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The Differentiable Manifold of Spherical Deltoids: Their Classification

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Abstract. We introduce the concepts of spherical deltoid of type I and spherical deltoid of type II, describing geometrical methods to construct both types. It is shown that any spherical deltoid is congruent to a spherical deltoid of type I and to a spherical deltoid of type II.
We classify spherical deltoids taking into account the relative positions of the spherical moons containing their sides. This allows us to conclude that the class of all spherical deltoids is a differentiable manifold of dimension three.

Key Words: spherical geometry, applications of spherical trigonometry

MSC 2010: 51E12, 51K05

1. Introduction

Let S^2 be the unit 2-sphere. A spherical deltoid D (Figure 1) is a convex spherical quadrangle with two congruent pairs of adjacent sides, but distinct from each other. Let us denote by a and b, with $a < b$, the length sides of D. The internal angles of D are denoted by $(\alpha_1, \alpha_2, \alpha_1', \alpha_3)$, in cyclic order.

Some of the obtained results in the paper are based in spherical trigonometry formulas. The cosine rules states that the angles α, β and γ of a (convex) spherical triangle satisfy

$$\cos\alpha = \frac{\cos c - \cos d\,\cos e}{\sin d\,\sin e} \quad \text{and} \quad \cos c = \frac{\cos\alpha + \cos\beta\,\cos\gamma}{\sin\beta\,\sin\gamma}\,, \tag{1}$$

where c, d and e are the lengths of the edges opposite to α, β and γ, respectively. For a detailed discussion on spherical trigonometry see [2].

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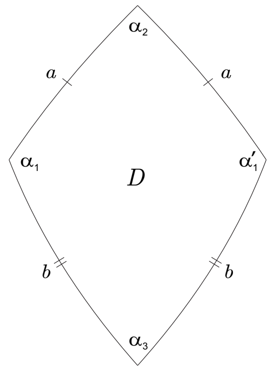

Figure 1: A spherical deltoid D

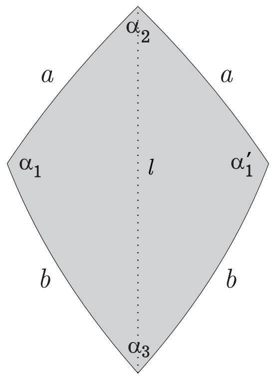

Figure 2: Diagonal through α_2 and α_3

Remark. With the above terminology, we have

$$\alpha_1 = \alpha_1' \quad \text{and} \quad \alpha_2 > \alpha_3.$$

In fact, if we consider the diagonal l of D through α_2 and α_3 (Figure 2), we get two congruent triangles, T and T', since they have equal length sides: a, b and l. And so $\alpha_1 = \alpha_1'$. Now, as $a < b$, by the cosine rules, we also obtain $\alpha_2 > \alpha_3$.

In this paper we introduce the notions of spherical deltoid of type I and spherical deltoid of type II.

2. Spherical deltoids

By a *spherical deltoid of type I* (*SDI*) we mean a spherical quadrangle D arising from the intersection of two spherical moons L_I^1, with vertices $v_1 = (0, \sin\phi, \cos\phi)$ and $-v_1$, and L_I^2, with vertices $v_2 = (0, -\sin\phi, \cos\phi)$ and $-v_2$, with $\phi \in \left(0, \frac{\pi}{2}\right)$ (see Figure 3).

We have used the following notation:

- θ is the angle measure of the spherical moons L_I^1 and L_I^2, $\theta \in (0, \pi)$;
- α_1, α_2, α_1', α_3, and a, a', b, b' are, respectively, the internal angles and the edge lengths (in cyclic order) of $D = L_I^1 \cap L_I^2$;
- ϕ is the oriented angle between $N = (0, 0, 1)$ and the vertex v_1, with $\phi \in \left(0, \frac{\pi}{2}\right)$;
- λ is the oriented angle between the line connecting v_1 (or v_2) and $C = (1, 0, 0)$, and the bisector of L_I^1. One has $\lambda \in \left(0, \frac{\pi - \theta}{2}\right)$.

