

$$uvx+(u^2-w^2)y-vwz=0\,;$$

it is easy to see that this line passes through $(w,0,u)$ and $(u,-v,w)$.
In general the polar of a point with respect to K is found by multiplying

$$\mathcal{K}=\begin{pmatrix}0 & w & v\\ w & 0 & u\\ v & u & 0\end{pmatrix}$$

by the matrix of coordinates of the point; when multiplied by $(w,0,-u)$ this gives the coordinates for the line M up to a factor. We can also show that the pole of the line L with respect to an irreducible conic K lies on M by using the inverse of $\mathcal{K}$. The pole is $\mathcal{K}^{-1}$ multiplied by (u,v,w). The pole of L is

$$\big(u(v^2+w^2-u^2),\ v(u^2+w^2-v^2),\ w(u^2+v^2-w^2)\big)\,.$$

It is easy to see that this point lies on M. □

Corollary 2. *The three Gauss-Newton lines of the complete quadrangle pass through the pole Z of the line L with respect to K_L.*

Proof: Using the proof of the Proposition, we determined the Gauss-Newton line with respect to the point B; applying the methods similarly for A and C we obtain three Gauss-Newton lines for the quadrangle. The pole

$$Z = \big(u(v^2 + w^2 - u^2),\ v(u^2 + w^2 - v^2),\ w(u^2 + v^2 - w^2)\big)$$

of L with respect to K_L lies on each of these Gauss-Newton lines. □

References

[1] R.C. Alperin: *The Poncelet Pencil of Rectangular Hyperbolas.* Forum Geometricorum **10**, 15–20 (2010).

[2] D. Pedoe: *Geometry: A Comprehensive Course.* Dover, 1988.

[3] P. Samuel: *Projective Geometry.* Springer-Verlag, 1988.

Received January 6, 2012; final form October 11, 2012