Journal for Geometry and Graphics
Volume 17 (2013), No. 2, 163–175.

On Equiform Stewart Gough Platforms with Self-motions

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Abstract. A Stewart Gough (SG) manipulator, where the platform is similar to the base, is called equiform SG manipulator. It is well known that these SG manipulators with planar platform and planar base only have self-motions, if they are architecturally singular; i.e., the anchor points are located on a conic section. Therefore this study focuses on the non-planar case. We prove that an equiform SG manipulator has translational self-motions, if and only if it is a so-called reflection-congruent one. Moreover we give a necessary geometric property of non-planar equiform SG platforms for possessing non-translational self-motions by means of bond theory. We close the paper by discussing some non-planar equiform SG platforms with non-translational self-motions, where also a set of new examples is presented.

Key Words: Stewart Gough platform, Self-motion, Bond theory, Cylinder of revolution

MSC 2010: 53A17, 68T40

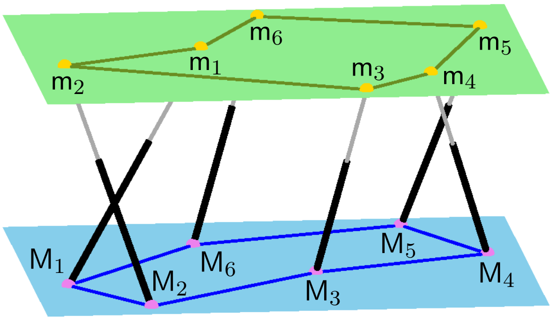
1. Introduction

The geometry of a *Stewart Gough* (SG) *platform* is given by the six *base anchor points* M_i with coordinates $\mathbf{M}_i := (A_i, B_i, C_i)^T$ with respect to the fixed system and by the six *platform anchor points* m_i with coordinates $\mathbf{m}_i := (a_i, b_i, c_i)^T$ with respect to the moving system (for $i = 1, \ldots, 6$). Each pair $(\mathsf{M}_i, \mathsf{m}_i)$ of corresponding anchor points is connected by a *SPS-leg*, where only the prismatic joint (P) is active and the spherical joints (S) are passive (cf. Fig. 1a).

If the geometry of the manipulator is given as well as the leg lengths, the SG platform is generically rigid. But, under particular conditions, the manipulator can perform a n-dimensional motion ($n > 0$) which is called *self-motion*.

Note that self-motions are also solutions to the still unsolved problem posed by the French Academy of Science for the *"Prix Vaillant"* of the year 1904, which is also known as *Borel Bricard problem* (cf. [1, 2, 7]) and reads as follows:

ISSN 1433-8157/\$ 2.50



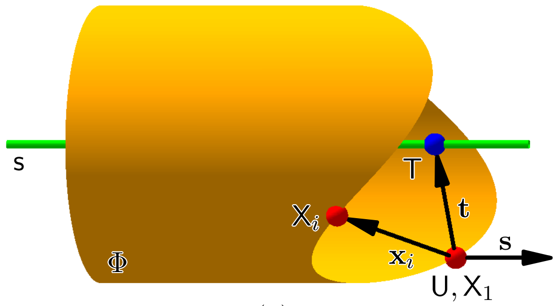


Figure 1: (a) SG manipulator with planar platform and planar base (= planar SG manipulator). (b) Notation used for the computation of cylinders of revolution.

"Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths."

In this article we study so-called equiform[1] SG manipulators, which can be defined as follows:

Definition 1. A SG manipulator is called *equiform*, if an equiform motion[2]

$$\mu: \ \mathsf{m}_i \mapsto \mu(\mathsf{m}_i) = \mathsf{M}_i \quad \text{for} \quad i = 1, \ldots, 6 \tag{1}$$

exists, which does not belong to the subset SE(3) of orientation preserving congruence transformations. If Eq. (1) holds for $\mu \in SE(3)$, then the SG manipulator is called *congruent*. Moreover, if Eq. (1) holds for an orientation reversing congruence transformation μ, then the non-planar equiform SG platform is called *reflection-congruent*.[3]

Without loss of generality (w.l.o.g.) we can choose Cartesian coordinate systems in the platform and base of an equiform SG platform in a way that

$$A_i = \rho a_i, \quad B_i = \rho b_i, \quad C_i = \rho c_i, \tag{2}$$

holds for $i = 1, \ldots 6$, where $\rho \in \mathbb{R} \setminus \{0, 1\}$ denotes the similarity factor (cf. Footnote 2). Note that for $\rho = 1$ we get a congruent SG manipulator and that $\rho = 0$ has to be excluded, as otherwise the base collapses into a single point. In this context it should also be mentioned that ρ equals -1 for reflection-congruent SG manipulators.

Moreover we can assume for the remainder of this article that all platform anchor points are distinct, as otherwise two legs coincide due to the similarity of the platform and the base.

1.1. Cylinders of revolution

In this section we review some results on cylinders of revolution, as they play a central role in the study of non-planar congruent/equiform SG manipulators with non-translational self-motions (cf. Theorems 1 and 3).

[1] This notation was introduced by Karger in [8].

[2] An equiform motion is a composition of an Euclidean motion and a similarity transformation.