Observe that any spherical deltoid of type I is a spherical deltoid (according to the initial definition). In fact, it is enough to see that the triangles X and X' (Figure 3) are congruent. And so $a = a'$. Analogously, $b = b'$. Now, it follows that $\alpha_1 = \alpha_1'$.

Note that

$$\cos\phi = \cos\frac{\alpha_2}{2} \sec\frac{\theta + 2\lambda}{2} \quad \text{and} \quad \cos(\pi - \phi) = -\cos\frac{\alpha_3}{2} \sec\frac{\theta - 2\lambda}{2},$$

implying that

$$\cos\frac{\alpha_3}{2} - \cos\frac{\alpha_2}{2} = 2\sin\frac{\theta}{2} \sin\lambda \cos\phi > 0$$

for all $\theta \in (0, \pi)$, $\lambda \in \left(0, \frac{\pi - \theta}{2}\right)$, $\phi \in \left(0, \frac{\pi}{2}\right)$, $\alpha_2, \alpha_3 \in (0, \pi)$, and so $\alpha_2 > \alpha_3$.

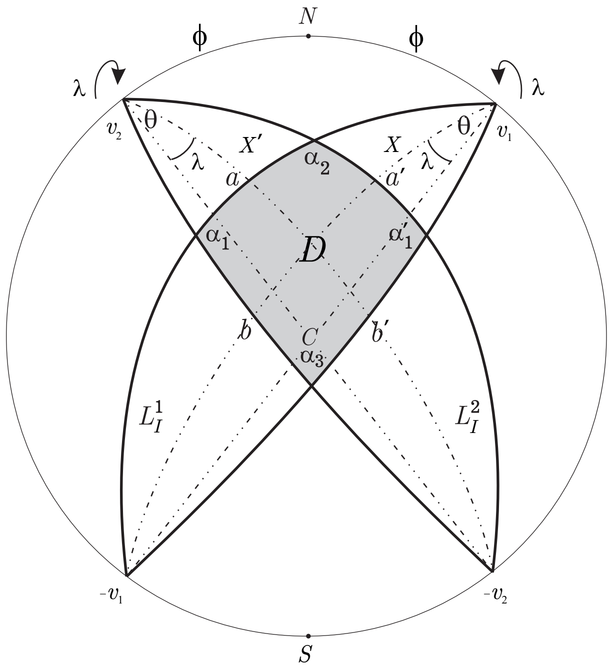

Figure 3: A spherical deltoid of type I

By a *spherical deltoid of type II* ($SDII$) we mean a spherical quadrangle D arising from the intersection of a well-centered spherical moon (a spherical moon whose vertices belong to the great circle $x = 0$, and whose bisecting semi-great circle contains the point $C = (1, 0, 0)$, see [1]), L_{II}^1, with vertices N and $S = -N$, and a spherical moon, L_{II}^2, with vertices in the great circle containing N and C, say $v = (\sin\phi, 0, \cos\phi)$ and $-v$, with $\phi \in (0, \pi)$ (see Figure 4). Observe that in this case a spherical deltoid of type $SDII$ is also a spherical deltoid.

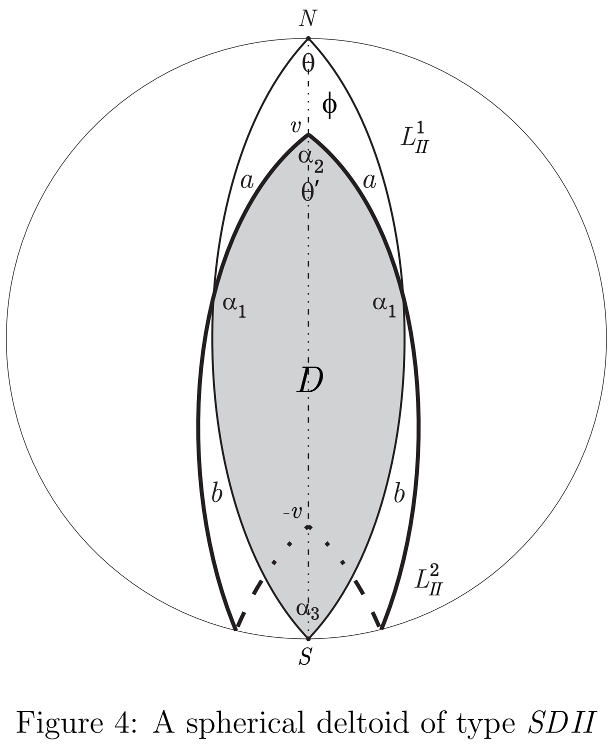

Figure 4: A spherical deltoid of type $SDII$

We have used the following notation:

- θ and θ' are the angle measures of the spherical moons L^1_{II} and L^2_{II}, with θ, $\theta' \in (0, \pi)$ and $\theta < \theta'$;
- α_1, α_2, α_3, and a, b are, respectively, the internal angles and the edge lengths of $D = L^1_{II} \cap L^2_{II}$;
- ϕ is the oriented angle between N and the vertex v, with $\phi \in (0, \pi)$.

Proposition 2.1. *Let D be a spherical deltoid with edge lengths (a, a, b, b), $a < b$, and internal angles $(\alpha_1, \alpha_2, \alpha_1, \alpha_3)$. Then, any two of these five parameters are completely determined by the remaining three.*

Proof. Let D be a spherical deltoid as described. We shall show how to determine α_1 and a as functions of α_2, α_3 and b. Other cases are treated in a similar way.
Let l be the diagonal of D through the angles of length α_1, as illustrated in Figure 5.

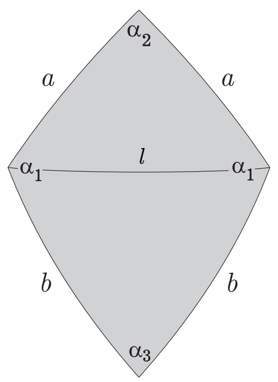

Figure 5: A spherical deltoid

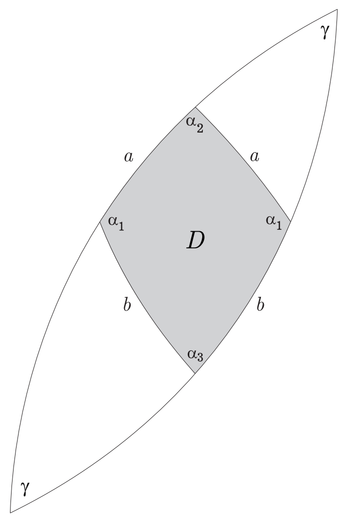

Figure 6: A spherical moon obtained by extending a pair of opposite sides of D

By (1), we have

$$\cos \alpha_2 = \frac{\cos l - \cos^2 a}{\sin^2 a} \quad \text{and} \quad \cos \alpha_3 = \frac{\cos l - \cos^2 b}{\sin^2 b},$$

and so

$$\sin a = \sin \frac{\alpha_3}{2} \csc \frac{\alpha_2}{2} \sin b\,. \tag{2}$$

Now, extending the sides a and b of D one gets a spherical moon as shown in Figure 6. Let γ be its angle measure. Using (1) again, one gets

$$\cos a = \frac{\cos \gamma + \cos \alpha_1 \cos \alpha_2}{\sin \alpha_1 \sin \alpha_2} \quad \text{and} \quad \cos b = \frac{\cos \gamma + \cos \alpha_1 \cos \alpha_3}{\sin \alpha_1 \sin \alpha_3},$$

and so

$$\cot \alpha_1 = \frac{\cos a \sin \alpha_2 - \cos b \sin \alpha_3}{\cos \alpha_2 - \cos \alpha_3}. \tag{3}$$

From Eqs. (2) and (3) we may obtain α_1 and a as functions of α_2, α_3 and b. Therefore, α_1 and a are completely determined when α_2, α_3 and b are fixed values. □

Proposition 2.2. *Any spherical deltoid is congruent to a SDI. Besides, its sides and angles are completely determined by the three parameters θ, ϕ and λ defined in Figure 3.*

Proof. Suppose that D is a spherical deltoid with internal angles $(\alpha_1, \alpha_2, \alpha_1, \alpha_3)$, and edge lengths (a, b, a, b), $a < b$. The extension of the two pairs of opposite sides of D give rise to spherical moons L_1 and L_2, such that $D = L_1 \cap L_2$. It is a straightforward exercise to show that there is a spherical isometry σ such that $\sigma(L_1)$ and $\sigma(L_2)$ are spherical moons such that their vertices belong to the great circle $x = 0$, i.e., $\sigma(L_1) = L_I^1$ and $\sigma(L_2) = L_I^2$ (Figure 3). It also follows that D is congruent to a *SDI*. By Proposition 2.1, the knowledge of α_2, α_3 and b determines α_1 and a.