[3] Note that the notation "reflection-congruent" only makes sense for non-planar equiform SG platforms, as in the planar case the composition of μ with the reflection on the carrier plane of the anchor points yields an element of SE(3).

A cylinder of revolution Φ equals the set of all points, which have equal distance to its rotation axis s (finite line). Under the assumption that Φ has at least one real point, we can distinguish the following four cases:

1. s is real and Φ is not reducible: Φ is a cylinder of revolution over $\mathbb{R}$.
2. s is real and Φ is reducible: Φ equals a pair of isotropic planes[4] γ_1 and γ_2, which are conjugate complex. Trivially s carries the only real points of Φ.
3. s is imaginary and Φ is not reducible: Φ is a cylinder of revolution over $\mathbb{C}$. The real points of Φ are located on the 4th order intersection curve of Φ and its conjugate $\overline{\Phi}$.
4. s is imaginary and Φ is reducible: In this case Φ equals a pair of isotropic planes γ_1 and γ_2, which are not conjugate complex. Moreover Φ contains two real lines g_i $(i = 1, 2)$, which are the intersections of γ_i and its isotropic conjugate $\overline{\gamma}_i$.

Note that due to our restriction not all cylinders of revolution appear as solution, e.g. imaginary cylinders (real axis and imaginary radius).

Remark 1. It is a well known fact from projective geometry that the axis s is the line, where the tangent planes γ_1 and γ_2 through s onto Φ are isotropic. ◇

Now we focus on the determination of all cylinders of revolution through a given set of real points $\mathsf{X}_1, \ldots, \mathsf{X}_n$. There exist many papers on this well studied problem (see e.g. [3, 13, 14] and the references therein). In the following we want to use the computational approach of Schaal [13], which was furthered by Zsombor-Murray and El Fashny in [14]. They pointed out that this problem is equivalent with the solution of the following system of equations, if X_1 equals the origin U of the reference frame:

$$\mathbf{s}^2 = 1, \tag{3}$$

$$\Upsilon: \quad \mathbf{s} \cdot \mathbf{t} = 0, \tag{4}$$

$$\Omega_i: \quad (\mathbf{x}_i \times \mathbf{s})^2 - 2\mathbf{s}^2(\mathbf{x}_i \cdot \mathbf{t}) = 0, \tag{5}$$

for $i = 2, \ldots, n$, where $\mathbf{x}_i$ is the coordinate vector of the point X_i, $\mathbf{s} := (s_1, s_2, s_3)^T$ the direction vector of the rotation axis s, and $\mathbf{t} := (t_1, t_2, t_3)^T$ is coordinate vector of the footpoint T on s with respect to $\mathsf{U} = \mathsf{X}_1$ (cf. Fig. 1b).

The rough procedure for solving this system of equations is as follows: In the first step, one solves the equations $\Upsilon, \Omega_2, \ldots, \Omega_n$, which already gives the solutions up to a common factor; i.e., we get $s_1 : s_2 : s_3 : t_1 : t_2 : t_3$. In the second step, we normalize these 6-tuples with respect to the normalizing condition given in Eq. (3). This normalization is always possible as the axis cannot be isotropic[5], because it is the intersection of two isotropic planes (cf. Remark 1).

Remark 2. For $n = 5$ there exist in general six cylinders of revolution over $\mathbb{C}$ (e.g. [14]). There even exist examples, where all six cylinders are real (e.g. [3]). For $n > 5$ no solution exists, if $\mathsf{X}_1, \ldots, \mathsf{X}_n$ are in general configuration. ◇

[4]A plane is called *isotropic*, if its ideal line is tangent to the absolute conic.
[5]A line is called *isotropic*, if its ideal point is located on the absolute quadric.

1.2. Bond theory

In this section we give a short introduction into the theory of bonds for SG manipulators presented in [11], which was motivated by the bond theory of overconstrained closed linkages with revolute joints given by Hegedüs, Schicho and Schröcker in [4] (see also [5]). We start with the direct kinematic problem of parallel manipulators of SG type and further with the definition of bonds.

Due to the result of Husty [6], it is advantageous to work with *Study parameters* $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$ for solving the forward kinematics. Note that the first four homogeneous coordinates $(e_0 : e_1 : e_2 : e_3)$ are the so-called *Euler parameters*. Now all real points of the 7-dimensional Study parameter space P^7, which are located on the so-called *Study quadric* $\Psi : \sum_{i=0}^{3} e_i f_i = 0$, correspond to an Euclidean displacement, with exception of the 3-dimensional subspace E of Ψ given by $e_0 = e_1 = e_2 = e_3 = 0$, as its points cannot fulfill the condition $N \neq 0$ with $N = e_0^2 + e_1^2 + e_2^2 + e_3^2$. The translation vector $\mathbf{v} := 2(v_1, v_2, v_3)^T$ and the rotation matrix $\mathbf{R} := (r_{ij})$ of the corresponding Euclidean displacement $\mathbf{Rx} + \mathbf{v}$ are given by:

$$\begin{aligned} v_1 &= e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2, \\ v_2 &= e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3, \\ v_3 &= e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1, \end{aligned}$$

and

$$\mathbf{R} = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_1 e_2 + e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}, \tag{6}$$

if the normalizing condition $N = 1$ is fulfilled. All points of the complex extension of P^7, which cannot fulfill this normalizing condition, are located on the so-called *exceptional cone* $N = 0$ with vertex E.

By using the Study parametrization of Euclidean displacements the condition that the point m_i is located on a sphere centered in M_i with radius R_i, is a quadratic homogeneous equation according to Husty [6]. This so-called *sphere condition* Λ_i has the following form:

$$\begin{aligned} \Lambda_i\colon\ & (a_i^2 + b_i^2 + c_i^2 + A_i^2 + B_i^2 + C_i^2 - R_i^2)N - 2(a_i A_i + b_i B_i + c_i C_i)e_0^2 - 2(a_i A_i - b_i B_i - c_i C_i)e_1^2 \\ & + 2(a_i A_i - b_i B_i + c_i C_i)e_2^2 + 2(a_i A_i + b_i B_i - c_i C_i)e_3^2 + 4(c_i B_i - b_i C_i)e_0 e_1 - 4(c_i A_i - a_i C_i)e_0 e_2 \\ & + 4(b_i A_i - a_i B_i)e_0 e_3 - 4(b_i A_i + a_i B_i)e_1 e_2 + 4(a_i - A_i)(e_0 f_1 - e_1 f_0) + 4(a_i + A_i)(e_3 f_2 - e_2 f_3) \\ & - 4(c_i A_i + a_i C_i)e_1 e_3 - 4(c_i B_i + b_i C_i)e_2 e_3 + 4(b_i - B_i)(e_0 f_2 - e_2 f_0) + 4(b_i + B_i)(e_1 f_3 - e_3 f_1) \\ & + 4(c_i - C_i)(e_0 f_3 - e_3 f_0) + 4(c_i + C_i)(e_2 f_1 - e_1 f_2) + 4(f_0^2 + f_1^2 + f_2^2 + f_3^2) = 0. \end{aligned} \tag{7}$$

Now the solution of the direct kinematics over $\mathbb{C}$ can be written as the algebraic variety V of the ideal $\mathcal{I}$ spanned by $\Psi, \Lambda_1, \ldots, \Lambda_6$, and $N = 1$. In general V consists of a discrete set of points with a maximum of 40 elements.

We consider the *algebraic motion* of the mechanism, which are the points on the Study quadric that the constraints define; i.e., the common points of the seven quadrics $\Psi, \Lambda_1, \ldots, \Lambda_6$. If the manipulator has a n-dimensional self-motion then the algebraic motion also has to be of this dimension. Now the points of the algebraic motion with $N \neq 0$ equal the kinematic image of V. But we can also consider the points of the algebraic motion, which belong to the exceptional cone $N = 0$. An exact mathematical definition of these so-called bonds can be given as follows (cf. Remark 5 of [11]):

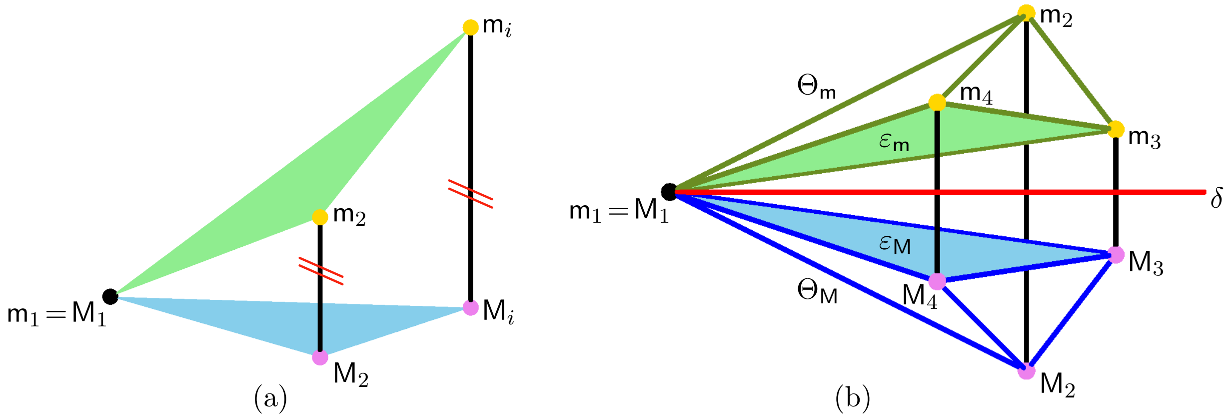


Figure 2: (a) Illustration of the condition given in Eq. (8) with $\mathsf{m}_1 = \mathsf{M}_1$. (b) The tetrahedra Θ_m and Θ_M are symmetric with respect to the plane δ, which is projecting in this sketch.

Definition 2. For a SG manipulator the *set $\mathcal{B}$ of bonds* is defined as:

$$\mathcal{B} := ZarClo(V^\star) \ \cap \ \{(e_0 : \ldots : f_3) \in P^7 \mid \Psi, \Lambda_1, \ldots, \Lambda_6, N = 0\},$$

where $V^\star$ denotes the variety V after the removal of all components, which correspond to pure translational motions. Moreover $ZarClo(V^\star)$ is the Zariski closure of $V^\star$, i.e., the zero locus of all algebraic equations that also vanish on $V^\star$.