Using the labelling of Figure 3, one gets the following system of equations in the three variables θ, ϕ and λ.

$$\begin{cases} \cos\alpha_2 &= 2\cos^2\phi\,\cos^2\dfrac{\theta+2\lambda}{2} - 1\,, \\ \cos\alpha_3 &= 2\cos^2\phi\,\cos^2\dfrac{\theta-2\lambda}{2} - 1\,, \\ \cos b &= \dfrac{\cos\theta + \cos\alpha_1\cos\alpha_3}{\sin\alpha_1\sin\alpha_3}\,. \end{cases}$$

Therefore, we obtain the expressions of θ, ϕ and λ from the equivalent 3×3 system of equations,

$$\begin{cases} \cos\theta &= \cos b\sin\alpha_1\sin\alpha_3 - \cos\alpha_1\cos\alpha_3\,, \\ \cos^2\phi &= \dfrac{2+\cos\alpha_2+\cos\alpha_3 - 4\cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2}\cos\theta}{2\sin^2\theta}\,, \\ \cos^2\lambda &= \left(\cos\dfrac{\alpha_2}{2}+\cos\dfrac{\alpha_3}{2}\right)^2 \dfrac{1-\cos\theta}{2+\cos\alpha_2+\cos\alpha_3-4\cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2}\cos\theta}\,, \end{cases}$$

where $\alpha_1 = \operatorname{arccot}\dfrac{\cos a\sin\alpha_2 - \cos b\sin\alpha_3}{\cos\alpha_2 - \cos\alpha_3}$. □

Proposition 2.3. *Any spherical deltoid is congruent to a SDII. Besides, its sides and angles are completely determined by the three parameters θ, θ' and ϕ defined in Figure 4.*

Proof. Suppose that D is a spherical deltoid with internal angles, $(\alpha_1, \alpha_2, \alpha_1, \alpha_3)$, and edge lengths (a, b, a, b), $a < b$. The extension of the two pairs of sides of D, with lengths a and b, respectively, give rise to spherical moons L_1 and L_2, such that $D = L_1 \cap L_2$.

Now, it follows that there is a spherical isometry σ such that $\sigma(L_1)$ is a well-centered spherical moon with vertices N and S and $\sigma(L_2)$ is a spherical moon with vertices in the great circle containing C and N. And so D is congruent to a *SDII*. By Proposition 2.1, the knowledge of α_2, α_3 and b determines α_1 and a.

Using the labelling of Figure 4, one gets the following system of equations,

$$\begin{cases} \alpha_2 &= \theta', \\ \alpha_3 &= \theta\,, \\ \cos b &= \dfrac{\cos\frac{\theta'}{2} + \cos\alpha_1\cos\frac{\theta}{2}}{\sin\alpha_1\sin\frac{\theta}{2}}\,, \end{cases}$$

and so

$$\begin{cases} \theta &= \alpha_3\,, \\ \theta' &= \alpha_2\,, \\ \phi &= \pi - \arccos \dfrac{\cos\alpha_1 + \cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2}}{\sin\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2}}\,, \end{cases}$$

where $\alpha_1 = \operatorname{arccot} \dfrac{\cos a \sin\alpha_2 - \cos b \sin\alpha_3}{\cos\alpha_2 - \cos\alpha_3}$. □

Let $\mathcal{D}$ be the class of all spherical deltoids.

Corollary 2.1. *Either of the previous propositions (the degree of freedom given by the three parameters θ, ϕ and λ in the first case and θ, θ' and ϕ in the last case) allows us to conclude that $\mathcal{D}$ is a differentiable manifold of dimension three.*

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