We have to restrict to non-translational motions for the following reason: A component of V, which corresponds to a pure translational motion, is projected to a single point O (with $N \neq 0$) of the Euler parameter space P^3 by the elimination of $f_0, \ldots, f_3$. Therefore the intersection of O and $N = 0$ equals $\varnothing$. Clearly, the kernel of this projection equals the group of translational motions. Moreover it is important to note that the set of bonds only depends on the geometry of the manipulator and not on the leg lengths (cf. Theorem 1 of [11]). For more details please see [11].

Due to Theorem 2 of [11] a SG platform possesses a pure translational self-motion, if and only if the platform can be rotated about the center $\mathsf{m}_1 = \mathsf{M}_1$ into a pose, where the vectors $\overrightarrow{\mathsf{M}_i\mathsf{m}_i}$ for $i = 2, \ldots, 6$ fulfill the condition

$$rk(\overrightarrow{\mathsf{M}_2\mathsf{m}_2}, \ldots, \overrightarrow{\mathsf{M}_6\mathsf{m}_6}) \leq 1. \tag{8}$$

Moreover all 1-dimensional self-motions are circular translations, which can easily be seen by considering a normal projection of the SG manipulator in direction of the parallel vectors $\overrightarrow{\mathsf{M}_i\mathsf{m}_i}$ for $i = 2, \ldots, 6$. If all these five vectors are zero-vectors, the platform and the base are congruent and therefore we get a congruent SG manipulator (cf. [12]), which has a well known 2-dimensional translational self-motion $\mathcal{T}$, if all legs have equal (non-zero) length.

2. Review and preliminary results

As congruent SG platforms can be seen as a special case of equiform manipulators, we start this section with a detailed review of their known self-motional behavior.

2.1. Congruent SG manipulators

In the case of planar platform and planar base there only exist translational self-motions, if the anchor points are not located on a conic section (cf. [9] and [10]). If the anchor points are located on a conic section, the manipulator is a so-called architecturally singular[6] one. Moreover, it is well known that architecturally singular manipulators possess self-motions in each pose over $\mathbb{C}$.

In the non-planar case the manipulator can only have non-translational self-motions beside the above-mentioned 2-dimensional translational self-motion $\mathcal{T}$. The geometric characterization for these non-planar congruent SG manipulators with non-translational self-motions is given in the following theorem, which will be proven by the author at the *16th International Conference on Geometry and Graphics* [12] by means of bond theory:

Theorem 1. *A non-planar congruent SG manipulator can have a real non-translational self-motion only if the six base (resp. platform) anchor points have equal distance to a finite line* s*, i.e., they are located on a cylinder of revolution of type 1, 3 or 4 listed in Section 1.1. Moreover this condition is also sufficient for the existence of self-motions over* $\mathbb{C}$*.*

Remark 3. Note that the cylinders of revolution of type 2 are missing in Theorem 1, as they violate the non-planarity condition. Although this result is known, a complete list of all possible non-translational self-motions of congruent SG platforms is still missing. Moreover a restriction of the sufficiency condition with respect to $\mathbb{R}$ also remains open. ◇

In this paper we are interested in an extension of Theorem 1 to equiform SG manipulators, for which the following is known until now:

2.2. Equiform SG manipulators

Equiform SG manipulators with planar platform and planar base are special cases of so-called *planar affine* SG manipulators, which were already discussed in detail by the author in [10]. Due to Remark 2 of [10] and the work [8] of Karger, it is well known that planar equiform SG manipulators only have self-motions, if the anchor points are located on a conic section; i.e., in the case of architecturally singularity. Therefore we can focus on the non-planar case, for which the following lemma gives information about the architecture singularity:

Lemma 1. *A non-planar equiform SG platform is architecturally singular, if and only if four anchor points are collinear. These manipulators possess self-motions in each pose over* $\mathbb{C}$*.*

As this lemma has exactly the same proof as Lemma 2 of [12], we can proceed with the following theorem on equiform SG manipulators with pure translational self-motions:

Theorem 2. *A non-planar equiform SG platform has translational self-motions, if and only if it is reflection-congruent. Moreover all these translational self-motions are 1-parametric circular translations.*

Proof: As the manipulator is non-planar, there exist four corresponding pairs of anchor points, which span a tetrahedron Θ_{m} and Θ_{M} in the platform and the base, respectively. After a perhaps necessary reindexing we can assume w.l.o.g. that these anchor points are $\mathsf{m}_1, \ldots, \mathsf{m}_4$ and $\mathsf{M}_1, \ldots, \mathsf{M}_4$, respectively (cf. Fig. 2b).

[6] A SG platform is called architecturally singular, if it is singular in every possible configuration.

If an equiform SG manipulator has a translational self-motion there has to exist an orientation of the platform with $rk\left(\overrightarrow{\mathsf{M}_2\mathsf{m}_2},\ldots,\overrightarrow{\mathsf{M}_6\mathsf{m}_6}\right)=1$ and $\mathsf{m}_1=\mathsf{M}_1$, as congruent SG platforms are excluded (cf. last two paragraphs of Section 1.2). We assume that the manipulator is in such a pose.

Due to our assumptions $\mathsf{m}_i\neq\mathsf{M}_i$ has to hold for at least one $i\in\{2,3,4\}$, as otherwise $\Theta_{\mathsf{m}}=\Theta_{\mathsf{M}}$ holds, which implies a congruent SG manipulator (a contradiction). W.l.o.g. we can assume that $i=2$ holds. As a consequence we can denote the ideal point of the line $[\mathsf{m}_2,\mathsf{M}_2]$ by P. There exist at least one face ε_{m} (resp. ε_{M}) of Θ_{m} (resp. Θ_{M}) through $\mathsf{m}_1=\mathsf{M}_1$, which does not contain P (cf. Fig. 2b). Therefore the linear mapping κ, which maps the points x of ε_{m} to points X of ε_{M} by :

$$\kappa:\ \mathsf{x}\mapsto\mathsf{X}:=\varepsilon_{\mathsf{M}}\cap[\mathsf{x},\mathsf{P}],$$

is well-defined. As $rk\left(\overrightarrow{\mathsf{M}_2\mathsf{m}_2},\ldots,\overrightarrow{\mathsf{M}_6\mathsf{m}_6}\right)=1$ has to hold, κ has to map the triangular face of Θ_{m} located in ε_{m} to the corresponding triangular face of Θ_{M} located in ε_{M}. By these three corresponding point pairs the affinity κ is uniquely determined.

As $\mathsf{m}_1=\mathsf{M}_1$ holds, the two planes ε_{m} and ε_{M} either intersect each other along a line g through $\mathsf{m}_1=\mathsf{M}_1$ or are identical. In the first case all points of g are fixed under κ and in the second case all points of the plane are fixed under κ. Therefore ρ can only equal -1 in both cases, as 1 is excluded due to Def. 1.

For $\rho=-1$ the reflection on the plane δ (cf. Fig. 2b) orthogonal to the line $[\mathsf{m}_2,\mathsf{M}_2]$ through $\mathsf{m}_1=\mathsf{M}_1$ maps the platform to the base in a way that each of the vectors $\overrightarrow{\mathsf{M}_i\mathsf{m}_i}$ for $i=3,\ldots,6$ either point in the direction of P or equals the zero-vector. This proves the first sentence of the theorem. The second one follows immediately from the last paragraph of Section 1.2. □

3. Non-translational self-motions

In the following we show that the necessary condition of non-planar equiform SG platforms for possessing non-translational self-motions is the same one as for the congruent case (cf. Theorem 1).

Theorem 3. *A non-planar equiform SG manipulator can have a real non-translational self-motion only if the six base (resp. platform) anchor points have equal distance to a finite line* s*, i.e., they are located on a cylinder of revolution of type 1, 3 or 4 listed in Section 1.1.*

Proof: This theorem can be proven similarly (but not analogously) as Theorem 1 by using the following fact: If a non-translational self-motion exists, the bond-set has to be non-empty. Therefore we have to determine the conditions for which the set of bonds consists of at least one element. The computation of these conditions is outlined next.

W.l.o.g. we can specify the coordinate systems of Eq. (2) by setting $a_1=b_1=b_2=c_1=c_2=c_3=0$. Moreover we choose the scale in a way that the distance from m_1 to m_2 equals the unit length; i.e., $a_2=1$. Finally we can assume (after a possible necessary reindexing of anchor points) that the first four points are not coplanar; i.e., $b_3c_4\neq 0$.

According to [11] the set of bonds can be computed as follows: We calculate $\Delta_{j,i}:=\Lambda_j-\Lambda_i$, which is only linear in the Study parameters $f_0,\ldots,f_3$. Under the assumption that the motion is real and that the following two conditions are not fulfilled simultaneously[7]

$$e_0 = 0, \quad \rho = -1, \tag{9}$$

we can solve the linear system of equations $\Psi, \Delta_{2,1}, \Delta_{3,1}, \Delta_{4,1}$ for f_0, f_1, f_2, f_3 w.l.o.g.. We plug the obtained expressions for f_0, f_1, f_2, f_3 into $\Lambda_1, \Delta_{5,1}, \Delta_{6,1}$ and consider their numerators, which are homogeneous polynomials P_1, P_5 and P_6, respectively. P_1 is of degree six in the Euler parameters in contrast to P_5 and P_6 which determine quadrics in the Euler parameters space.

We eliminate e_0 from P_i and $N = 0$ by computing the resultant Q_i of these two expressions for $i = 1, 5, 6$. Now Q_i can only vanish without contradiction, if the following factor F_i vanishes:

$$F_1 = \sum_{j+k+l=3} g_{jkl} e_1^j e_2^k e_3^l \quad \text{for} \quad j,k,l \in \{0,1,2,3\}$$

with

$$\begin{aligned}
&g_{210} = -b_3^2 c_4, \quad g_{111} = -2b_3 b_4(a_3 - a_4), \quad g_{003} = b_3 b_4(b_3 - b_4) + b_4 a_3(a_3 - 1) - b_3 a_4(a_4 - 1),\\
&g_{120} = b_3 c_4(2a_3 - 1), \quad g_{201} = b_3(b_3 b_4 - b_4^2 - c_4^2), \quad g_{021} = b_4 a_3(a_3 - 1) - b_3 a_4(a_4 - 1) - b_3 c_4^2,\\
&g_{300} = 0, \quad g_{102} = b_3 c_4(2a_4 - 1), \quad g_{012} = -c_4(a_3^2 - a_3 + b_3^2 - 2b_3 b_4), \quad g_{030} = -a_3 c_4(a_3 - 1),
\end{aligned}$$

and

$$F_t = \sum_{j+k+l=2} g_{jkl} e_1^j e_2^k e_3^l \quad \text{for} \quad j,k,l \in \{0,1,2\}, \quad t \in \{5,6\}$$

with

$$\begin{aligned}
&g_{002} = a_t b_3 c_4(a_t - 1) - b_t c_4(a_3^2 + b_3^2 - b_3 b_t - a_3) + b_3 c_t(a_4 - a_4^2 - b_4^2) - b_4 c_t(a_3 - a_3^2 - b_3^2),\\
&g_{020} = a_t b_3 c_4(a_t - 1) - a_3 c_4 b_t(a_3 - 1) + a_3 b_4 c_t(a_3 - 1) - a_4 b_3 c_t(a_4 - 1) - b_3 c_4 c_t(c_4 - c_t),\\
&g_{200} = b_3 c_t(c_4 c_t - c_4^2 - b_4^2 + b_3 b_4) - b_3 b_t c_4(b_3 - b_t), \quad g_{011} = 2b_3 c_4 c_t(b_4 - b_t),\\
&g_{110} = 2b_3 b_t c_4(a_3 - a_t) - 2b_3 b_4 c_t(a_3 - a_4), \quad g_{101} = 2b_3 c_4 c_t(a_4 - a_t).
\end{aligned}$$

Remark 4. One has to check as well whether Q_i can always be computed by means of resultant. This is the case, if the coefficient K_i of the highest exponent of e_0 in P_i does not vanish. As the bonds do not depend on the leg lengths, K_i has to vanish independently from $R_1, \ldots, R_6$. It can easily be seen that this cannot be the case without contradicting our assumptions. ◇

Now the necessary condition for the existence of a bond is that the cubic F_1 and the two conics F_5 and F_6 in the projective plane spanned by e_1, e_2, e_3 have a point in common. Due to the number of variables and the degree of the involved equations, the corresponding algebraic conditions for the existence of a common point cannot be computed explicitly (e.g. by applying a resultant based elimination method), and therefore it seems that we cannot prove the theorem.

But due to Theorem 1, we conjecture that bonds can only exist, if the six anchor points are located on a cylinder of revolution. Therefore we consider the system of equations $\Upsilon, \Omega_2, \ldots, \Omega_6$ given in Eqs. (4) and (5) with respect to the six anchor points. We distinguish three cases:

[7]The exceptional case given in Eq. (9) is discussed separately in Section 3.4.

3.1. General case: $s_3 e_3 \neq 0$

W.l.o.g. we can solve $\Upsilon, \Omega_2, \Omega_3$, which are linear in t_1, t_2, t_3 for these unknowns. We plug the obtained expressions into $\Omega_4, \Omega_5, \Omega_6$ and consider their numerators, which are homogeneous polynomials G_4, G_5, G_6. After the substitution s_i by e_i for $i = 1, 2, 3$ the polynomials G_4, G_5, G_6 are denoted by H_4, H_5, H_6. These three polynomials are related with F_1, F_5, F_6 as follows:

$$F_1 = H_4, \quad F_5 = (c_5 H_4 - c_4 H_5)/e_3, \quad F_6 = (c_6 H_4 - c_4 H_6)/e_3.$$

Therefore the existence of a cylinder of revolution with $s_3 \neq 0$ through the six anchor points implies the existence of a bond with $e_3 \neq 0$ and vice versa.

3.2. Special case: $s_3 = e_3 = 0$ and $s_2 e_2 \neq 0$

W.l.o.g. we can solve $\Upsilon, \Omega_2, \Omega_4$ for t_1, t_2, t_3. We plug the obtained expressions into $\Omega_3, \Omega_5, \Omega_6$ and consider their numerators, which are homogeneous polynomials G_3, G_5, G_6. After the substitution s_i by e_i for $i = 1, 2$ the polynomials G_3, G_5, G_6 are denoted by H_3, H_5, H_6. These three polynomials are related with F_1, F_5, F_6 as follows:

$$F_1 = e_2 c_4 H_3, \quad F_5 = (b_4 c_5 - b_5 c_4) H_3 + b_3 H_5, \quad F_6 = (b_4 c_6 - b_6 c_4) H_3 + b_3 H_6.$$

Therefore the existence of a cylinder of revolution with $s_3 = 0$, $s_2 \neq 0$ through the six anchor points implies the existence of a bond with $e_3 = 0$, $e_2 \neq 0$ and vice versa.

3.3. Very special case: $s_2 = s_3 = e_2 = e_3 = 0$

If $e_1 = 0$ holds, the platform has the same orientation during the whole self-motion. As a consequence we can only end up with a translational self-motion; a contradiction. Therefore we can assume $e_1 \neq 0$.

Moreover we can also assume $s_1 \neq 0$, because otherwise the direction vector of the cylinder axis s equals the zero-vector (a contradiction). W.l.o.g. we can solve $\Upsilon, \Omega_3, \Omega_4$ for t_1, t_2, t_3. If we plug the obtained expression into Ω_2, we see that it is fulfilled identically. Therefore we consider the numerators of Ω_5, Ω_6, which are homogeneous polynomials G_5, G_6. After the substitution s_1 by e_1 the polynomials G_5, G_6 are denoted by H_5, H_6. As for $e_2 = e_3 = 0$ the polynomial F_1 is already fulfilled identically, we get the following relation between H_5, H_6 and F_5, F_6:

$$F_5 = b_3 H_5, \quad F_6 = b_3 H_6.$$

Therefore the existence of a cylinder of revolution with $s_2 = s_3 = 0$, $s_1 \neq 0$ through the six anchor points implies the existence of a bond with $e_2 = e_3 = 0$, $e_1 \neq 0$ and vice versa.

3.4. Exceptional case

Due to the above given study, we are left with the exceptional case of Eq. (9). We distinguish the following two cases:

- $e_1 \neq 0$: Under this assumption we can solve the linear system of equations $\Psi, \Delta_{2,1}$ for f_0, f_1 w.l.o.g.. We plug the obtained expressions for f_0, f_1 into $\Delta_{3,1}, \Delta_{4,1}$ and consider their numerators, which are homogeneous polynomials P_3 and P_4, respectively.
  We eliminate e_3 from P_i and $N = 0$ by computing the resultant Q_i of these two expressions for $i = 3, 4$. Now Q_3 can only vanish without contradiction for:

$$(a_3 e_1 + b_3 e_2)(a_3 e_1 - e_1 + b_3 e_2) = 0.$$

In both cases we can solve the linear equation for e_2 w.l.o.g.. If we plug the obtained expression into Q_4 we see that e_1^6 factors out and that the remaining expression, which only depends on the design parameters, decomposes in two quadratic factors with respect to a_4. The computation of a_4 from each of these factors can be done w.l.o.g. and shows that none of the obtained solutions for a_4 can be real. Therefore no bond exists; thus there cannot be a non-translational self-motion in this case.

- $e_1 = 0$: If $e_2 = 0$ holds, the platform has the same orientation during the whole self-motion. As a consequence we can only end up with a translational self-motion, which has to be a 1-dimensional circular translation due to Theorem 2. Therefore we can assume $e_2 \neq 0$.
  Under this assumption we can solve the linear system of equations $\Psi, \Delta_{3,1}$ for f_0, f_2 w.l.o.g.. We plug the obtained expressions for f_0, f_2 into $\Delta_{4,1}$ and consider its numerator, which is a homogeneous polynomial P_4. Now we eliminate e_3 by computing the resultant Q_4 of P_4 and N, which equals
  $$16b_3^2e_2^6(b_4^2 + c_4^2)[(b_3 - b_4)^2 + c_4^2].$$
  This resulting expression cannot vanish without contradiction over $\mathbb{R}$, thus also this case cannot yield a non-translational self-motion.

One also has to check in this exceptional case that Q_i can always be computed by means of resultant. It can easily be verified that Remark 4 (with respect to e_3 instead of e_0) also holds for the exceptional case, which closes the proof of Theorem 3. □

Finally it should be noted that in contrast to non-planar congruent SG platforms (cf. Theorem 1) nothing is known about the sufficiency of this common necessary condition (cf. Theorem 3) for the equiform case.

4. Examples

As translational self-motions of reflection-congruent SG manipulators are trivial (circular translations), we focus on equiform SG manipulators with non-translational self-motions. Until now only the following examples are known to the author, which are the equiform analogues (and therefore generalizations) of the examples given in Section 5 of [12]:

- Four anchor points are located on a line (architecture singular case). In this case the self-motions are the motions of the 5-legged manipulator, which results from the removal of one of the four legs, whose anchor points are collinear (cf. Lemma 1). For the corresponding cylinders of revolution please see Section 4.3 of [12].
- The anchor points split up into two triples of collinear points. In this case the self-motions are butterfly motions. For the corresponding cylinders of revolution please see Sections 4.2 and 5.1 of [12].
- The manipulator is plane-symmetric; i.e., the fourth, fifth and sixth anchor point are obtained by reflecting the first, second and third one on a plane ε. Therefore there always exists a cylinder of revolution Φ of type 1 with generators orthogonal to ε.
  W.l.o.g. we can assume that ε is the xy-plane and that the rotation axis of Φ is the z-axis. Moreover we can choose the scale in a way that the radius of Φ equals 1. Finally we can rotate the coordinate system about the z-axis that the first and second anchor point have

the same y-coordinate, which results in the following coordinatization:

$$a_1 = a_4 = \sin(\mu), \qquad a_2 = a_5 = \sin(-\mu), \qquad a_3 = a_6 = \sin(\lambda),$$
$$b_1 = b_4 = \cos(\mu), \qquad b_2 = b_5 = \cos(\mu), \qquad b_3 = b_6 = \cos(\lambda),$$

$c_1 = -c_4 \neq 0$, $c_2 = -c_5 \neq 0$, $c_3 = -c_6 \neq 0$ and the angles $\mu \in (0, \pi)$and $\lambda \in [0, 2\pi)$. The coordinates of the corresponding base anchor points are determined by Eq. (2). For the corresponding cylinders of revolution beside Φ please see Section 5.2 of [12].

These plane-symmetric equiform SG manipulators have the following non-translational self-motions characterized by $e_3 = 0$, which are new to the best knowledge of the author:

We compute the unknowns f_0, f_1, f_2, f_3 from $\Psi, \Delta_{2,1}, \Delta_{3,1}, \Delta_{4,1}$. If we plug the obtained expressions into $\Delta_{5,1}$, it can easily be seen that it vanishes for

$$R_5^2 = \frac{c_2}{c_1}(R_4^2 - R_1^2) + R_2^2.$$

Moreover, if additionally

$$R_6^2 = \frac{c_3}{c_1}(R_4^2 - R_1^2) + R_3^2$$

holds, $\Delta_{6,1}$ is fulfilled identically. Therefore only the condition $\Lambda_1 = 0$ remains, which is a homogeneous equation of degree 6 in the Euler parameters e_0, e_1, e_2. Hence for given five design parameters $c_1, c_2, c_3, \mu, \lambda$, this sextic implies a 4-parametric set of self-motions, as it depends on the four leg lengths R_1, R_2, R_3, R_4.

We close the paper by giving the following concrete example.

Example 1. The geometry of the plane-symmetric equiform SG manipulator is determined by

$$\mu = \pi/4, \quad \lambda = -3\pi/4, \quad c_1 = c_2 = c_3 = -1.$$

For the following choice of leg lengths[8]

$$R_1^2 = 6, \quad R_2^2 = 4, \quad R_3^2 = 6, \quad R_4^2 = 9, \quad R_5^2 = 7, \quad R_6^2 = 9,$$

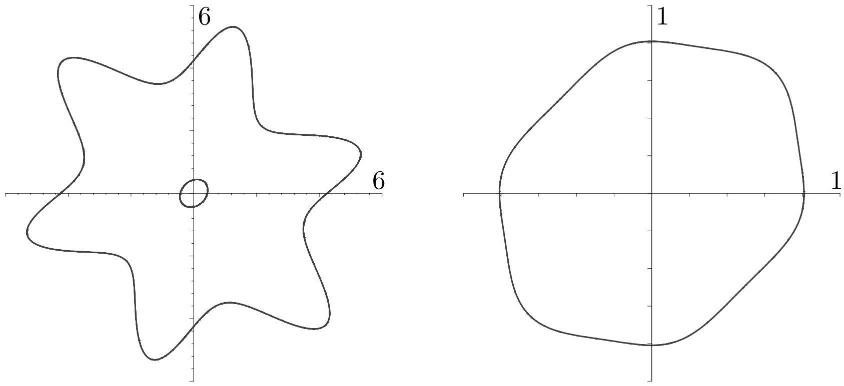
the sextic is displayed for $\rho = -1$ and $\rho = 2$ in Fig. 3. Animations of the corresponding self-motions can be downloaded as supplementary data from the author's homepage (cf. Footnote 8). ◇

5. Conclusions and outlook

In this paper we showed that the necessary condition of non-planar congruent SG manipulators for possessing non-translational self-motions (cf. Theorem 1) also holds for non-planar equiform SG manipulators (cf. Theorem 3). In contrast to non-planar congruent SG platforms nothing is known about the sufficiency of this common geometric characterization for the equiform case. This problem remains open and is dedicated to future research.

All known examples of equiform SG manipulators with non-translational self-motions are given in Section 4, where also a set of new self-motions is presented. Moreover we proved in

[8]Note that the input data $(\mu, \lambda, c_1, c_2, c_3, R_1, \ldots, R_6)$ is identical with the example given in the supplementary data (including animations) of the publication [12], which can be downloaded from the author's homepage `http://www.geometrie.tuwien.ac.at/nawratil`.



(a) (b)

Figure 3: We identify $e_0 = 0$ with the line at infinity and illustrate the affine part of the sextic; i.e., we set $e_0 = 1$ and plot e_1 horizontally and e_2 vertically for (a) $\rho = -1$ and (b) $\rho = 2$, respectively.

Theorem 2 that an equiform SG manipulator has translational self-motions, if and only if it is a so-called reflection-congruent one.

Finally it should be noted that we are interested in the generalization of this study with respect to the linear coupling of the non-planar platform and base. This problem is still open for the case where this mapping is an affinity or even a projectivity.

Acknowledgement

This research is funded by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project “Flexible polyhedra and frameworks in different spaces”, an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research. Moreover the author is supported by Grant No. P 24927-N25 for the FWF project “Stewart Gough platforms with self-motions”.

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Received October 2, 2013; final form November 20, 2013